Kybernetika

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Kybernetika, Vol. 54 (2018), No. 4, 718–735

Persistent URL: `http://dml.cz/dmlcz/147420`

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KYBERNETIKA — VOLUME 54 (2018), NUMBER 4, PAGES 718–735

DELAY-DEPENDENT STABILITY OF RUNGE–KUTTA METHODS FOR LINEAR NEUTRAL SYSTEMS WITH MULTIPLE DELAYS

Guang-Da Hu

In this paper, we are concerned with stability of numerical methods for linear neutral systems with multiple delays. Delay-dependent stability of Runge-Kutta methods is investigated, i. e., for delay-dependently stable systems, we ask what conditions must be imposed on the Runge–Kutta methods in order that the numerical solutions display stability property analogous to that displayed by the exact solutions. By means of Lagrange interpolation, Runge–Kutta methods can be applied to neutral differential systems with multiple delays. Based on the argument principle, sufficient conditions for delay-dependent stability of Runge–Kutta methods combined with Lagrange interpolation are presented. Numerical examples are given to illustrate the main results.

Keywords: neutral differential systems with multiple delays, delay-dependent stability, Runge–Kutta method, Lagrange interpolation, argument principle

Classification: 65L05,65L07,65L20

1. INTRODUCTION

We are concerned with linear neutral systems with multiple delays described by

$$\dot{u}(t) = Lu(t) + \sum_{j=1}^{m}[M_j u(t-\tau_j) + N_j \dot{u}_j(t-\tau_j)] \tag{1}$$

satisfying the condition

$$\sum_{j=1}^{m} \|N_j\| < 1, \tag{2}$$

where $u(t) \in \mathcal{R}^n$, $\tau_j > 0$ for $j = 1, \ldots, m$ and $\tau_m > \tau_{m-1} > \cdots > \tau_1$.

Delay differential equations are widely used for describing mathematical modelling of various processes and systems in different applied problems [10, 11, 14, 16]. Stability of delay and neutral systems can be divided into two categories according to its dependence upon the size of delays. The stability which does not depend on delays is called delay-independent, whereas it depends on delays is referred to as delay-dependent. The stability of numerical methods is also divided into delay-independent and delay-dependent

DOI: 10.14736/kyb-2018-4-0718

according to they are applied to delay-independently stable and delay-dependently stable neutral systems. In this paper, we are concerned with delay-dependent stability of Runge–Kutta methods for neutral differential systems with multiple delays.

Stability of numerical methods has been discussed in [1, 5, 7, 15, 17] for delay and neutral differential equations. Delay-independent stability of numerical methods for delay and neutral differential equations has been investigated in [1, 5, 15, 17]. For delay-dependent stability of numerical methods, only few works have been reported [1, 7]. Up to now, only one delay case is investigated for delay-dependent stability of numerical methods in literature, for instant, [1, 7]. In the presented paper, we are concerned with delay-dependent stability of Runge–Kutta methods for neutral differential systems with multiple delays, i. e., for delay-dependently stable systems (1), we ask what conditions must be imposed on the Runge–Kutta methods in order that the numerical solutions display stability property analogous to that displayed by the exact solutions.

This paper is organized as follows. Several lemmas are reviewed in section 2. In section 3, delay-dependent stability of Runge–Kutta methods are discussed. Numerical examples are provided to illustrate the main results in section 4. Conclusions are given in section 5.

Throughout the paper, $||A||$ stands for the matrix operator norm. The j^{th} eigenvalue of A is denoted by $\lambda_j(A)$. The symbol $\rho(A)$ represents the spectral radius. $\operatorname{Re} z$ and $\operatorname{Im} z$ stand for the real part and the imaginary part of a complex number z, respectively. The open left half-plane $\{s : \operatorname{Re} s < 0\}$ is denoted by $\mathbb{C}^-$ and the right half-plane $\{s : \operatorname{Re} s \geq 0\}$ by $\mathbb{C}^+$.

2. PRELIMINARIES

In this section, several lemmas are reviewed. They will be used in sections 3 and 4, respectively.

A function $P(s)$ is said to be meromorphic in a domain D if it is analytic throughout D except poles. The following Argument Principle is well-known.

Lemma 2.1. (e. g. Brown and Churchill [2]) Suppose that

(i) a function $P(s)$ is meromorphic in the domain interior to a positively oriented simple closed contour γ;

(ii) $P(s)$ is analytic and nonzero on γ;

(iii) counting multiplicities, Z is the number of zeros and Y is the number of poles of $P(s)$ inside γ.

Then

$$\frac{1}{2\pi}\triangle_\gamma \arg P(s) = Z - Y,$$

where $\triangle_\gamma \arg P(s)$ represents the change of the argument of $P(s)$ along γ.

Lemma 2.2. (e. g. Lancaster and Tismenetsky [13]) For a complex matrix F, if $\|F\| < 1$, then the matrix

$$(I - F)^{-1} \quad \text{exists and} \quad \|(I - F)^{-1}\| \leq \frac{1}{1 - \|F\|},$$

where the matrix norm $\|\cdot\|$ is some operator norm (e. g. 1-norm, 2-norm or ∞-norm).

Now we consider the asymptotic stability of system (1) satisfying condition (2). The characteristic equation of system (1) is

$$P(s) = \det[sI - L - \sum_{j=1}^{m}(M_j \exp(-\tau_j s) + sN_j \exp(-\tau_j s))] = 0. \tag{3}$$

The asymptotic stability of system (1) satisfying condition (2) is determined by the position of the characteristic roots. System (1) is asymptotically stable if and only if all characteristic roots lie in the open left complex half-plane [3].

Now we will review the results in [4] which are concerned with stability of (1) satisfying condition (2). For completeness, the details of the proofs are provided in the appendix section.

Lemma 2.3. (Hu [4]) For system (1), assume that condition (2) holds. Let $\tilde{s}$ be an unstable characteristic root of Eq. (3), then

$$|\tilde{s}| \leq \beta = \frac{\|L\| + \sum_{j=1}^{m} \|M_j\|}{1 - \sum_{j=1}^{m} \|N_j\|}, \tag{4}$$

where the matrix norm $\|\cdot\|$ is some operator norm.

We need the following definition to present a stability criterion of system (1).

Definition 2.4. Assume that the conditions of Lemma 2.3 hold. The set D is defined by

$$D = \{s : \operatorname{Re} s \geq 0 \quad \text{and} \quad |s| \leq \beta\},$$

and its boundary is denoted by C. Here β is given by (4) in Lemma 2.3. See Figure 1, where $d_1 = i\beta$ and $d_2 = -i\beta$.

The following lemma will exclude all the unstable characteristic root of Eq. (3) from the set D. A necessary and sufficient condition for asymptotic stability of system (1) satisfying condition (2) is given by the argument principle.

Lemma 2.5. (Hu [4]) System (1) satisfying condition (2) is asymptotically stable if and only if

$$P(s) \neq 0 \quad \text{for} \quad s \in C \tag{5}$$

and

$$\triangle_C \arg P(s) = 0 \tag{6}$$

hold, where $\arg P(s)$ stands for the argument of $P(s)$ and $\triangle_C \arg P(s)$ change of the argument of $P(s)$ along the curve C.

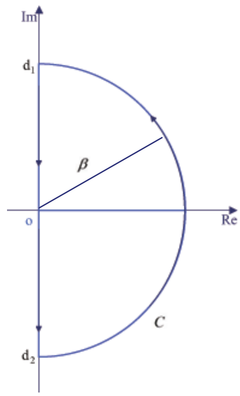

Fig. 1. Set D.

Remark 2.6. Since the spectrum of the neutral system tend to the spectrum of the associated difference operator at high frequencies, the neutral system may have infinitely many unstable roots. The stability of the neutral system may be very sensitive to small changes in delays. In order to discuss the sensitiveness of the stability, the concept of the strong stability has been introduced (e. g.[3]). The inequality (2) is a sufficient condition for the strong stability of the associated difference operator. Another sufficient condition, $\rho(\sum_{j=1}^{m} |N_j|) < 1$ has been given in [6], where $|W|$ denotes the nonnegative matrix with elements $|w_{jk}|$ for $W = \{w_{jk}\}$. Based on the condition $\rho(\sum_{j=1}^{m} |N_j|) < 1$, a bound for the unstable eigenvalues is derived [6] which is similar to β in Lemma 2.3. When the condition (2) does not hold, it is possible that the condition $\rho(\sum_{j=1}^{m} |N_j|) < 1$ holds. A further discussion on strong stability is given in [3]. If the associated difference operator is strongly stable, the neutral system has at most finitely many unstable roots [3, 4, 6, 14, 16]. If the neutral system is stable and the associated difference operator is strongly stable, the stability of the neutral system is insensitive to small changes in delays [3, 14, 16].

Now we describe an algorithm to check the delay-dependent stability of analytical solutions due to Lemma 2.5.

Algorithm 1

Step 0. Compute β by (4) and determine the curve C which consists of two parts, i. e., the segment $\{s = it;\ -\beta \le t \le \beta\}$ and the half-circle $\{s;\ |s| = \beta$ and $-\pi/2 \le \arg s \le \pi/2\}$.

Step 1. Take a sufficiently large integer $n \in \mathbb{N}$ and distribute n node points $\{s_j\}$ $(j = 0, 1, \ldots, n)$ on C as uniformly as possible. See Figure 1. Let

$$R(s) = sI - L - \sum_{i=1}^{m} (M_i \exp(-\tau_i s) + sN_i \exp(-\tau_i s)), \tag{7}$$

then

$$P(s) = \det R(s). \tag{8}$$

From (8), for each s_j $(j = 0, 1, \ldots, n)$, we have

$$P(s_j) = \det R(s_j), \tag{9}$$

where

$$R(s_j) = s_j I - L - \sum_{i=1}^{m} (M_i \exp(-\tau_i s_j) - s_j N_i \exp(-\tau_i s_j)). \tag{10}$$

Since $R(s_j)$ is a numerical matrix for each j, we can evaluate $P(s_j)$ by calculating the determinant of matrix $R(s_j)$ through the elementary row (or column) operations. Also we decompose $P(s_j)$ into its real and imaginary parts.

Step 2. We examine whether $P(s_j) = 0$ holds for each s_j $(j = 0, 1, \ldots, n)$ by checking its magnitude satisfies $|P(s_j)| \leq \delta$ with the preassigned tolerance δ. If it holds, i.e., $s_j \in C$ is a root of $P(s)$, then the neutral system is not asymptotically stable and stop the algorithm. Otherwise, go to the next step.

Step 3. We compute the change of the argument along the sequence

$$\{P(s_0), P(s_1), \ldots, P(s_n)\}.$$

If it is zero, then the system is asymptotically stable, otherwise not asymptotically stable.

Remark 2.7. Algorithm 1 does not evaluate the characteristic function $P(s)$, it calculates the determinant of numerical matrix $R(s_\ell)$ through the elementary row (or column) operations which are relatively efficient ways [8].

3. DELAY-DEPENDENT STABILITY OF RK METHODS

In this section, delay-dependent stability of Runge–Kutta (RK) methods for linear delay differential systems of neutral type is investigated, i.e., for delay-dependently stable systems (1), we ask what conditions must be imposed on the Runge–Kutta methods in order that the numerical solutions display stability property analogous to that displayed by the exact solutions. Based on the argument principle, sufficient conditions for delay-dependent stability of RK methods are presented.

First, we assume the numerical solution we are now discussing gives a sequence of approximate values $\{u_0, u_1, \ldots, u_n, \ldots\}$ of $\{u(t_0), u(t_1), \ldots, u(t_n), \ldots\}$ of (1) on certain equidistant step-values $\{t_n (= nh)\}$ with the step-size $h > 0$.

For the initial value problem of ordinary differential equations (ODEs)

$$\dot{y}(t) = f(t, y(t)), \quad \text{for} \quad t \geq 0 \tag{11}$$

an s-stage RK method for ODEs (11) is defined (e.g., [12]) by

$$k_i = f(t_n + c_i h, y_n + h \sum_{j=1}^{s} a_{ij} k_j) \quad (i = 1, 2, \ldots, s) \tag{12}$$

with

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i k_i. \tag{13}$$

We associate the s-square matrix A and the s-dimensional vector $\boldsymbol{b}$ as

$$A \stackrel{\text{def}}{=} (a_{ij}) \quad \text{and} \quad \boldsymbol{b} \stackrel{\text{def}}{=} (b_i).$$

We will analyse delay-dependent stability of RK method which is extended to apply to the neutral system. In order to solve numerically an asymptotically stable system (1) satisfying (2), we want to determine a step-size h such that the resulting difference system from the RK method is asymptotically stable, i. e.,

$$u_n \to 0 \quad \text{as} \quad n \to \infty$$

for any initial function $u(t) = \phi(t)$ with $-\tau_m \le t \le 0$. Denote the stage values of the s-stage RK formula by $K_{n,i}$. We can obtain the RK scheme for system (1) as follows [5].

$$\begin{aligned} K_{n,i} \quad &= \quad hL\left(u_n + \sum_{j=1}^{s} a_{ij} K_{n,j}\right) + h\sum_{k=1}^{m} M_k\left(u_{n-l_k+\delta_i} + \sum_{j=1}^{s} a_{ij} K_{n-l_k+\delta_k,j}\right) \\ &+ \quad \sum_{k=1}^{m} N_k K_{n-l_k+\delta_k,i}, \quad (i = 1, 2, \dots, s), \end{aligned} \tag{14}$$

and

$$u_{n+1} = u_n + \sum_{i=1}^{s} b_i K_{n,i}. \tag{15}$$

Here $i = 1, 2, \dots, s, a_{ij}$ and b_i stand for the parameters of the underlying RK method, $l_i = [\tau_i h^{-1}], \delta_i = l_i - \tau_i h^{-1}, 0 \le \delta_i < 1$ for $i = 1, \dots, m$ and $l_m \ge l_{m-1} \ge \cdots \ge l_1 \ge q+1$, where $[\sigma]$ denotes the smallest integer that is greater than or equal to $\sigma \in \mathcal{R}$, for $n = 1, 2, \dots,$

$$u_{n-l_i+\delta_i} = \phi((n - l_i + \delta_i)h),$$

and

$$K_{n-l_i+\delta_i,j} = \dot{\phi}((n - l_i + \delta_i)h)$$

with $n - l_i + \delta_i \le 0$, and $u_{n-l_i+\delta_i}$ and $K_{n-l_i+\delta_i,j}$ with $n - l_i + \delta_i \ge 0$ are defined by the following Lagrange interpolations, respectively.

$$u_{n-l_i+\delta_i} = \sum_{p=-r}^{q} L_p(\delta_i) u_{n-l_i+p}, \tag{16}$$

$$K_{n-l_i+\delta_i,j} = \sum_{p=-r}^{q} L_p(\delta_i) K_{n-l_i+p,j}, \tag{17}$$

for $0 \leq \delta_i < 1, i = 1\ldots, m, j = 1\ldots, s$ and

$$L_p(\delta) = \prod_{k=-r, k\neq p}^{q} \frac{\delta - k}{p - k}, \tag{18}$$

where $r, q \geq 0$ are integers and $r \leq q \leq r+2$.

The above scheme is a natural RK method combined with Lagrange interpolation, see [1]. In [10] a natural RK method combined with Lagrange interpolation for delay differential equations is discussed. In the sequel, we only consider the natural RK method combined with Lagrange interpolation. The following lemma is useful to prove the main results in this paper.

Lemma 3.1. The characteristic polynomial $P_{RK}(z)$ of the resulting difference system (14) with (15) from the natural RK scheme combined with Lagrange interpolation is given by

$$\begin{aligned} P_{RK}(z) = \det \Bigg\{ & \begin{bmatrix} I_{sd} - h(A \bigotimes L) & 0 \\ -b^T \bigotimes I_d & I_d \end{bmatrix} z^{l_m+r+1} - \begin{bmatrix} 0 & h(e \bigotimes L) \\ 0 & I_d \end{bmatrix} z^{l_m+r} \\ & - \sum_{i=1}^{m} \begin{bmatrix} (h(A \bigotimes M_i) + I_s \bigotimes N_i) z & h(e \bigotimes M_i) \\ 0 & 0 \end{bmatrix} \sum_{p=-r}^{q} L_p(\delta_i) z^{l_m - l_i + r + p} \Bigg\}. \end{aligned} \tag{19}$$

where the s-dimensional vector e is defined by $e = (1, 1, \ldots, 1)^T$.

Proof. The difference system (14) with (15) can be expressed with the Kronecker product as follows:

$$\begin{aligned} & \begin{bmatrix} I_{sd} - h(A \bigotimes L) & 0 \\ -b^T \bigotimes I_d & I_d \end{bmatrix} \begin{bmatrix} K_n \\ u_{n+1} \end{bmatrix} - \begin{bmatrix} 0 & h(e \bigotimes L) \\ 0 & I_d \end{bmatrix} \begin{bmatrix} K_{n-1} \\ u_n \end{bmatrix} \\ & - \sum_{i=1}^{m} \begin{bmatrix} h(A \bigotimes M_i) + I_s \bigotimes N_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sum_{p=-r}^{q} L_p(\delta_i) K_{n-l_i+p} \\ \sum_{p=-r}^{q} L_p(\delta_i) u_{n-l_i+1+p} \end{bmatrix} \\ & - \sum_{i=1}^{m} \begin{bmatrix} 0 & h(e \bigotimes M_i) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sum_{p=-r}^{q} L_p(\delta_i) K_{n-l_i-1+p} \\ \sum_{p=-r}^{q} L_p(\delta_i) u_{n-l_i+p} \end{bmatrix} = 0, \end{aligned} \tag{20}$$

where the (ds)-dimensional vector K_n means

$$K_n = [K_{n,1}^T, K_{n,2}^T, \ldots, K_{n,s}^T]^T.$$

Hence the dimension of the vector $\begin{bmatrix} K_n \\ u_{n+1} \end{bmatrix}$ becomes $(s+1)d$.

Taking z-transform to (20) and introducing as $Z\left\{ \begin{bmatrix} K_{n-l_m-1-r} \\ u_{n-l_m-r} \end{bmatrix} \right\} = V(z)$, we obtain that

$$\left\{\left[\begin{array}{cc} I_{sd}-h(A\bigotimes L) & 0 \\ -b^T\bigotimes I_d & I_d \end{array}\right] z^{l_m+r+1} - \left[\begin{array}{cc} 0 & h(e\bigotimes L) \\ 0 & I_d \end{array}\right] z^{l_m+r} \right.$$
$$\left. -\sum_{i=1}^{m}\left(\left[\begin{array}{cc} h(A\bigotimes M_i)+I_s\bigotimes N_i & 0 \\ 0 & 0 \end{array}\right] z - \left[\begin{array}{cc} 0 & h(e\bigotimes M_i) \\ 0 & 0 \end{array}\right]\right)\sum_{p=-r}^{q} L_p(\delta_i) z^{l_m-l_i+r+p}\right\}$$
$$V(z)=0.$$

Hence, the characteristic polynomial of difference system is given as desired. □

For delay-dependent stability of an explicit RK method, we have the following result.

Theorem 3.2. For an explicit RK method, assume that

(i) system (1) satisfying condition (2) is asymptotically stable for given matrices L, M_i, N_i and delays τ_i for $i=1,\dots,m$, i. e., Lemma 2.5 holds;

(ii) the underlying RK method is of s-stage and natural;

(iii) for a step-size h, the rational function $Q_{RK}(z)$ never vanishes on the unit circle $\mu=\{z:|z|=1\}$ and its change of argument satisfies

$$\frac{1}{2\pi}\triangle_\mu \arg Q_{RK}(z) = d(s+1), \tag{21}$$

where $Q_{RK}(z)$ relates to the RK scheme and is defined by

$$Q_{RK}(z)=\det\left\{\left[\begin{array}{cc} I_{sd}-h(A\bigotimes L) & 0 \\ -b^T\bigotimes I_d & I_d \end{array}\right] z - \left[\begin{array}{cc} 0 & h(e\bigotimes L) \\ 0 & I_d \end{array}\right]\right.$$
$$\left. -\sum_{i=1}^{m}\left[\begin{array}{cc} (h(A\bigotimes M_i)+I_s\bigotimes N_i)z & h(e\bigotimes M_i) \\ 0 & 0 \end{array}\right]\sum_{p=-r}^{q} L_p(\delta_i) z^{-l_i+p}\right\}. \tag{22}$$

Then the resulting difference system (14) with (15) from the RK method combined with Lagrange interpolation is asymptotically stable.

P r o o f. The difference system (14) with (15) is asymptotically stable if and only if all the characteristic root of $P_{RK}(z)=0$ lie in the inner of the unit circle, i. e.,

$$P_{RK}(z)=0 \quad\Rightarrow\quad |z|<1. \tag{23}$$

Notice that the coefficient matrix of the term z^{l_m+1+r} in $P_{RK}(z)=0$ is

$$\left[\begin{array}{cc} I_{sd}-h(A\bigotimes L) & 0 \\ -b^T\bigotimes I_d & I_d \end{array}\right].$$

Since the underlying RK method is explicit, $a_{ij}=0$ for $i\le j$. Hence $\lambda_i(A)=0$ holds for $i=1,\dots,s$. It means that all the eigenvalues of matrix $h(A\bigotimes L)$ vanish because of

$$\text{eigenvalue of } (hA\bigotimes L)=h\lambda_i(A)\lambda_j(L)=0$$

with $i = 1, \ldots, s$ and $j = 1, \ldots, d$. Thus that matrix $I_{sd} - h(A \bigotimes L)$ is nonsingular by Lemma 2.2 and the matrix

$$\begin{bmatrix} I_{sd} - h(A \bigotimes L) & 0 \\ -b^T \bigotimes I_d & I_d \end{bmatrix}$$

is also nonsingular. It means that the degree of $P_{RK}(z)$ is $d(s+1)(l_m+r+1)$ and $P_{RK}(z) = 0$ has $d(s+1)(l_m+r+1)$ roots in total by counting their multiplicity. The relationship between $P_{RK}(z)$ and $Q_{RK}(z)$ is as follows.

$$P_{RK}(z) = z^{(l_m+r)d(s+1)} Q_{RK}(z). \tag{24}$$

By (24)

$$\arg P_{RK}(z) = \arg z^{(l_m+r)d(s+1)} Q_{RK}(z) = \arg z^{(l_m+r)d(s+1)} + \arg Q_{RK}(z). \tag{25}$$

By (25), we have

$$\frac{1}{2\pi} \triangle_\mu \arg P_{RK}(z) = \frac{1}{2\pi} \triangle_\mu \arg z^{(l_m+r)d(s+1)} + \frac{1}{2\pi} \triangle_\mu \arg Q_{RK}(z). \tag{26}$$

According to (21) and (26),

$$\begin{aligned} &\frac{1}{2\pi} \triangle_\mu \arg P_{RK}(z) = \frac{1}{2\pi} \triangle_\mu \arg z^{(l_m+r)d(s+1)} + \frac{1}{2\pi} \triangle_\mu \arg Q_{RK}(z) \\ &= d(l_m+r)(s+1) + d(s+1) = d(s+1)(l_m+r+1). \end{aligned} \tag{27}$$

By (24) and condition (iii),

$$\arg P_{RK}(z) \neq 0 \quad \text{for} \quad z \in \mu. \tag{28}$$

Since (27) and (28) hold, by means of Lemma 2.1, all the roots of $P_{LM}(z) = 0$ lie in the inside of the unit circle $|z| = 1$. The proof completes. □

Remark 3.3. Assume that the orders of the underlying RK method and the Lagrange interpolation are p and q, respectively, then the order of the natural RK method with the Lagrange interpolation is $\min\{p, q\}$. See [1] for neutral differential equations and [10] for delay differential equations.

For an implicit RK method applied to the neutral system, we derive the following result.

Theorem 3.4. For an implicit RK method, assume that

(i) the conditions (i),(ii) and (iii) in Theorem 3.2 hold;

(ii) the product $h\lambda_i(A)\lambda_j(L)$ never equals to unity for all i $(1 \le i \le s)$ and j $(1 \le j \le d)$.

Then the resulting difference system (14) with (15) from the RK method combined with Lagrange interpolation is asymptotically stable.

Proof. The proof can be carried out similarly to that of Theorem 3.2. Notice that the condition (ii) ensures the matrix

$$\begin{bmatrix} I_{sd} - h(A \bigotimes L) & 0 \\ -b^T \bigotimes I_d & I_d \end{bmatrix}$$

is nonsingular since the matrix $I_{sd} - h(A \bigotimes L)$ is nonsingular. Thus the degree of the polynomial $P_{RK}(z)$ becomes $d(s+1)(l_m + r + 1)$. □

Now we can describe an algorithm to check the conditions of Theorems 3.2 and 3.4.

Algorithm 2

Step 1. Taking a sufficiently big $n \in \mathbb{N}$, we compute $n+1$ nodes $\{z_0, z_1, \dots, z_n\}$ upon the unit circle μ of z-plane so as $\arg z_\ell = (2\pi)\ell/n$. For each $z_\ell (\ell = 0, 1 \dots n)$, we evaluate the rational function $Q_{RK}(z)$ related to the RK scheme. That is, we evaluate it by computing the determinant

$$Q_{RK}(z_\ell) = \det G(z_\ell) \tag{29}$$

where the numerical matrix $G(z_\ell)$ is defined by

$$G(z_\ell) = \begin{bmatrix} I_{sd} - h(A \bigotimes L) & 0 \\ -b^T \bigotimes I_d & I_d \end{bmatrix} z_\ell - \begin{bmatrix} 0 & h(e \bigotimes L) \\ 0 & I_d \end{bmatrix}$$
$$- \sum_{i=1}^{m} \begin{bmatrix} (h(A \bigotimes M_i) + I_s \bigotimes N_i) z_\ell & h(e \bigotimes M_i) \\ 0 & 0 \end{bmatrix} \sum_{p=-r}^{q} L_p(\delta_i) z_\ell^{-l_i + p}.$$

Also we decompose $Q_{RK}(z_\ell)$ into its real and imaginary parts.

Step 2. We examine whether $Q_{RK}(z_\ell) = 0$ holds for each z_ℓ $(\ell = 0, 1, \dots, n)$ by checking its magnitude satisfies $|Q_{RK}(z_\ell)| \le \delta$ with the preassigned small positive tolerance δ. If it holds, i. e., $z_\ell \in \mu$ is a root of $Q_{RK}(z)$, then the numerical scheme for the neutral system is not asymptotically stable and stop the algorithm. Otherwise, to go to the next step.

Step 3. We compute the change of the argument along the sequence $\{Q_{RK}(z_\ell)\}$. If it equals to $d(s+1)$, then the numerical scheme for the neutral system is asymptotically stable, otherwise not stable.

Remark 3.5. From the above theorems, in order to solve numerically an asymptotically stable delay differential system of neutral type by a RK method combined with Lagrange interpolation, it is enough for us to choose a step-size h such that the resulting difference system is asymptotically stable.

Remark 3.6. Both Schur-Cohn and Jury stability criteria [9] need information of all the coefficients of the characteristic polynomial $P_{RK}(z)$. It is an ill-posed problem to compute all the coefficients of the characteristic polynomial for a high dimensional matrix [8]. Although Schur-Cohn and Jury stability criteria can be applied to the resulting difference systems from RK methods in theoretical sense, they can not work well in practice when l_m or d are big. Algorithm 2 does not compute the coefficients of the characteristic polynomial $P_{RK}(z)$, it evaluates the determinant of numerical matrix $G(z_\ell)$ through the elementary row (or column) operations which are relatively efficient ways [8].

Remark 3.7. For a linear multi-step method combined with Lagrange interpolation which is applied to an asymptotically stable system (1) satisfying condition (2), similar results can be derived. This is a current research topic of ours.

4. NUMERICAL EXAMPLES

In this section, two numerical examples are given to demonstrate the main results in section 3. The 2-matrix norm, $||F||_2 = \sqrt{\lambda_{\max}(F^T F)}$ is used. The classical fourth-order RK formulas for ODEs (11) is as follows.

$$\begin{aligned} k_1 &= f(t_n, y_n), \\ k_2 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1), \\ k_3 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2), \\ k_4 &= f(t_{n+1}, y_n + hk_3), \\ y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4). \end{aligned}$$

This is the underlying scheme of the natural RK for (1). Let $r = q = 2$ for Lagrange interpolation (18).

Example 4.1. Consider the two-dimensional linear neutral system (1) with five delays. The parameter matrices of the system are as follows.

$$L = \begin{bmatrix} -5 & 2 \\ 3 & -3 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 2 & -1.9 \\ 0.9 & 1.3 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} -0.2 & 0.1 \\ 0.4 & 0.3 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0.2 & -0.1 \\ 0.6 & 0.7 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 0.6 & 0.1 \\ 0.3 & -0.5 \end{bmatrix}, \quad M_5 = \begin{bmatrix} 0.02 & -0.01 \\ 0.03 & 0.01 \end{bmatrix},$$

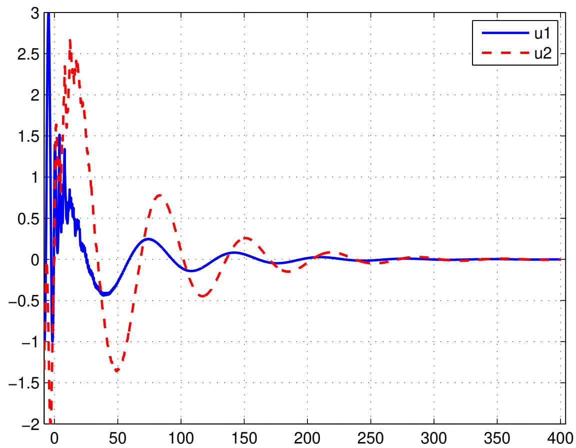

Fig. 2. Numerical solutions are asymptotically stable when $h = 0.1$ for $\tau_1 = 4, \tau_2 = 5, \tau_3 = 6, \tau_4 = 7, \tau_5 = 8$ in Example 4.1.

$$N_1 = \begin{bmatrix} 0.2 & 0.1 \\ -0.1 & 0 \end{bmatrix}, \; N_2 = \begin{bmatrix} 0.1 & 0.2 \\ -0.1 & 0.2 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & -0.1 \end{bmatrix}, \quad N_4 = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad \text{and} \quad N_5 = \begin{bmatrix} 0.06 & -0.04 \\ 0 & -0.03 \end{bmatrix}.$$

Let the initial vector function be

$$u(t) = \begin{bmatrix} 2\sin t + 1 \\ \cos t - 1 \end{bmatrix} \quad \text{for} \quad t \in [-\tau_5, 0].$$

Since $\Sigma_{j=1}^{5} \|N_j\| = 0.9273 < 1$, the condition (2) holds. We have $\beta = 160.5469$.

The case of $\tau_1 = 4, \tau_2 = 5, \tau_3 = 6, \tau_4 = 7, \tau_5 = 8$. First we analyze the stability of the system by Lemma 2.5. We have that $P(s) \neq 0$ for $s \in C$ and $\triangle_C \arg P(s) = 0$ along the curve C. Lemma 2.5 tells that the system with the given parameter matrices is asymptotically stable. Now we investigate the numerical stability of the RK scheme combined with Lagrange interpolation by Algorithm 2 with $n = 3.2 \times 10^5$. When $h = 0.1$, we obtain that $Q_{RK}(z) \neq 0$ for $z \in \mu$ and $\triangle_\mu \arg Q_{RK}(z) = d(s+1) = 2 \times (4+1) = 10$. Theorem 3.2 asserts the RK method for the system is asymptotically stable. The numerical solution is converging to 0, is depicted in Figure 2. Conversely, when $h = 0.3$, we obtain that $Q_{RK}(z) \neq 0$ for $z \in \mu$ and $\triangle_\mu \arg Q_{RK}(z) = 8 \neq d(s+1) = 10$, the theorem does not hold. The numerical solution is divergent and its behaviour is shown in Figure 3.

The case of $\tau_1 = 10, \tau_2 = 11, \tau_3 = 12, \tau_4 = 13, \tau_5 = 14$.. First we analyze the stability of the system by Lemma 2.5. We have that $P(s) \neq 0$ for $s \in C$ and $\triangle_C \arg P(s) = 2 \neq 0$

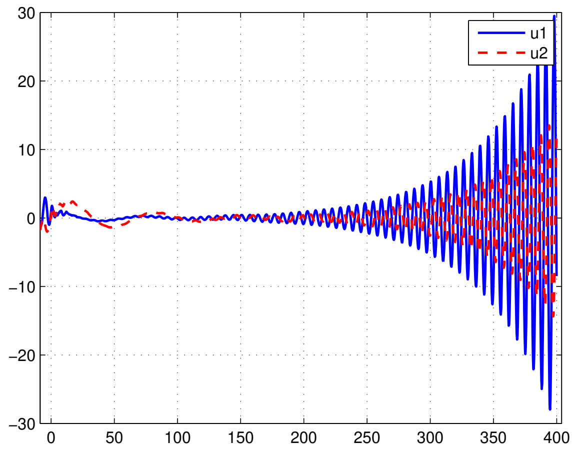

Fig. 3. Numerical solutions are not stable when $h = 0.3$ for $\tau_1 = 4, \tau_2 = 5, \tau_3 = 6, \tau_4 = 7, \tau_5 = 8$ in Example 4.1.

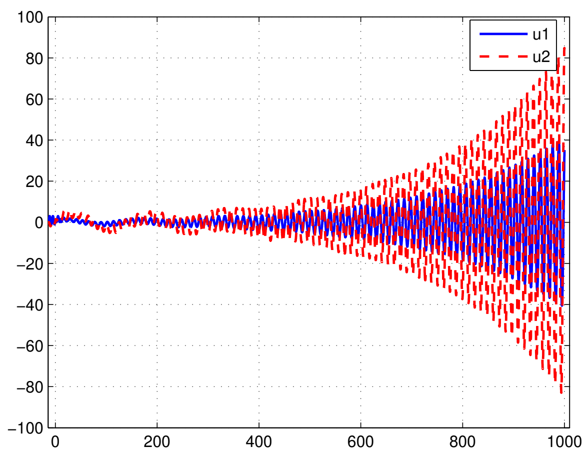

Fig. 4. Numerical solutions are not stable when $h = 0.1$ for $\tau_1 = 10, \tau_2 = 11, \tau_3 = 12, \tau_4 = 13, \tau_5 = 14$ in Example 4.1.

along the curve C. Lemma 2.5 tells that the system with the given parameter matrices is not asymptotically stable. Then the assumptions of Theorem 3.2 do not hold and the numerical solution is not guaranteed to be asymptotically stable. In fact, its figure given in Figure 4 shows a divergence for $h = 0.1$. We also carry out experiments for $h = 0.01$ and $h = 0.001$, respectively, the numerical solutions are still divergent.

Example 4.2. We take the three-dimensional linear neutral system (1) with three delays. The parameter matrices of the system are as follows.

$$L = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{bmatrix}, \quad M_1 = \begin{bmatrix} -1 & 0 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & -1 & 2 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 0.1 & 0.1 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.2 & 0.1 & 0 \\ 0.1 & 0 & 0.1 \end{bmatrix} \quad \text{and} \quad N_3 = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.3 & 0 \\ 0.1 & 0 & 0.1 \end{bmatrix}.$$

Let the initial vector function be

$$u(t) = \begin{bmatrix} 2\sin t + 1 \\ \cos t - 1 \\ \sin 2t + 1 \end{bmatrix} \quad \text{for} \quad t \in [-\tau_3, 0].$$

Since $\Sigma_{j=1}^{3} \|N_j\| = 0.7618 < 1$, the condition (2) holds. We have $\beta = 51.4434$.

The case of $\tau_1 = 0.7, \tau_2 = 0.8, \tau_3 = 0.9$. First we analyze the stability of the system by Lemma 2.5. We have that $P(s) \neq 0$ for $s \in C$ and $\triangle_C \arg P(s) = 0$ along the curve C. Lemma 2.5 tells that the system with the given parameter matrices is asymptotically stable. Now we investigate the numerical stability of the RK scheme combined with Lagrange interpolation by Algorithm 2 with $n = 3.2 \times 10^5$. When $h = 0.1$, we obtain that $Q_{RK}(z) \neq 0$ for $z \in \mu$ and $\triangle_\mu \arg Q_{RK}(z) = d(s+1) = 3\times(4+1) = 15$. Theorem 3.2 asserts the RK method for the system is asymptotically stable. The numerical solution is converging to 0, is depicted in Figure 5. Conversely, when $h = 0.5$, we obtain that $Q_{RK}(z) \neq 0$ for $z \in \mu$ and $\triangle_\mu \arg Q_{RK}(z) = 13 \neq d(s+1) = 15$, the theorem does not hold. The numerical solution is divergent and its behaviour is shown in Figure 6.

The case of $\tau_1 = 0.9, \tau_2 = 2, \tau_3 = 3$. First we analyze the stability of the system by Lemma 2.5. We have that $P(s) \neq 0$ for $s \in C$ and $\triangle_C \arg P(s) = 2 \neq 0$ along the curve C. Lemma 2.5 tells that the system with the given parameter matrices is not asymptotically stable. Then the assumptions of Theorem 3.2 do not hold and the numerical solution is not guaranteed to be asymptotically stable. In fact, its figure given in Figure 7 shows a divergence for $h = 0.1$. We also carry out experiments for $h = 0.01$ $h = 0.001$, respectively, the numerical solutions are still divergent.

Remark 4.3. The two numerical examples show that the main results are valid for actual computation. The main results explain that our following experiences are reasonable: In order to solve numerically an asymptotically stable system (1), if one chooses a small step-size h, it is possible that the resulting difference system from RK method combined with Lagrange interpolation is asymptotically stable.

Remark 4.4. When an explicit RK formula is unstable for system (1), an implicit RK scheme may be considered.

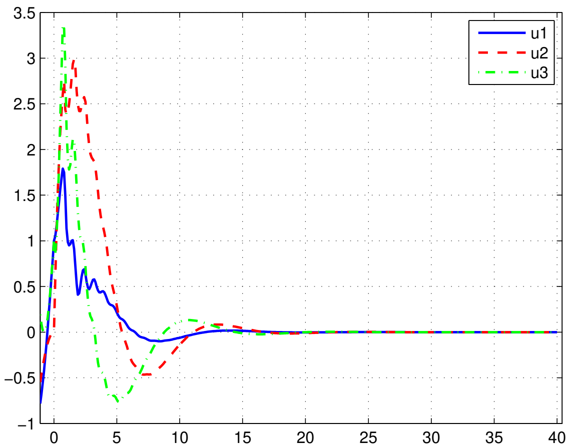

Fig. 5. Numerical solutions are asymptotically stable when $h = 0.1$ for $\tau_1 = 0.7, \tau_2 = 0.8, \tau_3 = 0.9$ in Example 4.2.

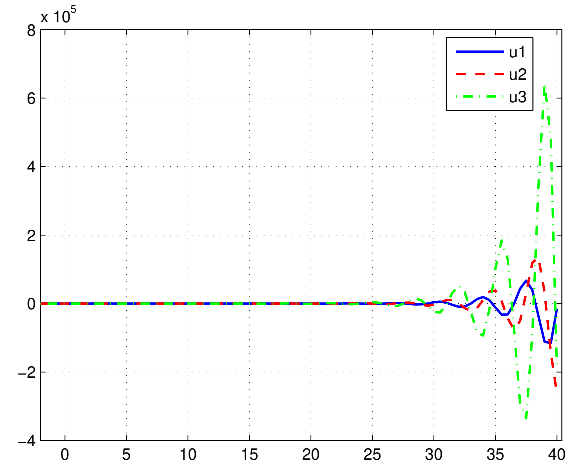

Fig. 6. Numerical solutions are not stable when $h = 0.5$ for $\tau_1 = 0.7, \tau_2 = 0.8, \tau_3 = 0.9$ in Example 4.2.

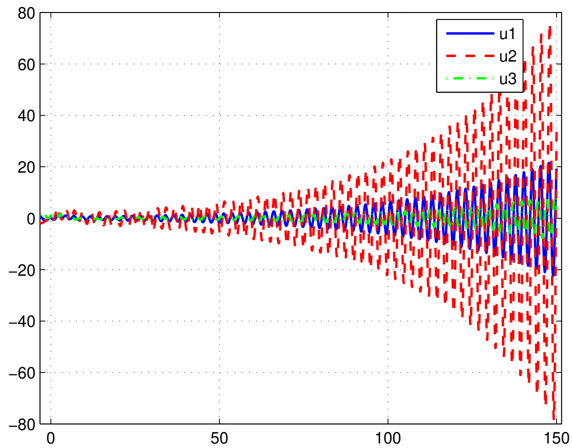

Fig. 7. Numerical solutions are not stable when $h = 0.1$ for $\tau_1 = 0.9, \tau_2 = 2, \tau_3 = 3$ in Example 4.2.

5. CONCLUSIONS

Theorems 3.2 and 3.4 provide sufficient conditions for delay-dependent stability of RK methods. They show that it is possible that there is a step-size h such that the resulting difference system from a RK method combined with Lagrange interpolation is asymptotically stable for an asymptotically stable system (1) satisfying (2). The theorems are useful for practical computation.

6. APPENDIX

P r o o f of Lemma 2.3. Let $\tilde{s}$ be an unstable characteristic root of Eq.(3). As we are discussing an unstable root, we assume $\mathrm{Re}\, \tilde{s} \geq 0$ throughout the proof. Since

$$\| \sum_{j=1}^{m} N_j \exp(-\tau_j \tilde{s}) \| \leq \sum_{j=1}^{m} \|N_j\| < 1$$

which assures the existence of $(I - \sum_{j=1}^{m} N \exp(-\tau_j \tilde{s}))^{-1}$ by Lemma 2.2. Introduction of

$$W(s) = (I - \sum_{j=1}^{m} N_j \exp(-\tau_j s))^{-1} (L + \sum_{j=1}^{m} M_j \exp(-\tau_j s))$$

for $\operatorname{Re} s \geq 0$, rewrites $P(s)$ as

$$\begin{aligned} P(s) &= \det[sI - L - \sum_{j=1}^{m} M_j \exp(-\tau_j s) - s\sum_{j=1}^{m} N_j \exp(-\tau_j s)] \\ &= \det\left([I - \sum_{j=1}^{m} N_j \exp(-\tau_j s)][sI - W(s)]\right). \end{aligned}$$

Since $\rho(\sum_{j=1}^{m} N_j \exp(-\tau_j \tilde{s})) \leq \sum_{j=1}^{m} \|N_j\| < 1$, $P(\tilde{s}) = 0$ implies the equation

$$\det[\tilde{s}I - W(\tilde{s})] = 0 \tag{30}$$

must hold. This implies the $\tilde{s}$ is an eigenvalue of the matrix $W(\tilde{s})$ and there exists an integer j $(1 \leq j \leq d)$ such that

$$\tilde{s} = \lambda_j(W(\tilde{s})). \tag{31}$$

According to (31) and Lemma 2.2, we have that

$$\begin{aligned} |\tilde{s}| &= |\lambda_j(W(\tilde{s}))| \leq \|W(\tilde{s})\| = \|(I - \sum_{j=1}^{m} N_j \exp(-\tau_j \tilde{s}))^{-1}(L + \sum_{j=1}^{m} M_j \exp(-\tau_j \tilde{s}))\| \\ &\leq \|(I - \sum_{j=1}^{m} N_j \exp(-\tau_j \tilde{s}))^{-1}\| \|L + \sum_{j=1}^{m} M_j \exp(-\tau_j \tilde{s})\| \\ &\leq \frac{\|L + \sum_{j=1}^{m} M_j \exp(-\tau_j \tilde{s})\|}{1 - \|\sum_{j=1}^{m} N_j \exp(-\tau_j \tilde{s})\|} \leq \frac{\|L\| + \sum_{j=1}^{m} \|M_j\|}{1 - \sum_{j=1}^{m} \|N_j\|} \stackrel{\text{def}}{=} \beta. \end{aligned}$$

Thus the proof is completed. $\square$

Proof of Lemma 2.5. Suppose the system is asymptotically stable. All zeros of $P(s)$ are located on the left half plane $\mathbb{C}^-$. It means that $P(s) \neq 0$ when $\operatorname{Re} s \geq 0$. By the argument principle, we have that (5) and (6) hold.

Conversely, assume that the conditions (5) and (6) hold. According to Lemma 2.3, it means that $P(s)$ never vanishes for $\operatorname{Re} s \geq 0$. Hence (5) and (6) imply system (1) is asymptotically stable. Thus the proof completes. $\square$

ACKNOWLEDGEMENT

This work was supported by the National Natural Science Foundation of China (11471217).

The author wishes to thank Miss Fangyuan Zhao for implementing the numerical experiments. The author also wishes to thank the associate editor and the anonymous reviewers who kindly gave the detailed and valuable comments which are helpful to improve this manuscript.

(Received December 6, 2016)

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