Mathematica Bohemica

Rasoul Aghalary; Jafar Kazemzadeh

Some properties of certain subclasses of bounded Mocanu variation with respect to $2k$-symmetric conjugate points

Mathematica Bohemica, Vol. 144 (2019), No. 2, 191–202

Persistent URL: http://dml.cz/dmlcz/147759

SOME PROPERTIES OF CERTAIN SUBCLASSES OF BOUNDED MOCANU VARIATION WITH RESPECT TO $2k$-SYMMETRIC CONJUGATE POINTS

Rasoul Aghalary, Jafar Kazemzadeh, Urmia

Received December 18, 2017. Published online September 4, 2018.
Communicated by Grigore Sălăgean

Abstract. We introduce subclasses of analytic functions of bounded radius rotation, bounded boundary rotation and bounded Mocanu variation with respect to $2k$-symmetric conjugate points and study some of its basic properties.

Keywords: $2k$-symmetric conjuqate points; bounded Mocanu variation; bounded radius rotation; bounded boundary rotation

MSC 2010: 30C45, 30C80

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f defined on the unit disc $E=\{z\in\mathbb{C}\colon\ |z|<1\}$, normalized by $f(0)=f'(0)-1=0$ and of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,\quad z\in E. \tag{1.1}$$

Also, let S, K, S^* and C denote the subclasses of $\mathcal{A}$ which are univalent, close-to-convex, starlike and convex in E, respectively. Let $P_m(\gamma)$ be the class of functions $p(z)$ analytic in the unit disc E satisfying the properties $p(0)=1$ and for $z=re^{\mathrm{i}\theta}$, $m\geqslant 2$,

$$\int_0^{2\pi}\Big|\mathrm{Re}\,\frac{p(z)-\gamma}{1-\gamma}\Big|\,\mathrm{d}\theta\leqslant m\pi,\quad 0\leqslant\gamma<1. \tag{1.2}$$

The class $P_m(\gamma)$ for $\gamma=0$ and $0\leqslant\gamma<1$ has been introduced and investigated by Pinchuk in [6], and Padmanabhan and Parvatham in [5], respectively. We note that

DOI: 10.21136/MB.2018.0141-17

$P_m(0) = P_m$ and $P_2(\gamma) = P(\gamma)$ is the class of analytic functions with positive real part greater than γ. For $m = 2$ and $\gamma = 0$ we have the class P of functions with positive real part. We can write (1.2) as

$$p(z) = \frac{1}{2}\int_0^{2\pi} \frac{1 + (1 - 2\gamma)z\mathrm{e}^{-\mathrm{i}t}}{1 - z\mathrm{e}^{-\mathrm{i}t}}\,\mathrm{d}\mu(t), \tag{1.3}$$

where $\mu(t)$ is a function with bounded variation on $[0, 2\pi]$ such that

$$\int_0^{2\pi} \mathrm{d}\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |\mathrm{d}\mu(t)| \leqslant m. \tag{1.4}$$

Also, for $p \in P_m(\gamma)$ we can write from (1.2)

$$p(z) = \Big(\frac{m}{4} + \frac{1}{2}\Big)p_1(z) - \Big(\frac{m}{4} - \frac{1}{2}\Big)p_2(z), \quad p_1, p_2 \in P_2(\gamma),\ z \in E. \tag{1.5}$$

It is known [3] that $P_m(\gamma)$ is a convex set. Also $p \in P_m(\gamma)$ is in $P_2(\gamma) = P(\gamma)$ for $|z| < r_1$, where

$$r_1 = \frac{1}{2}\big(m - \sqrt{m^2 - 4}\big). \tag{1.6}$$

The classes $V_m(\gamma)$ of functions of bounded boundary rotation of order γ and $R_m(\gamma)$ of functions of bounded radius rotation of order γ are closely related with $P_m(\gamma)$. A function $f \in \mathcal{A}$ is in $V_m(\gamma)$ if and only if $(zf'(z))'/f'(z) \in P_m(\gamma)$. Also

$$f \in R_m(\gamma) \Leftrightarrow \frac{zf'(z)}{f(z)} \in P_m(\gamma). \tag{1.7}$$

It is clear that

$$f \in V_m(\gamma) \Leftrightarrow zf'(z) \in P_m(\gamma). \tag{1.8}$$

When $m = 2, \gamma = 0$, then $V_2(0)$ coincides with the class C and $R_2(0) = S^*$. Wang et al. in [9] introduced and investigated class $S_s^{(k)}(\varphi)$, which satisfies the inequality:

$$\frac{zf'(z)}{f_k(z)} \prec \varphi(z), \quad z \in E,$$

where $\varphi(z) \in P$, $k \geqslant 2$ is a fixed positive integer and $f_k(z)$ is defined by the following equality:

$$f_k(z) = \frac{1}{k}\sum_{\upsilon=0}^{k-1} \varepsilon^{-\upsilon} f(\varepsilon^{\upsilon} z), \quad \varepsilon = \exp\frac{2\pi\mathrm{i}}{k},$$

and a function $f(z) \in E$ is in the class $C_s^{(k)}(\varphi)$ if and only if $zf'(z) \in S_s^{(k)}(\varphi)$. Also Wang and Gao (see [9]) introduced and investigated two classes $S_{sc}^{(k)}(\varphi)$ and $C_{sc}^{(k)}(\varphi)$ of functions starlike and convex with respect to $2k$-symmetric conjugate points. Noor and Mustafa in [2] introduced and investigated class $R_s^k(\gamma)$ of analytic functions which are of bounded radius rotation of order γ with respect to symmetrical points if and only if

$$\frac{2zf'(z)}{f(z)-f(-z)} \in P_k(z), \quad z \in E.$$

We now define the following.

Definition 1.1. Let $f \in \mathcal{A}$. Then f is said to be of bounded radius rotation of order γ with respect to $2k$-symmetric conjugate points if and only if

$$\frac{zf'(z)}{f_{2k}(z)} \in P_m(\gamma), \quad z \in E, \tag{1.9}$$

where $k \geqslant 1$ is a fixed positive integer and $f_{2k}(z)$ is defined as

$$f_{2k}(z) = \frac{1}{2k}\sum_{\upsilon=0}^{k-1}(\varepsilon^{-\upsilon}f(\varepsilon^{\upsilon}z) + \varepsilon^{\upsilon}\overline{f(\varepsilon^{\upsilon}\overline{z})}), \quad \varepsilon = \exp\frac{2\pi \mathrm{i}}{k}. \tag{1.10}$$

We shall denote the class of such functions as $R_m^{s-2k}(\gamma)$. We note that $R_2^{s-2}(\gamma)$ is the class S_s^* of univalent functions starlike with respect to symmetrical points defined by Sakaguchi (see [8]). Also we define the class $V_m^{s-2k}(\gamma)$ as follows.

Definition 1.2.

$$f \in V_m^{s-2k}(\gamma) \Leftrightarrow zf' \in R_m^{s-2k}(\gamma), \quad z \in E. \tag{1.11}$$

Motivated by the above-mentioned classes we now define the following subclasses of analytic functions.

Definition 1.3. Let $f \in \mathcal{A}$ and $f(z)f'(z)z^{-1} \neq 0$ for $z \in E$. Then f is said to be of bounded Mocanu variation of order γ with respect to $2k$-symmetric conjugate points if and only if

$$\alpha\frac{zf'(z)}{f_{2k}(z)} + (1-\alpha)\frac{(zf'(z))'}{f_{2k}'(z)} \in P_m(\gamma), \quad z \in E, \tag{1.12}$$

where $0 \leqslant \alpha \leqslant 1$ and $k \geqslant 1$ is a fixed positive integer and $f_{2k}(z)$ is defined by (1.10). We shall denote the class of such functions as $\mathcal{M}_m^{s-2k}(\alpha,\gamma)$.

Definition 1.4. Let $f \in \mathcal{A}$ and $f(z)f'(z)z^{-1} \neq 0$ for $z \in E$. Then f belongs to the class $\mathcal{H}^{s-2k}_{m,m_1}(\alpha,\gamma)$ if

$$\alpha\frac{zf'(z)}{g_{2k}(z)} + (1-\alpha)\frac{(zf'(z))'}{g'_{2k}(z)} \in P_m(\gamma), \tag{1.13}$$

where $0 \leqslant \alpha \leqslant 1$ and $k \geqslant 1$ is a fixed positive integer and $g_{2k}(z)$ is defined as

$$g_{2k}(z) = \frac{1}{2k}\sum_{\upsilon=0}^{k-1}(\varepsilon^{-\upsilon}g(\varepsilon^{\upsilon}z) + \varepsilon^{\upsilon}\overline{g(\varepsilon^{\upsilon}\overline{z})}), \quad \varepsilon = \exp\frac{2\pi \mathrm{i}}{k} \tag{1.14}$$

with $g \in \mathcal{M}^{s-2k}_{m_1}(\alpha,\gamma)$.

For simplicity, we write $\mathcal{H}^{s-2k}_{m,m}(\alpha,\gamma) =: \mathcal{H}^{s-2k}_{m}(\alpha,\gamma)$.

In our investigation of the classes $R^{s-2k}_m(\gamma)$, $V^{s-2k}_m(\gamma)$, $\mathcal{M}^{s-2k}_m(\alpha,\gamma)$ and $\mathcal{H}^{s-2k}_{m,m_1}(\alpha,\gamma)$ we need the following lemmas.

Lemma 1.1 ([1]). *Let p be an analytic function in the unit disc with $P(0) = a$, where $\operatorname{Re} a > 0$. Let $P\colon E \to \mathbb{C}$ be a function such that $\operatorname{Re} P(z) > 0$ for $z \in E$. Then*

$$\operatorname{Re}[p(z) + P(z)zp'(z)] > 0 \Rightarrow \operatorname{Re} p(z) > 0.$$

Lemma 1.2 ([1]). *Let $\beta, \gamma \in \mathbb{C}$ and h be convex and univalent function in E with*

$$h(0) = 1 \quad \text{and} \quad \operatorname{Re}(\beta h(z) + \gamma) > 0, \quad z \in E.$$

If p is analytic in E with $p(0) = 1$, then subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z)$$

implies that

$$p(z) \prec h(z).$$

2. Basic properties of $R_m^{s-2k}(\gamma)$, $V_m^{s-2k}(\gamma)$, $\mathcal{M}_m^{s-2k}(\alpha,\gamma)$ and $\mathcal{H}_{m,m_1}^{s-2k}(\alpha,\gamma)$

Theorem 2.1. *Let $f \in \mathcal{M}_m^{s-2k}(\alpha,\gamma)$. Then the function*

$$\psi(z) = f_{2k}(z) \tag{2.1}$$

belongs to $\mathcal{M}_m^{s-2k}(\alpha,\gamma)$.

P r o o f. Let $f \in \mathcal{M}_m^{s-2k}(\alpha,\gamma)$. Then from Definition 1.3 we have

$$\alpha\frac{zf'(z)}{f_{2k}(z)} + (1-\alpha)\frac{(zf'(z))'}{f'_{2k}(z)} \in P_m(\gamma), \quad z \in E,$$

or

$$\alpha\frac{zf'(z)}{f_{2k}(z)} + (1-\alpha)\frac{f'(z)+zf''(z)}{f'_{2k}(z)} \in P_m(\gamma), \quad z \in E. \tag{2.2}$$

Replacing z by $\varepsilon^{\upsilon}z$, $\upsilon = 0,1,2,\ldots,k-1$ in (2.2) leads to

$$\alpha\frac{\varepsilon^{\upsilon}zf'(\varepsilon^{\upsilon}z)}{f_{2k}(\varepsilon^{\upsilon}z)} + (1-\alpha)\frac{f'(\varepsilon^{\upsilon}z)+\varepsilon^{\upsilon}zf''(\varepsilon^{\upsilon}z)}{f'_{2k}(\varepsilon^{\upsilon}z)} \in P_m(\gamma). \tag{2.3}$$

We note that

$$\begin{gathered} f_{2k}(\varepsilon^{\upsilon}z) = \varepsilon^{\upsilon}f_{2k}(z), \quad f'_{2k}(\varepsilon^{\upsilon}z) = f'_{2k}(z), \\ \overline{f_{2k}(\varepsilon^{\upsilon}\overline{z})} = \varepsilon^{-\upsilon}f_{2k}(z), \quad \overline{f'_{2k}(\varepsilon^{\upsilon}\overline{z})} = f'_{2k}(z), \quad \psi_{2k}(z) = f_{2k}(z). \end{gathered} \tag{2.4}$$

Thus, in view of (2.3) and (2.4) we obtain

$$\alpha\frac{zf'(\varepsilon^{\upsilon}z)}{f_{2k}(z)} + (1-\alpha)\frac{f'(\varepsilon^{\upsilon}z)+\varepsilon^{\upsilon}zf''(\varepsilon^{\upsilon}z)}{f'_{2k}(z)} \in P_m(\gamma) \tag{2.5}$$

and

$$\alpha\frac{z\overline{f'(\varepsilon^{\upsilon}\overline{z})}}{f_{2k}(z)} + (1-\alpha)\frac{\overline{f'(\varepsilon^{\upsilon}\overline{z})}+\varepsilon^{-\upsilon}z\overline{f''(\varepsilon^{\upsilon}\overline{z})}}{f'_{2k}(z)} \in P_m(\gamma). \tag{2.6}$$

Since $P_m(\gamma)$ is a convex set, summing (2.5) and (2.6) leads to

$$\begin{aligned} &\alpha\frac{\frac{1}{2}z\big(f'(\varepsilon^{\upsilon}z)+\overline{f'(\varepsilon^{\upsilon}\overline{z})}\big)}{f_{2k}(z)} \\ &\quad + (1-\alpha)\frac{\frac{1}{2}\big(f'(\varepsilon^{\upsilon}z)+\overline{f'(\varepsilon^{\upsilon}\overline{z})}\big)+\frac{1}{2}z\big(\varepsilon^{\upsilon}f''(\varepsilon^{\upsilon}z)+\varepsilon^{-\upsilon}\overline{f''(\varepsilon^{\upsilon}\overline{z})}\big)}{f'_{2k}(z)} \in P_m(\gamma). \end{aligned} \tag{2.7}$$

Putting $\upsilon = 0, 1, 2, \ldots, k-1$ in (2.7) and summing the resulting equations yields

$$\alpha \frac{\frac{1}{2} z k^{-1} \sum_{\upsilon=0}^{k-1} \left(f'(\varepsilon^{\upsilon} z) + \overline{f'(\varepsilon^{\upsilon} \overline{z})}\right)}{f_{2k}(z)} + (1-\alpha) \frac{\frac{1}{2} k^{-1} \sum_{\upsilon=0}^{k-1} \left(f'(\varepsilon^{\upsilon} z) + \overline{f'(\varepsilon^{\upsilon} \overline{z})} + z\left(\varepsilon^{\upsilon} f''(\varepsilon^{\upsilon} z) + \varepsilon^{-\upsilon} \overline{f''(\varepsilon^{\upsilon} \overline{z})}\right)\right)}{f'_{2k}(z)} \in P_m(\gamma)$$

and hence $\psi \in P_k(\gamma)$ in E. □

Putting $\alpha = 0, 1$ in Theorem 2.1 we have the following results for the classes $R_m^{s-2k}(\gamma)$ and $V_m^{s-2k}(\gamma)$.

Corollary 2.1. *Let $f \in R_m^{s-2k}(\gamma)$. Then the function $\psi(z) = f_{2k}(z)$ belongs to $R_m^{s-2k}(\gamma)$ in E.*

Corollary 2.2. *Let $f \in V_m^{s-2k}(\gamma)$. Then the function $\psi(z) = f_{2k}(z)$ belongs to $V_m^{s-2k}(\gamma)$ in E.*

In order to prove our next result we need the following lemma.

Lemma 2.1. *Let p and φ be analytic functions in E with $p(0) = 1$ and $\operatorname{Re} \varphi(z) > 0$ for $z \in E$. If*

$$p(z) + \varphi(z) z p'(z) \in P_m(\gamma),$$

then $p(z) \in P_m(\gamma)$.

Proof. From the definition of $P_m(\gamma)$ there exist $q_1, q_2 \in P_2(\gamma)$ such that

$$p(z) + \varphi(z) z p'(z) = m q_1(z) + (1-m) q_2(z). \tag{2.8}$$

Let p_1 and p_2 be the solutions of the Cauchy problems

$$p(z) + \varphi(z) z p'(z) = q_1(z), \quad p(0) = 1 \tag{2.9}$$

and

$$p(z) + \varphi(z) z p'(z) = q_2(z), \quad p(0) = 1, \tag{2.10}$$

respectively. In view of (2.9) and (2.10) we rewrite (2.8) as

$$p(z) + \varphi(z) z p'(z) = m(p_1(z) + \varphi(z) z p_1'(z)) + (1-m)(p_2(z) + \varphi(z) z p_2'(z)),$$

or equivalently,

$$(p(z) - mp_1(z) - (1-m)p_2(z)) + z\varphi(z)(p'(z) - mp'_1(z) - (1-m)p'_2(z)) = 0. \tag{2.11}$$

Now if we define $h(z) = p(z) - mp_1(z) - (1-m)p_2(z)$, then $h(0) = 0$ and (2.11) yields

$$h(z) + \varphi(z)zh'(z) = 0, \quad h(0) = 0. \tag{2.12}$$

But it is clear that Cauchy problem (2.12) has the only solution $h(z) = 0$. Hence $p(z) = mp_1(z) + (1-m)p_2(z)$. For completing the proof we show that $p_1, p_2 \in P_2(\gamma)$. Form equation (2.9) we can write

$$\frac{q_1(z) - \gamma}{1-\gamma} = \frac{p_1(z) - \gamma}{1-\gamma} + \frac{\varphi(z)}{1-\gamma} zp'_1(z).$$

Since $\operatorname{Re}(q_1(z) - \gamma)/(1-\gamma) > 0$ and $\operatorname{Re}\varphi(z) > 0$, applying Lemma 1.1 we obtain $\operatorname{Re} p_1(z) > \gamma$. Similarly, we have $\operatorname{Re} p_2(z) > \gamma$ and this means that $p \in P_m(\gamma)$ and the proof is complete. □

Theorem 2.2. *Let $0 < \alpha \leqslant 1$, $k \geqslant 1$ and $m \geqslant 2$. Then*

$$\mathcal{H}^{s-2k}_{m,2}(\alpha, \gamma, g) \subseteq \mathcal{H}^{s-2k}_{m,2}(1, \gamma, g).$$

Proof. Let $f \in \mathcal{H}^{s-2k}_{m,2}(\alpha, \gamma, g)$. Then by the definition of the class $\mathcal{H}^{s-2k}_{m,2}(\alpha, \gamma, g)$ and applying Theorem 2.1 we know that $g_{2k} \in \mathcal{M}^{s-2k}_2(\alpha, \gamma)$, i.e.

$$\alpha\frac{z\varphi'(z)}{\varphi(z)} + (1-\alpha)\frac{(z\varphi')'(z)}{\varphi'(z)} \in P(\gamma),$$

where $\varphi = g_{2k}$.

Or equivalently,

$$\alpha\frac{z\varphi'(z)}{\varphi(z)} + (1-\alpha)\frac{(z\varphi'(z))'}{\varphi'(z)} \prec h(z) := \frac{1 + (1-2\gamma)z}{1-z}. \tag{2.13}$$

Set

$$q(z) = \frac{z\varphi'(z)}{\varphi(z)},$$

then we can rewrite (2.13) as

$$\alpha\frac{z\varphi'(z)}{\varphi(z)} + (1-\alpha)\frac{(z\varphi')'(z)}{\varphi'(z)} = q(z) + \frac{(1-\alpha)zq'(z)}{q(z)} \prec h(z). \tag{2.14}$$

Since h is convex and univalent in E with $h(0) = 1$ and $\mathrm{Re}(h(z)/(1-\alpha)) > 0$, applying Lemma 1.2, we obtain

$$q(z) \prec h(z), \quad z \in E. \tag{2.15}$$

By Setting

$$p(z) = \frac{zf'(z)}{g_{2k}(z)},$$

we get

$$\begin{aligned} zp'(z) &= z\frac{(zf'(z))'g_{2k}(z) - g'_{2k}(z)zf'(z)}{g_{2k}^2(z)} = z\frac{(zf'(z))'}{g_{2k}(z)} - \frac{zf'(z)}{g_{2k}(z)}q(z) \\ &= \frac{(zf'(z))'}{g'_{2k}(z)}q(z) - \frac{zf'(z)}{g_{2k}(z)}q(z). \end{aligned} \tag{2.16}$$

Therefore in view of $f \in \mathcal{H}_{m,2}^{s-2k}(\alpha, \gamma, g)$ and (2.16) we conclude that

$$\alpha\frac{zf'(z)}{g_{2k}(z)} + (1-\alpha)\frac{(zf'(z))'}{g'_{2k}(z)} = p(z) + (1-\alpha)\frac{zp'(z)}{q(z)} \in P_m(\gamma).$$

Now from relation (2.15) it is clear that $\mathrm{Re}(q(z)/(1-\alpha)) > 0$, so applying Lemma 2.1, we get $p(z) \in P_m(\gamma)$ and the proof is complete. □

By Putting $m = 2$ and considering $g = f_{2k}$ in Theorem 2.2, we have the following corollary.

Corollary 2.3. *Let $0 < \alpha < 1$ and $k \geqslant 1$. Then*

$$\mathcal{M}_2^{s-2k}(\alpha, \gamma) \subseteq R_2^{s-2k}(\gamma) \subseteq K \subseteq S.$$

Theorem 2.3. *Let $0 \leqslant \alpha < 1$ and $f \in \mathcal{M}_m^{s-2k}(\alpha, \gamma)$. Then there exists a function $p \in P_m(\gamma)$ such that*

$$f_{2k}(z) = \left(\frac{1}{1-\alpha}\int_0^z u^{\alpha/(1-\alpha)} \exp\left(\frac{1}{1-\alpha}\int_0^u \frac{h(t)-1}{t}\,\mathrm{d}t\right)\mathrm{d}u\right)^{1-\alpha}, \tag{2.17}$$

where

$$h(z) = \frac{1}{2k}\sum_{\upsilon=0}^{k-1}\left(p(\varepsilon^\upsilon z) + \overline{p(\varepsilon^\upsilon \overline{z})}\right). \tag{2.18}$$

P r o o f. Since $f \in \mathcal{M}_m^{s-2k}(\alpha,\gamma)$, there exists a function $p \in P_m(\gamma)$ such that

$$\alpha\frac{zf'(z)}{f_{2k}(z)} + (1-\alpha)\frac{(zf'(z))'}{f'_{2k}(z)} = p(z). \tag{2.19}$$

Using similar arguments given in the proof of Theorem 2.1 to (2.19) we obtain

$$\alpha\frac{zf'_{2k}(z)}{f_{2k}(z)} + (1-\alpha)\frac{(zf'_{2k}(z))'}{f'_{2k}(z)} = \frac{1}{2k}\sum_{\upsilon=0}^{k-1}\big(p(\varepsilon^{\upsilon}z) + \overline{p(\varepsilon^{\upsilon}\overline{z})}\big) = h(z). \tag{2.20}$$

Let us define F as

$$\alpha\frac{zf'_{2k}(z)}{f_{2k}(z)} + (1-\alpha)\frac{(zf'_{2k}(z))'}{f'_{2k}(z)} = \frac{zF'(z)}{F(z)},$$

then

$$f_{2k}(z) = \left(\frac{1}{1-\alpha}\int_0^z \frac{(F(t))^{1/(1-\alpha)}}{t}\,\mathrm{d}t\right)^{1-\alpha} \tag{2.21}$$

and the function F is analytic with $F(0)=0$ and from (2.20) we can write

$$\frac{zF'(z)}{F(z)} = h(z).$$

Now by solving the last equation and putting its response into equality (2.21) we get the result and the proof is complete. □

Theorem 2.4. *Let $0 \leqslant \alpha < 1$ and $f \in \mathcal{M}_m^{s-2k}(\alpha,\gamma)$. Then there exists a function $p \in P_m(\gamma)$ such that*

$$f'(z) = \frac{1}{(1-\alpha)^{1-\alpha}}\frac{\int_0^1 u^{\alpha/(1-\alpha)}\exp((1-\alpha)^{-1}\int_0^{uz}(h(t)-1)t^{-1}\,\mathrm{d}t)p(u)\,\mathrm{d}u}{\big(\int_0^1 u^{\alpha/(1-\alpha)}\exp((1-\alpha)^{-1}\int_0^{uz}(h(t)-1)t^{-1}\,\mathrm{d}t)\,\mathrm{d}u\big)^{\alpha}}, \tag{2.22}$$

where h is given by (2.18).

P r o o f. Suppose that $f \in \mathcal{M}_m^{s-2k}(\alpha,\gamma)$, we can get

$$\alpha\frac{zf'(z)}{f_{2k}(z)} + (1-\alpha)\frac{(zf'(z))'}{f'_{2k}(z)} \in P_k(\gamma),$$

so there exists a function $p \in P_k(\gamma)$ such that

$$\alpha\frac{zf'(z)}{f_{2k}(z)} + (1-\alpha)\frac{(zf'(z))'}{f'_{2k}(z)} = p(z).$$

Taking $F(z) = zf'(z)$ and $G(z) = f_{2k}(z)$ in the above equation yields

$$\alpha \frac{F(z)}{G(z)} + (1-\alpha)\frac{F'(z)}{G'(z)} = p(z),$$

or

$$F'(z) + \frac{\alpha}{1-\alpha}\frac{G'(z)}{G(z)}F(z) = \frac{p(z)G'(z)}{1-\alpha}. \tag{2.23}$$

Now solving Cauchy problem (2.23) and considering (2.17) we get our result and the proof is complete. □

Theorem 2.5. *Let $f, g \in \mathcal{M}_2^{s-2k}(\alpha, \gamma)$ and suppose that F is defined by*

$$F(z) = \frac{1}{\delta z^{1/\delta - 1}} \int_0^z t^{1/\delta - 2}(f_{2k}(t))^{\beta/(1+\beta)}(g_{2k}(t))^{1/(1+\beta)}\,\mathrm{d}t, \tag{2.24}$$

where $z \in E$, $\delta > 0$, $\beta \geqslant 0$ and $\gamma + \delta^{-1} - 1 > 0$. Then F belongs to $\mathcal{M}_2^{s-2k}(1, \gamma)$.

Proof. Since $f, g \in \mathcal{M}_2^{s-2k}(\alpha, \gamma)$, by applying Theorem 2.1 and Corollary 2.3 we obtain $f_{2k}, g_{2k} \in \mathcal{M}_2^{s-2k}(1, \gamma)$. Differentiating (2.24) logarithmically and setting $p(z) = zF'(z)/F(z)$, we have

$$p(z) + \frac{zp'(z)}{p(z) + \delta^{-1} - 1} = \frac{\beta}{1+\beta}\frac{zf'_{2k}(z)}{f_{2k}(z)} + \frac{1}{1+\beta}\frac{zg'_{2k}(z)}{g_{2k}(z)}. \tag{2.25}$$

Since the functions $zf'_{2k}(z)/f_{2k}(z)$ and $zg'_{2k}(z)/g_{2k}(z)$ belong to $P_2(\gamma)$ in E, and $P_2(\gamma)$ is a convex set,

$$\frac{\beta}{1+\beta}\frac{zf'_{2k}(z)}{f_{2k}(z)} + \frac{1}{1+\beta}\frac{zg'_{2k}(z)}{g_{2k}(z)} \in P_2(\gamma).$$

We now apply Lemma 1.2 to obtain $p(z) \in P_2(\gamma)$ and the proof is complete. □

Let $L(r, f)$ denote the length of the image of the circle $|z| = r$ under f. We prove the following.

Theorem 2.6. *Let $f \in \mathcal{H}_2^{s-2k}(1, \gamma)$. Then for $0 < r < 1$,*

$$L(r, f) \leqslant \frac{4\pi(1-\gamma)}{(1-r)^{(k+2)/k}}. \tag{2.26}$$

Proof. Using Theorem 2.2 and in view of the definition of class $\mathcal{H}_2^{s-2k}(1,\gamma)$ there exists a function $g \in \mathcal{M}_2^{s-2k}(1,\gamma)$ such that

$$zf'(z) = \psi(z)h(z), \quad \psi = g_{2k} \in S^*(\gamma),\ h \in P_2(\gamma). \tag{2.27}$$

Since $\psi \in S^*(\gamma)$ and ψ is a k-fold symmetric function, there exists a k-fold symmetric function $\psi_1(z)$ such that

$$\psi(z) = z\Big(\frac{\psi_1(z)}{z}\Big)^{1-\gamma}.$$

Now for $z = re^{i\theta}$ we have

$$\begin{aligned} L(r,f) &= \int_0^{2\pi} |zf'(z)|\,d\theta \\ &= \int_0^{2\pi} \Big|z\Big(\frac{\psi_1(z)}{z}\Big)^{1-\gamma} h(z)\Big|\,d\theta = r^\gamma \int_0^{2\pi} |(\psi_1(z))^{1-\gamma}h(z)|\,d\theta, \end{aligned}$$

and so, using Hölder's inequality, we obtain

$$L(r,f) \leqslant 2\pi r^\gamma \Big(\frac{1}{2\pi}\int_0^{2\pi} |\psi_1(z)|^2\,d\theta\Big)^{1/2} \Big(\frac{1}{2\pi}\int_0^{2\pi} |h(z)|^2\,d\theta\Big)^{1/2}. \tag{2.28}$$

For $h \in P_2(\gamma)$, from the Parseval's identity it is easy to see that

$$\frac{1}{2\pi}\int_0^{2\pi} |h(z)|^2\,d\theta \leqslant \frac{1+(4(1-\gamma)^2-1)r^2}{1-r^2}. \tag{2.29}$$

Also for k-fold symmetric function ψ_1 it is known that (see [4])

$$|\psi_1(z)| \leqslant \frac{|z|}{(1-|z|^k)^{2/k}}. \tag{2.30}$$

Using (2.29) and (2.30) in (2.28), it follows that

$$L(r,f) \leqslant 2\pi r^\gamma \Big(\frac{1+(4(1-\gamma)^2-1)r^2}{1-r^2}\Big)^{1/2} \frac{r}{(1-r^k)^{2/k}} \leqslant \frac{4\pi(1-\gamma)}{(1-r)^{1+2/k}}.$$

This completes the proof. □

Theorem 2.7. *Let $f \in \mathcal{H}_2^{s-2k}(1,\gamma)$. Then for $0 < r < 1$,*

$$|a_n| \leqslant 4\pi(1-\gamma)n^{2/k}. \tag{2.31}$$

P r o o f. Since with $z = r\mathrm{e}^{i\theta}$ Cauchy Theorem gives

$$na_n = \frac{1}{2\pi r^n}\int_0^{2\pi} zf'(z)\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta,$$

then

$$n|a_n| \leqslant \frac{1}{2\pi r^n}\int_0^{2\pi} |zf'(z)|\,\mathrm{d}\theta = \frac{1}{2\pi r^n}L(r,f).$$

Using Theorem 2.6 and putting $r = 1 - n^{-1}$, $n \to \infty$, we obtain the required result. $\square$

A c k n o w l e d g e m e n t. The authors would like to thank the referee for many useful suggestions.

References

[1] *P. Eenigenberg, S. S. Miller, P. T. Mocanu, M. O. Reade*: On a Briot-Bouquet differential subordination. General Inequalities 3 (E. F. Beckenbach et al., eds.). International Series of Numerical Mathematics 64. Birkhäuser, Basel, 1983, pp. 339–348. zbl MR doi
[2] *I. Graham, G. Kohr*: Geometric Function Theory in One and Higher Dimensions. Pure and Applied Mathematics 255. Marcel Dekker, New York, 2003. zbl MR doi
[3] *S. S. Miller, P. T. Mocanu*: Differential Subordinations: Theory and Applications. Pure and Applied Mathematics 225. Marcel Dekker, New York, 2000. zbl MR doi
[4] *K. I. Noor*: On subclasses of close-to-convex functions of higher order. Int. J. Math. Math. Sci. *15* (1992), 279–289. zbl MR doi
[5] *K. S. Padmanabhan, R. Parvatham*: Properties of a class of functions with bounded boundary rotation. Ann. Pol. Math. *31* (1976), 311–323. zbl MR doi
[6] *B. Pinchuk*: Functions with bounded boundary rotation. Isr. J. Math. *10* (1971), 6–16. zbl MR doi
[7] *K. Sakaguchi*: On a certain univalent mapping. J. Math. Soc. Japan. *11* (1959), 72–75. zbl MR doi
[8] *Z.-G. Wang, C.-Y. Gao*: On starlike and convex functions with respect to $2k$-symmetric conjugate points. Tamsui Oxf. J. Math. Sci. *24* (2008), 277–287. zbl MR
[9] *Z.-G. Wang, C.-Y. Gao, S.-M. Yuan*: On certain subclasses of close-to-convex and quasi-convex functions with respect to k-symmetric points. J. Math. Anal. Appl. *322* (2006), 97–106. zbl MR doi

Authors' address: *Rasoul Aghalary, Jafar Kazemzadeh*, Department of Mathematics, Faculty of Sciences, Urmia University, 11km SERO Road, Urmia City 5756151818, Iran, e-mail: `raghalary@yahoo.com`, `r.aghalary@urmia.ac.ir`, `j.kazemzadeh@urmia.ac.ir`.