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ON FINITE TIME STABILITY WITH GUARANTEED COST CONTROL OF UNCERTAIN LINEAR SYSTEMS

Atif Qayyum and Alfredo Pironti

This paper deals with the design of a robust state feedback control law for a class of uncertain linear time varying systems. Uncertainties are assumed to be time varying, in one-block norm bounded form. The proposed state feedback control law guarantees finite time stability and satisfies a given bound for an integral quadratic cost function. The contribution of this paper is to provide a sufficient condition in terms of differential linear matrix inequalities for the existence and the construction of the proposed robust control law. In particular, the construction of the feedback control law is brought back to a feasibility problem which can be solved inside the convex optimization framework. The effectiveness of the proposed approach is shown by means of the results obtained on a numerical and a physical example.

Keywords: differential LMIs, finite time stability, guaranteed cost control, robust control, state feedback control

Classification: 93B50,93B51,93D15

1. INTRODUCTION

The concept of finite time stability (FTS) dates back to the seminal paper by Dorato [14], and has been revamped in recent years (see the book [5] and the references therein), because it allows control system designers to overcome some limitations intrinsic to the classic Lyapunov stability concept.Indeed, FTS is not concerned about the asymptotic behavior of the state trajectory of a system, but consider its transient behavior over a given finite time interval, this is important for application where the system state is required to operate in a fixed region for a given time-frame. In a general setting, a system is FTS, if assuming that the initial conditions are inside a prescribed region (which usually is limited by an ellipsoid), then the state trajectories do not cross a given outer boundary (which usually is again an ellipsoid) for a specified finite time interval. FTS attracts control system designers, because it guarantees that the system transient performances satisfy given bounds. In contrast, the classical Lyapunov approach only deals with the steady state behavior of the state trajectory without giving quantitative bound on the state trajectories.

The paper is focused on uncertain linear time-varying systems, with uncertainties in the so-called norm bounded one-block form (for more details see [13] and the references

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therein). The aim is to describe some sufficient conditions allowing to solve the simultaneous problem of guaranteeing FTS and an integral quadratic cost constraint. Both the analysis and state feedback synthesis problems are considered.

Among others, some fundamental results on FTS of linear system have been extensively discussed in [1, 3, 9], the robustness problem has been considered in [4] for uncertain linear systems, whereas stochastic systems are studied in [6, 31, 32]. FTS for nonlinear systems have been comprehensively addressed in [18]. The concept is also dealt with impulsive dynamical linear systems (IDLS) in [2, 8, 10, 33]. In these papers the problem of analysis (determine when a not controlled system verifies an FTS condition) and/or synthesis (design a state or output feedback controller making the closed loop system to verify an FTS condition) have been considered and solved for various class of systems. Unfortunately in the design of a control system it is, often, necessary to consider various performance index, so as multi-objective control design techniques are desirable from an engineering point of view. From this point of view, integral quadratic performance index are among the ones more considered in the literature, since the birth of the linear quadratic optimal control theory [11].

In the context of robust control, the idea of considering an integral quadratic cost constraint for linear systems can be found in the initial work of Petersen [20, 21, 22], whereas the problem of impulsive switched systems is considered in [30]. More recent work can be found in [12, 17, 28].

To the authors best knowledge, the idea of solving a guaranteed cost control (GCC) problem, together with a FTS constraint, has been considered for the first time in [26]. In this paper we extend the work done in [26] by considering the presence of uncertainties in the considered system. With respect to the existing literature the novel contribution of this paper is hence to consider a multi-objective design problem, where the closed loop system has to simultaneously satisfy the FTS conditions and the guaranteed cost control constraints, in the presence of time-varying uncertainties.

A similar multi-objective problem has been considered in [7], but in the context of Input Output Finite Time Stability (IO-FTS). In this case initial conditions of the controlled system are zero, but the system is subject to a disturbance input; IO-FTS and a $\mathcal{H}_\infty$ bound are then simultaneously considered. Here we consider the different case where the initial conditions are not zero, and instead of an $\mathcal{H}_\infty$ constraint we consider a quadratic integral cost constraint. Moreover differently from [7], our approach allows to explicitly consider actuator constraints. Note that, although not directly discussed in the paper, the presence of external disturbances can be taken into account in our design methodology, if these disturbances can be generated by an autonomous exo-system (it will be sufficient to apply our design methodology to a suitable augmented system).

Summarizing, we can say that the novel contribution of this paper consists in establishing a framework for linear time-varying system to design a state feedback control law which guarantees FTS, while simultaneously satisfying an integral quadratic cost constraint,with the main aim of contributing to the solution of practical problems defined over a finite time interval. Our solution will be characterized as a feasibility problem for a set of suitable differential linear matrix inequalities (DLMIs), that can be solved by means of well know convex optimization techniques.

Note that in the literature, a different concept of FTS can also be found, where it is required that, given an initial condition, the state trajectories converge to the zero equilibrium in a finite time interval, see [16, 23, 24, 25]; based on Lyapunov stability criteria, the asymptotic convergence is transformed to finite time convergence. Recent results achieved, considering the control problem of finite horizon estimation in the presence of noise for linear time-varying systems [27]. Applications of this theory to the control of nonlinear systems are reported in [15, 19, 29]. However, this concept is an independent one, and it is not linked with the content of this paper.

The paper is organized as follows. In Section 2 we give some definitions and state some preliminary results which are valid for linear time-varying systems. In Section 3 we precisely state the problem we deal with, and we describe the state feedback design methods. These results lead to the consider the solution of a suitable DLMI feasibility problem. In Section 4 we consider two different case studies. The first one is a numerical example, whereas the second one considers the problem of controlling an inverted pendulum on a cart by means of a state feedback control law. Finally in Section 5, some concluding remarks are given.

2. PRELIMINARIES

We shortly recall the definitions related to FTS of a linear time-varying system (see [1]).

Definition 2.1. (FTS) Given $t_0 \in \mathbb{R}$, a scalar $T > 0$, a positive definite matrix $R \in \mathbb{R}^{n\times n}$ and a positive definite matrix-valued function $\Gamma(\cdot) \in \mathbb{R}^{n\times n}$, defined in $\Omega = [t_0, t_0 + T]$, such that, $\Gamma(t_0) < R$, the system,

$$\dot{x}(t) = A(t)x(t)\,, \quad x(t_0) = x_0\,, \tag{1}$$

is said to be finite-time stable with respect to $(\Omega, R, \Gamma(\cdot))$ if,

$$x_0^T R x_0 \le 1 \Rightarrow x(t)^T \Gamma(t) x(t) < 1\,, \tag{2}$$

where $A(t) \in \mathbb{R}^{n\times n}$ and $t \in \Omega$.

In general terms, a system is categorized as FTS, if for all initial conditions within a given inner bound, its state trajectories remain inside a prescribed outer bound over the time interval of interest.

Remark 2.2. Since quadratic functions are used in condition (2), the FTS property results in ellipsoidal bounds on initial conditions and state trajectory. Different type of function can be used to obtain other type of bounds (for example polytopic ones see [1, chap 6] for more details).

Considering the presence of uncertainties, the concept of FTS is readily extended to robust finite time stability (RFTS). Here we consider the case where uncertainties, in the so-called norm bounded one-block form, are added to the system (1).

Definition 2.3. (RFTS) Consider $\big(\Omega, R, \Gamma(\cdot)\big)$, as in Definition 2.1, and the uncertain system

$$\dot{x}(t) = \Big(A(t) + F(t)\Delta(t)E(t)\Big)x(t)\,, \quad x(t_0) = x_0\,, \tag{3}$$

where $\Delta(\cdot)$ is an uncertain matrix satisfying the norm bound

$$\Delta(t)^T \Delta(t) \leq I \Leftrightarrow \|\Delta(t)\|_2 \leq 1\,, \tag{4}$$

and $F(\cdot)$, $E(\cdot)$ are weighting matrices of suitable dimensions. Then system (3) is said to be RFTS with respect to $\big(\Omega, R, \Gamma(\cdot)\big)$, if condition (2) is satisfied for all $\Delta(\cdot)$, such that $\|\Delta(t)\|_2 \leq 1$ and $t \in \Omega$.

When dealing with uncertain system (3), we resort to the concept of quadratic finite-time stability (QFTS).

Definition 2.4. (QFTS) Given $\big(\Omega, R, \Gamma(\cdot)\big)$, as in Definition 2.1, the uncertain system (3) is said to be QFTS with respect to $\big(\Omega, R, \Gamma(\cdot)\big)$, if and only if there exists a positive definite matrix-valued function $P(\cdot)$ which satisfies the following DLMI/LMI conditions:

$$\begin{aligned}
&\dot{P}(t) + \big(A(t) + \Delta A(t)\big)^T P(t) + P(t)\big(A(t) + \Delta A(t)\big) < 0\,,\\
&P(t) > \Gamma(t)\,,\\
&P(t_0) < R\,,
\end{aligned}$$

for any admissible uncertainty realization $\Delta(\cdot)$, and for all $t \in \Omega$.

Remark 2.5. The DLMIs involved in the definition of QFTS implies RFTS (see [1, Lemma 4.1]), therefore, it follows that QFTS is a stronger property than RFTS, in the sense that QFTS implies RFTS, whereas, the inverse may not be true in general.

In this paper, we consider the following uncertain linear time-varying system

$$\dot{x}(t) = A(t)x(t) + F(t)u_\Delta(t) + B(t)u(t)\,, \quad x(t_0) = x_0\,, \tag{5a}$$
$$u_\Delta(t) = \Delta(t)v_\Delta(t)\,, \tag{5b}$$
$$v_\Delta(t) = E_1(t)x(t) + E_2(t)u(t)\,, \tag{5c}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $F(t) \in \mathbb{R}^{n\times p}$, $E_1(t) \in \mathbb{R}^{q\times n}$ and $E_2(t) \in \mathbb{R}^{q\times m}$. The time-varying system matrices $A(t)$, $B(t)$,$E_1(t)$, $E_2(t)$, and $F(t)$ are piecewise continuous in the time interval Ω. The uncertain time-varying matrix $\Delta(t) \in \mathbb{R}^{p\times q}$ is assumed to be Lebesgue measurable and satisfying the spectral norm bound $\|\Delta(t)\|_2 \leq 1\,, \forall t \in \Omega$. A schematic of system (5) is shown in Figure 1. Note that we have not considered direct feedthrough matrix from u to u_Δ to simplify the main results; however, such assumption can be easily removed.

In this paper we extend the concept of FTS by introducing an integral quadratic cost bound on the trajectories of system (5). However, to simplify the discussion, we first consider the case without uncertainties, and present some preliminary results for system (1)

Definition 2.6. (FTS-GCB) System (1) is said to be finite time stable with a guaranteed cost bound (FTS-GCB) with respect to $\big(\Omega, R, \Gamma, Q\big)$, where R is a given positive

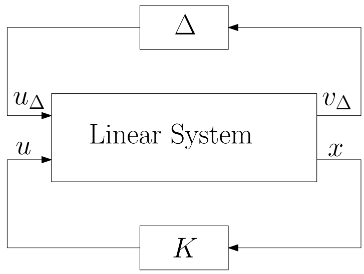

Fig. 1. Structure of the considered uncertain system with state feedback.

definite matrix, $\Gamma(\cdot)$ and $Q(\cdot)$ are given piecewise continuous positive definite matrix-valued functions of suitable dimensions defined in Ω, if

$$x_0^T R x_0 \le 1 \Rightarrow \begin{cases} x^T(t)\Gamma(t)x(t) < 1\,, \\ J_q \triangleq \displaystyle\int_\Omega x^T(t)Q(t)x(t)\mathrm{d}t < 1\,. \end{cases} \tag{6}$$

The property of guaranteeing an integral quadratic cost bound to the system trajectories has been introduced in [21] for the case of linear time-invariant system over infinite time horizon. In our case, the FTS-GCB property requires that, if the initial conditions for the system (1) are inside a given ellipsoid, then the trajectory of the states remain inside a given ellipsoid (whose boundary can be time-varying), and a bound on an integral quadratic cost function is satisfied.

Remark 2.7. The term *integral quadratic* cost is considered implicit in all subsequent mention of guaranteed cost in this paper to simplify the notation.

The following lemmas play a fundamental role in the derivation of the main results of this paper. These lemmas give sufficient conditions for FTS-GCB of system (1).

Remark 2.8. To formulate simple and concise expressions, we will often drop the time dependency, if this is not the cause of ambiguity.

Lemma 2.9. If there exist two positive definite and piecewise continuously differentiable matrix functions $P_1(t)$ and $P_2(t)$ such that, for $t \in \Omega$,

$$P_1(t_0) < R \tag{7a}$$
$$P_1 > \Gamma \tag{7b}$$
$$\dot{P}_1 + A^T P_1 + P_1 A < 0 \tag{7c}$$
$$P_2 > 0 \tag{7d}$$
$$P_2(t_0) < R \tag{7e}$$
$$\dot{P}_2 + A^T P_2 + P_2 A + Q < 0 \tag{7f}$$

then system (1) is FTS-GCB with respect to (Ω, R, Γ, Q).

Proof. Proof is done in [26]. For completeness it is also reported here.

Inequalities (7a) – (7c) have been proved as necessary and sufficient conditions for FTS of system (1) (see [1, Theorem 2.1]). The equivalent expression involving the state transition matrix is circumvented to avoid the computational complexity of determining state transition matrix and to facilitate the system design and stabilization process. We only need to show that inequalities (7d) – (7f) implies that the integral quadratic cost bound is guaranteed, i. e. if $x_0^T R x_0 \leq 1$, then

$$J_q = \int_{\Omega} x^T(t) Q(t) x(t) \mathrm{d}t < 1 \,,$$

is satisfied.

Consider a time-varying quadratic Lyapunov function $V_2(t,x) = x(t)^T P_2(t) x(t)$, then along the trajectory of (1), we have

$$\dot{V}_2 = x^T(\dot{P}_2 + A^T P_2 + P_2 A)x = -x^T(Q+H)x \,, \tag{8}$$

where, by inequality (7f), $H \in \mathbb{R}^{n\times n}$ is a positive definite matrix. Now integrating left side of (8), we get

$$\begin{aligned} \int_{t_0}^{t_0+T} \dot{V}_2 \mathrm{d}t &= V_2\big(x(t_0+T)\big) - V_2\big(x(t_0)\big) \\ &= x(t_0+T)^T P_2(t_0+T) x(t_0+T) - x_0^T P_2(t_0) x_0 \,, \end{aligned}$$

and integrating right side of (8), we get

$$\int_{t_0}^{t_0+T} \dot{V}_2 \mathrm{d}t = -\int_{t_0}^{t_0+T} x^T(Q+H)x \mathrm{d}t \,,$$

therefore, (8) can be written as

$$x_0^T P_2(t_0) x_0 = x(t_0+T)^T P_2(t_0+T) x(t_0+T) + \int_{t_0}^{t_0+T} x^T Q x \mathrm{d}t + \int_{t_0}^{t_0+T} x^T H x \mathrm{d}t \,.$$

P_2 and H being positive definite matrices, therefore we have

$$x_0^T P_2(t_0) x_0 > \int_{t_0}^{t_0+T} x^T Q x \mathrm{d}t \,.$$

Finally by using inequality (7e) and considering initial bounds $x_0^T R x_0 \leq 1$ we have

$$J_q = \int_{t_0}^{t_0+T} x^T Q x \mathrm{d}t < 1 \,.$$

This completes the proof. □

At the cost of some conservativeness, the search of two piecewise continuously differentiable matrices P_1 and P_2 satisfying the conditions of Lemma 2.9, can be reduced to discover the existence of only one such matrix using the following result.

Lemma 2.10. If there exists a positive definite piecewise continuously differentiable matrix function $P(\cdot)$, such that, for $t \in \Omega$,

$$P > 0 \tag{9a}$$

$$P(t_0) < R \tag{9b}$$

$$P > \Gamma \tag{9c}$$

$$\dot{P} + A^T P + PA + Q < 0 \tag{9d}$$

then the system (1) is FTS-GCB with respect to (Ω, R, Γ, Q).

P r o o f . The proof is easily obtained from Lemma 2.9 by letting $P_1 = P_2 = P$. □

In the next step we consider the case in which we want to make a given linear system FTS-GCB by means of state feedback. We start with the case of no uncertainties ($\Delta = 0$ in system (5)).

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)\,, \quad x(t_0) = x_0\,. \tag{10}$$

The control input $u(t)$ is driven by a memoryless state feedback matrix $K(t) \in \mathbb{R}^{m\times n}$, i. e.

$$u(t) = K(t)x(t)\,. \tag{11}$$

Our aim is to choose the state feedback matrix $K(t)$ in such a way to make system (10) FTS-GCB. Before addressing the problem and stating our main results, we give the following definition of finite-time stability with guaranteed cost control (FTS-GCC).

Definition 2.11. (FTS-GCC) Given a positive definite matrix R and positive definite piecewise continuous matrix-valued functions $\Gamma(\cdot)$, $Q(\cdot)$, $Y(\cdot)$ and $Z(\cdot)$ of suitable dimensions, possibly time-varying in Ω, then system (10), in the presence of the control law (11) is FTS-GCC with respect to $(\Omega, R, \Gamma, Q, Y, Z)$ for $t \in \Omega$, if

$$x_0^T R x_0 \leq 1\,, \tag{12}$$

implies

$$x^T(t)\Gamma(t)x(t) < 1\,, \tag{13a}$$

$$u^T(t)Y(t)u(t) < 1\,, \tag{13b}$$

$$J_{qz} \triangleq \int_\Omega \Big(x^T(t)Q(t)x(t) + u^T(t)Z(t)u(t)\Big)\mathrm{d}t < 1\,. \tag{13c}$$

If the control law (11) satisfies the FTS-GCC conditions (13), then the closed loop system

$$\dot{x}(t) = \Big(A(t) + B(t)K(t)\Big)x(t)\,, \quad x(t_0) = x_0\,, \tag{14}$$

is FTS-GCB with respect to $(\Omega, R, \Gamma_1, Q_1)$, where Γ_1 and Q_1 are suitable matrices for the closed loop system (14) as discussed in the proof of Lemma 2.9.

Problem 2.12. Given system (14), find the state feedback matrix-valued function $K(\cdot)$ in such a way to satisfy the FTS-GCC conditions (13) with respect to $(\Omega, R, \Gamma, Q, Y, Z)$.

In the sequel we assume that the matrix Γ, defining the outer bound ellipsoid, as described in Definition 2.1, can be written as

$$\Gamma = S^T S \,, \tag{15}$$

since $\Gamma > 0$, this is always possible.

A sufficient condition for the solution of Problem 2.12 is given by the following theorem.

Theorem 2.13. If there exist a positive definite piecewise continuously differentiable matrix $X(t)$ and a piecewise continuous matrix-valued function $L(t)$ such that

$$\begin{pmatrix} \Theta_{11} & X & L^T \\ X & -Q^{-1} & 0 \\ L & 0 & -Z^{-1} \end{pmatrix} < 0 \tag{16a}$$

$$SXS^T - I < 0 \tag{16b}$$

$$\begin{pmatrix} X & L^T \\ L & Y^{-1} \end{pmatrix} > 0 \tag{16c}$$

$$X(t_0) > R^{-1} \tag{16d}$$

$$X > 0 \tag{16e}$$

where,

$$\Theta_{11} \triangleq -\dot{X} + XA^T + AX + L^T B^T + BL$$

then Problem 2.12 is solved by letting $K = LX^{-1}$.

Proof. Proof is done in [26]. For completeness it is also reported here.

We exploit Lemma 2.10 for the closed loop system (14) to determine that it satisfies FTS-GCB definition. Indeed, if there exist a piecewise continuously differentiable matrix $P(\cdot)$ and a piecewise continuous matrix-valued function matrix $K(\cdot)$ such that substituting $(A+BK)$ and $(Q+K^TZK)$ in place of A and Q in the statement of Lemma 2.10, we obtain

$$P > 0 \tag{17a}$$

$$P(t_0) < R \tag{17b}$$

$$P > \Gamma \tag{17c}$$

$$\dot{P} + (A+BK)^T P + P(A+BK) + (Q + K^T ZK) < 0. \tag{17d}$$

The inequality (17d) can be written as

$$\dot{P} + A^T P + K^T B^T P + PA + PBK + Q + K^T ZK < 0 \,.$$

Let $P = X^{-1}$ and $L = KX$, we get

$$-\dot{X} + XA^T + AX + L^T B^T + BL + XQX + L^T ZL < 0 \,.$$

Using definition of Θ_{11}, we get

$$\Theta_{11} + XQX + L^T ZL < 0\,.$$

Using Schur complement, inequality (16a) is verified.

Using $P = X^{-1}$, inequality (17c) can be written as

$$\begin{aligned} X^{-1} - S^T S &> 0\,, \\ I - XS^T S &> 0\,. \end{aligned}$$

Therefore, inequality (16b) is verified.

The condition (13b) can be written as

$$x^T K^T Y K x < 1\,.$$

Using condition (13a) and let $L = KX$, we get

$$X - L^T Y L > 0\,.$$

Therefore, inequality (16c) is verified.

The proof is completed by noting that (16e) and (16d) are easily obtained from (17a) and (17b) respectively. □

Remark 2.14. The weighting matrices $Y(\cdot)$ and $Z(\cdot)$ provide constraints on the control input $u(t)$.

Before stating our main theorem, we end this section by giving definitions of robust finite time stability with guaranteed cost bound (RFTS-GCB) and robust finite time stability with guaranteed cost control (RFTS-GCC) for the uncertain system (3).

Definition 2.15. (RFTS-GCB) System (3) is said to be RFTS-GCB with respect to (Ω, R, Γ, Q), where R is a given positive definite matrix, $\Gamma(\cdot)$ and $Q(\cdot)$ are given piecewise continuous positive definite matrix-valued functions of suitable dimensions defined in Ω, if for every uncertain time-varying matrix $\Delta(t)$, satisfying the norm bound (4), the following condition hold

$$x_0^T R x_0 \le 1 \Rightarrow \begin{cases} x^T(t)\Gamma(t)x(t) < 1\,, \\ J_q \triangleq \displaystyle\int_\Omega x^T(t)Q(t)x(t)\mathrm{d}t < 1\,. \end{cases} \tag{18}$$

From the above definition, it follows that an uncertain system of type (3) is RFTS-GCB, if it is FTS-GCB for all possible realizations of the uncertain matrix $\Delta(t)$. However, to make an uncertain system RFTS, using a control input $u(t)$, we also introduce the following definition of RFTS-GCC.

Definition 2.16. (RFTS-GCC) Given a positive definite matrix R and positive definite piecewise continuous matrix-valued functions $\Gamma(\cdot)$, $Q(\cdot)$, $Y(\cdot)$ and $Z(\cdot)$ of suitable dimensions, possibly time-varying in Ω, then system (5), in the presence of the control law (11), is RFTS-GCC with respect to $(\Omega, R, \Gamma, Q, Y, Z)$ defined in Ω, if

$$x_0^T R x_0 \leq 1\,, \tag{19}$$

implies

$$x^T(t)\Gamma(t)x(t) < 1\,, \tag{20a}$$

$$u^T(t)Y(t)u(t) < 1\,, \tag{20b}$$

$$J_{qz} \triangleq \int_\Omega \Big(x^T(t)Q(t)x(t) + u^T(t)Z(t)u(t)\Big)\mathrm{d}t < 1\,, \tag{20c}$$

$\forall \Delta$ such that $\|\Delta(t)\|_2 \leq 1$.

Now we are ready to describe our main results, which deals with RFTS under GCB and GCC for uncertain linear time-varying systems.

3. MAIN RESULTS

In this section we precisely state the problem we deal with and state our main results. Considering the uncertain linear time-varying systems of the form (3) and (5), we transform the Lemma 2.10 and Theorem 2.13 of Section 2, into equivalent statements to make the uncertain systems (3) and (5) simultaneously RFTS and, respectively, GCB, and GCC. At first, we consider the unforced system (3).

Theorem 3.1. If there exist a positive definite piecewise continuously differentiable matrix-valued function $P(t)$, and a positive definite piecewise continuous scalar function $\lambda(t)$, such that, for $t \in \Omega$

$$\lambda > 0 \tag{21a}$$

$$P > 0 \tag{21b}$$

$$P(t_0) < R \tag{21c}$$

$$P > \Gamma \tag{21d}$$

$$\begin{pmatrix} \dot{P} + A^T P + PA + Q + \lambda E_1^T E_1 & PF \\ F^T P & -\lambda I \end{pmatrix} < 0 \tag{21e}$$

Then system (3) is RFTS-GCB.

Proof. Inequalities (21) already imply the QFTS conditions (see [1, Theorem 4.1]) and hence RFTS, so we have only to show that these also guarantee that if $x_0^T R x \leq 1$, then the bound on the integral quadratic cost function J_q (18) is satisfied. Define time-varying quadratic Lyapunov function

$$V(t,x) = x^T(t)P(t)x(t)\,.$$

Differentiating with respect to time t along the trajectories, we have

$$\begin{aligned} \dot{V} &= \dot{x}^T P x + x^T \dot{P} x + x^T P \dot{x} \\ \dot{V} &= x^T (A + F\Delta E_1)^T P x + x^T \dot{P} x + x^T P (A + F\Delta E_1) x \\ \dot{V} &= x^T (\dot{P} + A^T P + PA + E_1^T \Delta^T F^T P + PF\Delta E_1) x\,. \end{aligned}$$

Now, from [20, Theorem 2.2], for any $\lambda > 0$ we have that

$$E_1^T \Delta^T F^T P + PF\Delta E_1 \leq \lambda E_1^T E_1 + \frac{1}{\lambda} PFF^T P + Q\,,$$

therefore, we get,

$$\dot{V} \leq x^T(\dot{P} + A^T P + PA + \lambda E_1^T E_1 + \frac{1}{\lambda} PFF^T P + Q)x < 0\,.$$

From above, it follows that the matrix

$$H_\Delta = -(\dot{P} + A^T P + PA + E_1^T \Delta^T F^T P + PF\Delta E_1)$$

is positive definite, moreover

$$\dot{V} = -x^T(Q + H_\Delta)x\,.$$

The remaining part of the proof is completed by following the same steps as in the proof of Lemma 2.9 (by replacing H in (8) with H_Δ). □

Completing all the progressive steps, finally we refer back to the uncertain system (5). If the control law (11) satisfies the RFTS-GCC conditions (20), then the closed loop system

$$\dot{x} = \Big(A + BK + F\Delta(E_1 + E_2 K)\Big)x\,, \quad x(t_0) = x_0\,, \tag{22}$$

is RFTS-GCB with respect to $(\Omega, R, \Gamma_2, Q_2)$, where Γ_2 and Q_2 are suitable matrices for the closed loop system (22) satisfying Theorem 3.1.

Problem 3.2. Given system (22), find the state feedback matrix-valued function $K(\cdot)$ in such a way to satisfy the FTS-GCC conditions (20) with respect to $(\Omega, R, \Gamma, Q, Y, Z)$.

A sufficient condition to solve Problem 3.2 is given by the following theorem.

Theorem 3.3. If there exist a positive definite piecewise continuously differentiable matrix $X(t)$, a piecewise continuous matrix-valued function $L(t)$ and a positive definite scalar function $\beta(t)$, such that following conditions are satisfied,

$$\begin{pmatrix} \Phi_{11} & X & L^T & XE_1^T + L^T E_2^T \\ X & -Q^{-1} & 0 & 0 \\ L & 0 & -Z^{-1} & 0 \\ E_1 X + E_2 L & 0 & 0 & -\beta I \end{pmatrix} < 0 \tag{23a}$$

$$SXS^T - I < 0 \tag{23b}$$

$$\begin{pmatrix} X & L^T \\ L & Y^{-1} \end{pmatrix} > 0 \tag{23c}$$

$$X(t_0) > R^{-1} \tag{23d}$$

$$X > 0 \tag{23e}$$

where

$$\Phi_{11} \triangleq \Theta_{11} + \beta FF^T.$$

Then Problem 3.2 is solved with $K = LX^{-1}$.

Proof. From Theorem 3.1 we know that the system (22) is RFTS-GCB, if there exist a positive definite piecewise continuously differentiable matrix function $P(t)$ and a positive definite piecewise continuous scalar function $\lambda(t)$, such that,

$$\lambda > 0 \tag{24a}$$

$$P > 0 \tag{24b}$$

$$P(t_0) < R \tag{24c}$$

$$P > \Gamma \tag{24d}$$

$$\begin{pmatrix} \Psi_{11} & PF \\ F^T P & -\lambda I \end{pmatrix} < 0 \tag{24e}$$

where

$$\Psi_{11} \triangleq \dot{P} + (A+BK)^T P + P(A+BK) + (Q + K^T ZK) + \lambda(E_1 + E_2K)^T(E_1 + E_2K)\,.$$

After taking Schur complement, inequality (24e) can be written as,

$$\dot{P} + (A+BK)^T P + P(A+BK) + (Q + K^T ZK)$$
$$+\lambda(E_1 + E_2K)^T(E_1 + E_2K) + \frac{1}{\lambda} PFF^T P < 0\,.$$

Let $P = X^{-1}, L = KX$ and $\lambda = \beta^{-1}$, we get

$$\Theta_{11} + \beta FF^T + L^T ZL + XQX + \frac{1}{\beta}(E_1X + E_2L)^T(E_1X + E_2L) < 0\,.$$

Using definition of Φ_{11}, we get

$$\Phi_{11} + L^T ZL + XQX + \frac{1}{\beta}(E_1X + E_2L)^T(E_1X + E_2L) < 0\,.$$

Using Schur complement, the inequality (23a) is proved.

The remaining part of the proof is completed by noting that inequalities (23b) – (23e) have already been treated in the proof of Theorem 2.13. □

Solutions of Problem 2.12 and Problem 3.2 requires to solve a set of differential linear matrix inequalities over the interval Ω. As shown in [1], these DLMIs feasibility problem can be recast in terms of LMIs. This is obtained by splitting the total time interval Ω in a number of subintervals N, each of length T_s, and assuming a piecewise affine behavior (with respect to t) for the DLMIs variables. The resulting LMIs could be solved using off-the-shelf optimization softwares, such as YALMIP, SeDuMi, MATLAB LMI toolbox etc. For example DLMIs of the matrix-valued functions $P(\cdot)$ in (9), (21) and $X(\cdot)$ in (16), (23) are transformed to LMIs. These matrices are assumed to have piecewise affine structure as,

$$P(t) = \begin{cases} P_0 + P_1(t - t_0) \;: \forall t \in [t_0, t_0 + T_s] \\ P_0 + \sum_{h=1}^{j} P_h T_s + P_{j+1}(t - jT_s - t_0) \;: j = 1 \dots J \\ \forall t \in [t_0 + jT_s, t_0 + (j+1)T_s]\,. \end{cases}$$

Where P_0 and P_js are the optimization variables. The problem is adequately solved provided the length of T_s is sufficiently small in the domain Ω such that P_js can approximate a generic continuous matrix-valued $P(\cdot)$. Using similar structure, $X(\cdot)$ is also approximated as a piecewise continuous matrix-valued function.

4. EXAMPLES

In this section we apply the techniques developed in this paper to two illustrative examples; in particular, the second example, based on the model of an inverted pendulum on a cart, shows how to design a state feedback controller that satisfy prescribed bounds on the state trajectory in the presence of uncertainties.

4.1. Numerical example

Consider a fourth order LTI system, with system matrices defined by

$$A = \begin{pmatrix} -3.0000 & -1.0000 & -0.7500 & -0.5000 \\ 2.0000 & 0 & 0 & 0 \\ 0 & 2.0000 & 0 & 0 \\ 0 & 0 & 0.5000 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0.50 \\ 0.50 \\ 0.50 \\ 0.05 \end{pmatrix},$$

and with the following weighting matrices

$$R = 15I_n \,, \Gamma = 4I_n \,, Q = 2I_n \,, Y = 8 \,, Z = 4 \,.$$

We assume that the time interval of interest is $\Omega = [0, 10]$. Considering the open loop case (i. e. $K = 0$), since the DLMIs (9) are feasible, by Lemma 2.10 we can conclude that the system is FTS-GCB. As an example, Figure 2 shows that both the finite time stability and the guaranteed cost bound are respected when the initial condition $x_0 = \begin{pmatrix}0.05 & 0.05 & 0.05 & 0.05\end{pmatrix}^T$, satisfying $x_0^T R x_0 \leq 1$ is chosen.

Now we modify the system by introducing a norm bounded one-block from uncertainty, such that,

$$F = \begin{pmatrix}0.5 & 0.5 & 0.5 & 0.5\end{pmatrix}^T, E_1 = \begin{pmatrix}0.5 & 0.5 & 0.5 & 0.5\end{pmatrix}, E_2 = 0.1, \Delta = 0.6.$$

With the presence of uncertainties the property of finite time stability and of guaranteed cost bound are no more verified. Indeed as it is shown in Figure 3, if we choose $\Delta = 0.6$ the bounds on the state trajectory and on the quadratic cost are not satisfied (here again we choose $x_0 = \begin{pmatrix}0.05 & 0.05 & 0.05 & 0.05\end{pmatrix}^T$ as initial condition). It follows that our system is not RFTS-GCB, and indeed the DLMIs (21) result to be not feasible.

To make the system RFTS-GCB, we introduce a control input $u(t)$ as in (11) and look for a feedback matrix $K(\cdot)$, using the conditions (23) of Theorem 3.3. Since the DLMIs (23) are feasible, we are able to find a state feedback matrix $K(\cdot)$ such that makes the closed loop system RFTS-GCB. This is shown in Figures 4–6, where the behavior of the system for 100 randomly chosen realization of the uncertain parameter Δ is shown.

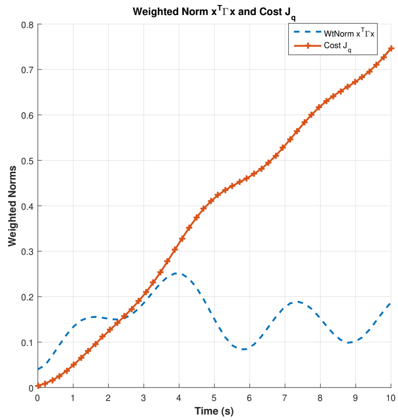

Fig. 2. Open loop behavior without uncertainties.

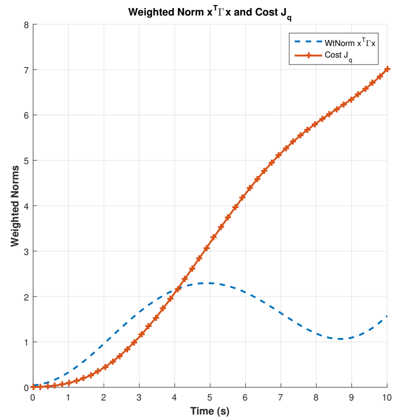

Fig. 3. Open loop behavior with uncertainties ($\Delta = 0.6$).

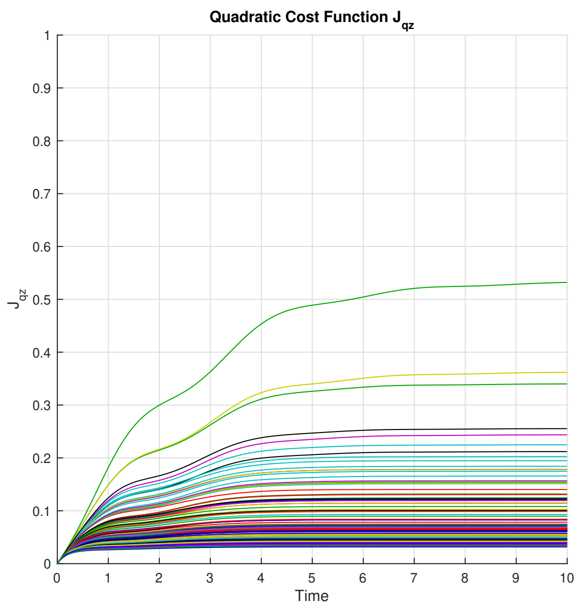

Fig. 4. Closed loop behavior of the quadratic cost functional for 100 randomly chosen realizations of Δ.

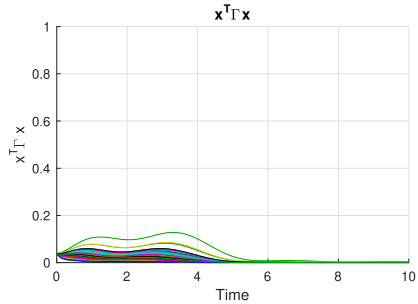

Fig. 5. Closed loop behavior of the $x^T \Gamma x$ for 100 randomly chosen realizations of Δ.

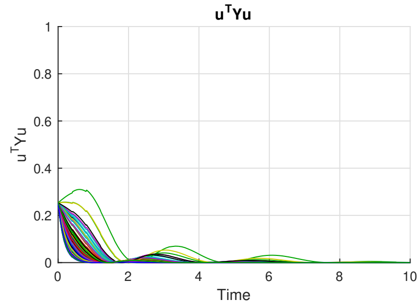

Fig. 6. Closed loop behavior of the $u^T Y u$ for 100 randomly chosen realizations of Δ.

4.2. Inverted pendulum on a cart

In this section we consider the problem of controlling an inverted pendulum on a cart as shown in Figure 7. The linearized model of the system is reported in equation (25), where the state vector is $x = \begin{pmatrix} \eta & \dot{\eta} & \theta & \dot{\theta} \end{pmatrix}^T$, with θ being the pendulum angle, and η being the horizontal position of the cart (see Figure 7). The parameters of the model are reported in Table 1.

m_c	cart mass	$0.5\ Kg$
m_p	pendulum mass	$0.2\ Kg$
L_p	pendulum length	$0.3\ m$
I_p	pendulum moment of inertia	$0.006\ Kg \cdot m^2$
b_c	coefficient of friction of the cart	$0.1\ N/(m \cdot s)$
η	cart horizontal position	
$\dot{\eta}$	cart horizontal velocity	
θ	pendulum angular position	
$\dot{\theta}$	pendulum angular velocity	
u	cart controlling input for stabilization	

Tab. 1. Inverted pendulum parameters.

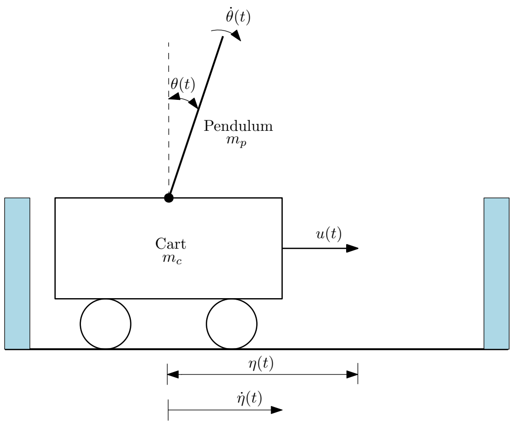

Fig. 7. Example of a cart with an inverted pendulum.

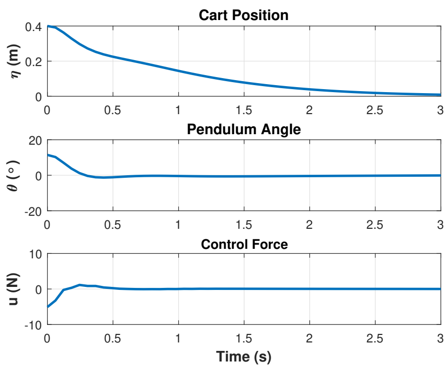

Fig. 8. Cart position η of pendulum, angle θ and control input $u(t)$.

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{b_c\left(I_p+m_pL_p^2\right)}{I_p(m_c+m_p)+m_pm_cL_p^2} & \frac{m_p^2L_p^2g}{I_p(m_c+m_p)+m_pm_cL_p^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{-b_pm_pL_p}{I_p(m_c+m_p)+m_pm_cL_p^2} & \frac{m_pL_pg(m_c+m_p)}{I_p(m_c+m_p)+m_pm_cL_p^2} & 0 \end{pmatrix} x(t)$$
$$+ \begin{pmatrix} 0 \\ \frac{I_p+m_pL_p^2}{I_p(m_c+m_p)+m_pm_cL_p^2} \\ 0 \\ \frac{mL}{I(M+m)+mML^2} \end{pmatrix} u(t)\,. \tag{25}$$

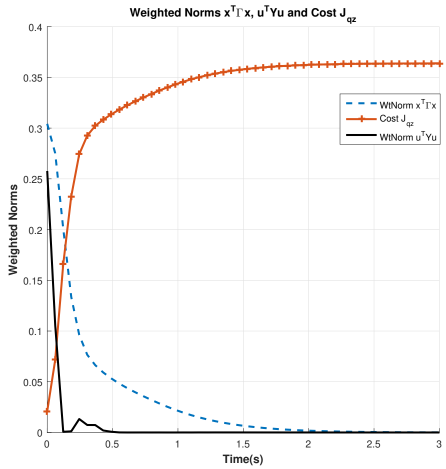

Fig. 9. Perfomance indices achieved by the RFTS-GCC state-feedback control for the inverted pendulum on a cart.

The considered time interval is $\Omega = [0, 3]$ and the respective weighting matrices are,

$$R = 8I_n\,, \Gamma = I_n\,, Q = 3I_n\,, Y = 1 \times 10^{-4}\,, Z = 0.5 \times 10^{-4}\,.$$

We also consider the presence of uncertainties in the model parameters, by considering a norm bounded one-block form uncertainty characterized by the following matrices,

$$\begin{aligned} F &= \begin{pmatrix} 0 & 0.9 & 0 & 0.9 \end{pmatrix}^T\,, \\ E_1 &= \begin{pmatrix} 0 & 0.05 & 0.3 & 0 \end{pmatrix}\,, \\ E_2 &= 0.012\,. \end{aligned}$$

We try to achieve the following FTS performance bound

$$\begin{aligned} |\eta| &< 0.5\ m\,, \\ |\theta| &< 12^\circ\,, \\ |u| &< 6\ N\,, \end{aligned}$$

in Ω when $x_0^T R x_0 \leq 1$. Figure 8 shows the obtained closed loop behavior when the system driven by the initial condition, $x_0 = [0.4, 0.1, 0.2, 0.1]^T$. The designed control law allows to satisfy the RFTS-GCC property as shown in Figure 9.

5. CONCLUSION

In this paper the finite-time stabilization problem for linear time-varying uncertain system is addressed. Sufficient conditions are given to design a state feedback controller which is able to achieve not only finite time stability of the plant, but also to guaranteed cost a given bound for an integral quadratic cost function. The design procedure requires the solution of a DLMI feasibility problem of DLMIs. Simulation results show the ease to analyze a system in the presence of uncertainties in the so-called norm bounded one-block form..

6. FUTURE RECOMMENDATIONS

Further enhancement in this line of research could be in establishing similar results in the output feedback case, and to extend the theory to impulsive and discrete time systems.

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