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CONTROLLABILITY OF LINEAR IMPULSIVE MATRIX LYAPUNOV DIFFERENTIAL SYSTEMS WITH DELAYS IN THE CONTROL FUNCTION

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In this paper, we establish the controllability conditions for a finite-dimensional dynamical control system modelled by a linear impulsive matrix Lyapunov ordinary differential equations having multiple constant time-delays in control for certain classes of admissible control functions. We characterize the controllability property of the system in terms of matrix rank conditions and are easy to verify. The obtained results are applicable for both autonomous (time-invariant) and non-autonomous (time-variant) systems. Two numerical examples are given to illustrate the theoretical results obtained in this paper.

Keywords: matrix Lyapunov systems, controllability, impulsive differential systems, delays

Classification: 34A37, 93B05, 93C05, 93C15

1. INTRODUCTION

It is known that many evolution processes, for example, the biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, stimulated neural networks, certain trajectories of missiles and aircrafts, frequency modulated systems etc. do exhibit the impulsive effects, that is, abrupt changes in their states, at certain moments of times. Such processes are modelled by impulsive differential equations. Readers can refer a monograph by Lakshmikantham et al. [25] to understand the impulsive systems in detail. In the past three decades, we observed a growing interest by many mathematicians in the study of controllability of impulsive dynamical systems due to their significance in both theory and applications. Roughly speaking, a dynamical system is said to be state-controllable over some space V on some finite time interval, if it is possible to steer that system from every initial state in V to every desired final state in V by using the set of admissible control functions. The controllability of impulsive systems was first studied by Leela et al. [26] and from then onwards many other mathematicians have contributed in the development of controllability of different types of impulsive systems, for example, refer [4, 11, 13, 14, 15, 29, 32, 33, 34] etc.

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Time-delay is one of the inevitable problem which arise in many practical applications, like, continuum mechanics, population dynamics, ecology, systems theory, viscoelasticity, biology, epidemics etc. Refer the book by Erneux [10] and references therein for the study on delay systems. A control system may experience time-delay either in state or in control or in both. Further these time-delays can be constant or variable. The controllability of systems possessing constant time-delays in control was studied by many authors since 1970's, for example [1, 6, 16, 19, 20, 22, 23, 28, 30, 31] etc, whereas the controllability results of systems having variable time-delays in control appeared in [2, 3, 7, 17, 18, 24] and references therein.

Murthy et al. [27] studied the controllability of the linear matrix Lyapunov systems. Later their work has been extended to the nonlinear systems in [8]. More recently, in [9], Dubey and George investigated the controllability of matrix Lyapunov impulsive differential systems of both linear and nonlinear type in a finite-dimensional space. Now, if such impulsive systems involves time-delays in control function, then finding the controllability conditions will become more complex due to the coexistence of impulses and delays in the systems.

In this article, we establish the controllability results of linear impulsive matrix Lyapunov systems having multiple constant time-delays in control, in terms of a matrix rank condition for certain classes of admissible control functions. Further, under each class of admissible control functions, we computed the corresponding steering control. The obtained controllability conditions are reduced for the corresponding system without impulses and with delays; system with impulses and without delays; system without impulses and without delays. The numerical example given in the last section of this paper, will help the reader to compare how the controlled trajectory and steering control behaves under different classes of admissible control functions.

The rest of this paper is organized as follows: in section 2, some preliminaries are given and the controllability problem for a class of linear impulsive matrix Lyapunov differential system with multiple constant time delays in control is formulated. In section 3, the matrix Lyapunov differential system is converted into vector differential system, by applying vector operator. Section 4 contains the main results of the paper, where we establish the controllability results of the system for certain classes of admissible control functions. Finally in section 5, we give the illustrative examples for an autonomous system to demonstrate our theoretical results. Further, the control function and controlled trajectory are plotted for the given classes of admissible control functions.

2. PRELIMINARIES AND SYSTEM DESCRIPTION

This section begins with functional settings required to establish the results of this paper. The natural space to work on the solvability of linear impulsive matrix Lyapunov differential control systems with multiple constant time-delays in control is the real Banach space given by

$$\mathcal{B}_1 := \Big\{ \mathbf{X}(\cdot) \,\Big|\, \mathbf{X}(\cdot) : [t_0, T] \to \mathbb{R}^{n\times n},\ \mathbf{X}(\cdot) \text{ is a continuous function on } [t_0,T] \setminus \{t_k : k = 1,$$
$$\ldots, M\} \text{ and differentiable a.e. on } [t_0,T] \text{ such that there exists } \mathbf{X}(t_k^-) := \lim_{t\uparrow t_k} \mathbf{X}(t)$$

$$\text{and } \mathbf{X}(t_k^+) := \lim_{t \downarrow t_k} \mathbf{X}(t) \text{ with } \mathbf{X}(t_k^-) = \mathbf{X}(t_k),\ \forall\, k \text{ and } \mathbf{X}(t_0) = \lim_{t \downarrow t_0} \mathbf{X}(t) \Big\},$$

endowed with the norm

$$\|\mathbf{X}(\cdot)\|_{\mathcal{B}_1} := \sup_{t \in [t_0, T]} \|\mathbf{X}(t)\|_{\mathcal{F}}.$$

Here a.e. stands for 'almost everywhere', which is defined as follows: A property $\mathcal{P}$ is said to holds a.e. on $[t_0, T]$, if the following conditions are satisfied:

(i) The property $\mathcal{P}$ holds on a subset S of $[t_0, T]$.

(ii) If the property $\mathcal{P}$ fails to satisfy on $[t_0, T] \setminus \mathrm{S}$, then the Lebesgue measure of the set $[t_0, T] \setminus \mathrm{S}$ is zero.

We also need the following real Banach space:

$$\mathcal{B}_2 := \Big\{ \mathbf{U}(\cdot) \,\Big|\, \mathbf{U}(\cdot) : [t_0, T] \to \mathbb{R}^{m \times n},\ \mathbf{U}(\cdot) \text{ is continuous and bounded function with finite number of discontinuity points on } [t_0, T] \Big\},$$

endowed with the norm

$$\|\mathbf{U}(\cdot)\|_{\mathcal{B}_2} := \sup_{t \in [t_0, T]} \|\mathbf{U}(t)\|_{\mathcal{F}}.$$

Throughout this paper, for any matrix $\mathbf{A} = (a_{rj})$, the Frobenius norm is defined as $\|\mathbf{A}\|_{\mathcal{F}} := \sqrt{\sum_{r,j=1} |a_{rj}|^2}$, the Euclidean norm (also called 2-norm or spectral norm) $\|\mathbf{A}\|_2 := \sqrt{\lambda_{\max}(\mathbf{A}^*\mathbf{A})}$, where $\lambda_{\max}$ is the maximum eigen value of the positive semidefinite matrix $\mathbf{A}^*\mathbf{A}$. Also for any operator $\mathcal{T}$, the Hermitian adjoint is denoted by $\mathcal{T}^*$, for any vector $\mathbf{v} \in \mathbb{R}^n$, the Euclidean norm by $\|\mathbf{v}\|_2$, the set of all continuous functions from set A to set B by $\mathcal{C}(A; B)$. Further, for the matrix $\mathbf{A} = (a_{rj})$, if

$$vec\,\mathbf{A} := \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \\ a_{12} \\ \vdots \\ a_{m2} \\ \vdots \\ a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}_{mn \times 1} \qquad \text{then } \|vec\,\mathbf{A}\|_2 = \|\mathbf{A}\|_{\mathcal{F}}.$$

We consider a dynamical control system modelled by the following linear impulsive matrix Lyapunov ordinary differential equations whose state matrix belongs to $\mathbb{R}^{n \times n}$, with multiple constant time-delays in the control function,

$$\left.\begin{aligned}\dot{\mathbf{X}}(t) &= \mathbf{A}_1(t)\mathbf{X}(t)+\mathbf{X}(t)\mathbf{A}_2(t)+\sum_{i=1}^{N}\mathbf{B}_i(t)\mathbf{U}(t-h_i),\ t\in[t_0,T]\setminus\{t_k : k=1,\dots,M\},\\ \mathbf{X}(t_0) &= \mathbf{X}_0,\\ \Delta\mathbf{X}(t_k) &:= \mathbf{X}(t_k^+)-\mathbf{X}(t_k) = D^k\mathbf{U}(t_k)\mathbf{X}(t_k),\\ \mathbf{U}(t) &= \mathbf{U}_0(t),\ \ t\in[t_0-h_N,t_0).\end{aligned}\right\} \quad (1)$$

We make the following assumptions on this system components:

(i) the state function $\mathbf{X}(\cdot)\in\mathcal{B}_1$ with a given initial state $\mathbf{X}(t_0)=\mathbf{X}_0\in\mathbb{R}^{n\times n}$,

(ii) the control function $\mathbf{U}(\cdot)\in\mathcal{B}_2$,

(iii) $\mathbf{A}_1(\cdot)$, $\mathbf{A_2}(\cdot)\in\mathcal{C}\big([t_0,T];\,\mathbb{R}^{n\times n}\big)$ and $\mathbf{B}_i(\cdot)\in\mathcal{C}\big([t_0,T];\,\mathbb{R}^{n\times m}\big)$,

(iv) $t_0\le t_1\le\cdots\le t_M<T$, t_k's are the fixed times at which the state function $\mathbf{X}(\cdot)$ experiences impulses and are state independent,

(v) $0\le h_1\le h_2\le\cdots\le h_N\le\min\big\{(t_1-t_0),(t_2-t_1),...,(t_M-t_{M-1}),(T-t_M)\big\}$, h_i's are the known constant time delays in the control function $\mathbf{U}(\cdot)$,

(vi) $\Delta(\mathbf{X}(t_k))$ is an impulse in the state function $\mathbf{X}(\cdot)$ at the time t_k,

(vii) $D^k\mathbf{U}(t_k):=\sum_{r=1}^{m}\sum_{j=1}^{n}d_{rj}^kU_{rj}(t_k)\mathbf{I}_n$, d_{rj}^k are the known constant real numbers, $\mathbf{I}_n$ is an $n\times n$ identity matrix, and the matrix $\mathbf{U}(t_k)=\big(U_{rj}(t_k)\big)$,

(viii) $\mathbf{U}_0(\cdot):[t_0-h_N,t_0)\to\mathbb{R}^{m\times n}$ denotes the known initial control function (and is assumed to be bounded and continuous on its domain) applied to the system (1), the subscripts $i=1,\dots,N$ and $k=1,\dots,M$.

The following definition for the controllability of the system (1) is adopted in this paper.

Definition 2.1. (Controllability) The system (1) is said to be controllable over $\mathbb{R}^{n\times n}$ on $[t_0,T]$, if for all $\mathbf{X}_0$, $\mathbf{X}_T\in\mathbb{R}^{n\times n}$ and for every continuous and bounded function $\mathbf{U}_0(\cdot):[t_0-h_N,t_0)\to\mathbb{R}^{m\times n}$, there exists at least one control function $\mathbf{U}(\cdot)\in\mathcal{B}_2$ such that, with this control function on $[t_0,T]$, the corresponding solution to the system (1) with $\mathbf{X}(t_0)=\mathbf{X}_0$, $\mathbf{U}(t)=\mathbf{U}_0(t)$, $t\in[t_0-h_N,t_0)$, satisfies the condition $\mathbf{X}(T)=\mathbf{X}_T$.

Remark 2.2. In the above definition of controllability, if $\mathbf{X}_T=\mathbf{O}$, then the system (1) is said to be null controllable over $\mathbb{R}^{n\times n}$ on $[t_0,T]$.

We recall the following definition of Kronecker product which is used in this paper.

Definition 2.3. (Kronecker product, Graham [12]) Let $\mathbf{A} = (a_{rj}) \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{p\times q}$ denotes any matrices, then the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$ is denoted and defined by the partitioned matrix

$$\mathbf{A}\otimes\mathbf{B} := \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}_{mp\times nq} \in \mathbb{R}^{mp\times nq}.$$

The Kronecker product satisfies the following properties:

(i) $(\mathbf{A}\otimes\mathbf{B})^* = (\mathbf{A}^*\otimes\mathbf{B}^*)$,

(ii) $(\mathbf{A}\otimes\mathbf{B})^{-1} = (\mathbf{A}^{-1}\otimes\mathbf{B}^{-1})$,

(iii) $(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D}) = (\mathbf{AC}\otimes\mathbf{BD})$, provided the dimension of the these matrices are compatible with the matrix product,

(iv) $\frac{d(\mathbf{A}(t)\otimes\mathbf{B}(t))}{dt} = \frac{d(\mathbf{A}(t))}{dt}\otimes\mathbf{B}(t) + \mathbf{A}(t)\otimes\frac{d(\mathbf{B}(t))}{dt}$,

(v) if $\mathbf{A}$ and $\mathbf{X}$ are the matrices of order $n\times n$, then

(vi) $vec(\mathbf{AX}) = (\mathbf{I}_n\otimes\mathbf{A})vec(\mathbf{X})$,

(vii) $vec(\mathbf{XA}) = (\mathbf{A}^*\otimes\mathbf{I}_n)vec(\mathbf{X})$,

(viii) $vec(\mathbf{AXB}) = (\mathbf{B}^*\otimes\mathbf{A})vec(\mathbf{X})$.

3. CONVERSION OF MATRIX LYAPUNOV DIFFERENTIAL SYSTEM INTO VECTOR DIFFERENTIAL SYSTEM

In this section, the given matrix Lyapunov differential system (1) is converted into vector differential system by applying the vector operator to it, as follows:

$$\begin{aligned} vec\,\dot{\mathbf{X}}(t) &= vec\Big(\mathbf{A}_1(t)\mathbf{X}(t) + \mathbf{X}(t)\mathbf{A}_2(t) + \sum_{i=1}^{N}\mathbf{B}_i(t)\mathbf{U}(t-h_i)\Big), \\ &\qquad \text{for } t\in[t_0,T]\setminus\{t_k : k=1,\dots,M\}, \\ vec\,\mathbf{X}(t_0) &= vec\,\mathbf{X}_0, \\ vec\,\Delta(\mathbf{X}(t_k)) &= vec\big(D^k\mathbf{U}(t_k)\mathbf{X}(t_k)\big), \\ vec\,\mathbf{U}(t) &= vec\,\mathbf{U}_0(t),\ t\in[t_0-h_N,t_0). \end{aligned}$$

That is,

$$\left.\begin{aligned} vec\,\dot{\mathbf{X}}(t) &= \big(\mathbf{I}_n\otimes\mathbf{A}_1(t) + \mathbf{A}_2^*(t)\otimes\mathbf{I}_n\big)vec\,\mathbf{X}(t) + \sum_{i=1}^{N}\big(\mathbf{I}_n\otimes\mathbf{B}_i(t)\big)vec\,\mathbf{U}(t-h_i), \\ &\qquad t\in[t_0,T]\setminus\{t_k : k=1,\dots,M\}, \\ vec\,\mathbf{X}(t_0) &= vec\,\mathbf{X}_0, \\ vec\,\Delta(\mathbf{X}(t_k)) &= \big(\mathbf{I}_n\otimes D^k\mathbf{U}(t_k)\big)vec\,\mathbf{X}(t_k), \\ vec\,\mathbf{U}(t) &= vec\,\mathbf{U}_0(t),\ t\in[t_0-h_N,t_0). \end{aligned}\right\} \quad (2)$$

Now we introduce the following notations:

$$\begin{aligned}
\mathbf{x}(t) &:= vec\,\mathbf{X}(t) \in \mathbb{R}^{n^2},\\
\mathbf{u}(t) &:= vec\,\mathbf{U}(t) \in \mathbb{R}^{mn},\\
\mathbf{x}_0 &:= vec\,\mathbf{X}_0 \in \mathbb{R}^{n^2},\\
\mathbf{A}(t) &:= \big(\mathbf{I}_n \otimes \mathbf{A}_1(t) + \mathbf{A}_2^*(t) \otimes \mathbf{I}_n\big)_{n^2\times n^2},\\
\mathbf{C}_i(t) &:= \big(\mathbf{I}_n \otimes \mathbf{B}_i(t)\big)_{n^2\times mn},\\
d^k\mathbf{u}(t_k) &:= \big(\mathbf{I}_n \otimes D^k\mathbf{U}(t_k)\big)_{n^2\times n^2} = \sum_{r=1}^{m}\sum_{j=1}^{n} d_{rj}^k U_{rj}(t_k)\mathbf{I}_{n^2} = \alpha_k \mathbf{I}_{n^2}, \text{ where}\\
\alpha_k &:= \sum_{r=1}^{m}\sum_{j=1}^{n} d_{rj}^k U_{rj}(t_k) \in \mathbb{R}.
\end{aligned}$$

With the above notations, the system (2) becomes,

$$\left.\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \sum_{i=1}^{N} \mathbf{C}_i(t)\mathbf{u}(t-h_i), \;\; t \in [t_0, T] \setminus \{t_k : k = 1, \dots, M\},\\
\mathbf{x}(t_0) &= \mathbf{x}_0,\\
\Delta\mathbf{x}(t_k) &= d^k\mathbf{u}(t_k)\mathbf{x}(t_k) = \alpha_k \mathbf{x}(t_k),\\
\mathbf{u}(t) &= \mathbf{u}_0(t), \;\; t \in [t_0 - h_N, t_0).
\end{aligned}\right\} \tag{3}$$

Proposition 3.1. The matrix Lyapunov differential system (1) is controllable over $\mathbb{R}^{n\times n}$ on the time interval $[t_0, T]$ if and only if the vector differential system (3) is controllable over $\mathbb{R}^{n^2}$ on the same time interval $[t_0, T]$.

The proof of this proposition is trivial, as the systems (1) and (3) are identical with each other under the vector operator.

4. CONTROLLABILITY RESULTS

In this section, the controllability results of the system (1) for certain classes of admissible control functions are obtained. First, we recall the following lemma [8].

Lemma 4.1. Let $\mathbf{\Phi}_1(t, t_0)$ and $\mathbf{\Phi}_2(t, t_0)$ be the state-transition matrices for $\mathbf{A}_1(t)$ and $\mathbf{A}_2^*(t)$, respectively. Then the state-transition matrix for $\mathbf{A}(t)$ is given by $\mathbf{\Phi}(t, t_0) = \mathbf{\Phi}_2(t, t_0) \otimes \mathbf{\Phi}_1(t, t_0)$, where $\mathbf{A}(t) = \big(\mathbf{I}_n \otimes \mathbf{A}_1(t) + \mathbf{A}_2^*(t) \otimes \mathbf{I}_n\big)_{n^2\times n^2}$.

The solution to the linear impulsive vector differential delay system (3) is given in the following theorem.

Theorem 4.2. The solution to the system (3) in the time interval $(t_k, t_{k+1}]$, $k = 1, 2, \dots, M$, with $t_{M+1} = T$ is given by,

$$\begin{aligned}
\mathbf{x}(t) = {} & \prod_{j=1}^{k}(1+\alpha_j)\mathbf{\Phi}(t,t_0)(\mathbf{x_0}+\mathbf{a_0}) + \int_{t_0}^{t_1-h_N} \prod_{j=1}^{k}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i) \\
& \times \mathbf{u}(s)\,\mathrm{d}s + \sum_{l=1}^{N-1}\int_{t_1-h_{l+1}}^{t_1-h_l}\left\{\prod_{j=1}^{k}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\right. \\
& \left. + \prod_{j=2}^{k}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\right\}\mathbf{u}(s)\,\mathrm{d}s \\
& + \sum_{q=1}^{k-1}\left\{\int_{t_q-h_1}^{t_{q+1}-h_N}\prod_{j=q+1}^{k}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\right. \\
& + \sum_{l=1}^{N-1}\int_{t_{q+1}-h_{l+1}}^{t_{q+1}-h_l}\left(\prod_{j=q+1}^{k}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\right. \\
& \left.\left. + \prod_{j=q+2}^{k}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\right)\mathbf{u}(s)\,\mathrm{d}s\right\} \\
& + \int_{t_k-h_1}^{t-h_N}\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s + \sum_{l=1}^{N-1}\int_{t-h_{l+1}}^{t-h_l}\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i) \\
& \times \mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s,
\end{aligned} \tag{4}$$

where $\prod_{j=k+1}^{k}(1+\alpha_j) = 1$, $\prod_{j=k+2}^{k}(1+\alpha_j) = 0$, for all $k = 1, 2, \dots, M$ and

$$\mathbf{a}_0 := \sum_{i=1}^{N}\int_{t_0-h_i}^{t_0}\mathbf{\Phi}(t_0,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}_0(s)\,\mathrm{d}s.$$

Proof. Let $\mathbf{\Phi}(t,t_0)$ be the state-transition matrix of $\mathbf{A}(t)$, and hence $\mathbf{x}(t) = \mathbf{\Phi}(t,t_0)\mathbf{x}_0$ is a unique solution to the homogeneous system $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$ with $\mathbf{x}(t_0) = \mathbf{x}_0$. The solution to the system (3) on $[t_0, t_1]$ is given by,

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{\Phi}(t,t_0)\mathbf{x}_0 + \int_{t_0}^{t}\mathbf{\Phi}(t,s)\sum_{i=1}^{N}\mathbf{C}_i(s)\mathbf{u}(s-h_i)\,\mathrm{d}s \\
&= \mathbf{\Phi}(t,t_0)\mathbf{x}_0 + \mathbf{\Phi}(t,t_0)\sum_{i=1}^{N}\int_{t_0-h_i}^{t_0}\mathbf{\Phi}(t_0,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}_0(s)\,\mathrm{d}s \\
&\quad + \sum_{i=1}^{N}\int_{t_0}^{t-h_i}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s.
\end{aligned}$$

Since

$$\sum_{i=1}^{N}\int_{t_0-h_i}^{t_0}\mathbf{\Phi}(t_0,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}_0(s)\,\mathrm{d}s=\mathbf{a}_0\in\mathbb{R}^{n^2},\tag{5}$$

therefore

$$\mathbf{x}(t)=\mathbf{\Phi}(t,t_0)(\mathbf{x}_0+\mathbf{a}_0)+\sum_{i=1}^{N}\int_{t_0}^{t-h_i}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s.\tag{6}$$

Now simplify the summation given in (6) as

$$\begin{aligned}\sum_{i=1}^{N}\int_{t_0}^{t-h_i}&\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\\&=\int_{t_0}^{t-h_N}\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\\&\quad+\sum_{l=1}^{N-1}\int_{t-h_{l+1}}^{t-h_l}\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s.\end{aligned}\tag{7}$$

Using (7) in (6), the solution to the system (3) on $[t_0,t_1]$ is given by,

$$\begin{aligned}\mathbf{x}(t)=\mathbf{\Phi}(t,t_0)(\mathbf{x}_0+\mathbf{a}_0)&+\int_{t_0}^{t-h_N}\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\\+\sum_{l=1}^{N-1}\int_{t-h_{l+1}}^{t-h_l}&\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s.\end{aligned}\tag{8}$$

Now as $\mathbf{x}(t_1^+)=\mathbf{x}(t_1)+\Delta\mathbf{x}(t_1)=(1+\alpha_1)\mathbf{x}(t_1)$, the solution to the system (3) on $(t_1,t_2]$ is given by

$$\begin{aligned}\mathbf{x}(t)&=\mathbf{\Phi}(t,t_1)\mathbf{x}(t_1^+)+\int_{t_1}^{t}\mathbf{\Phi}(t,s)\sum_{i=1}^{N}\mathbf{C}_i(s)\mathbf{u}(s-h_i)\,\mathrm{d}s\\&=(1+\alpha_1)\mathbf{\Phi}(t,t_1)\Bigg\{\mathbf{\Phi}(t_1,t_0)(\mathbf{x}_0+\mathbf{a}_0)+\int_{t_0}^{t_1-h_N}\sum_{i=1}^{N}\mathbf{\Phi}(t_1,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\\&\quad+\sum_{l=1}^{N-1}\int_{t_1-h_{l+1}}^{t_1-h_l}\sum_{i=1}^{l}\mathbf{\Phi}(t_1,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\Bigg\}+\sum_{i=1}^{N}\int_{t_1-h_i}^{t-h_i}\mathbf{\Phi}(t,s+h_i)\\&\quad\times\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\\&=(1+\alpha_1)\Bigg\{\mathbf{\Phi}(t,t_0)(\mathbf{x}_0+\mathbf{a}_0)+\int_{t_0}^{t_1-h_N}\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\\&\quad+\sum_{l=1}^{N-1}\int_{t_1-h_{l+1}}^{t_1-h_l}\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\Bigg\}\end{aligned}$$

$$+\sum_{l=1}^{N-1}\int_{t_1-h_{l+1}}^{t_1-h_l}\sum_{i=l+1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s+\int_{t_1-h_1}^{t-h_N}\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)$$
$$\times\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s+\sum_{l=1}^{N-1}\int_{t-h_{l+1}}^{t-h_l}\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s.$$

Therefore,

$$\begin{aligned}\mathbf{x}(t)&=\prod_{j=1}^{1}(1+\alpha_j)\mathbf{\Phi}(t,t_0)(\mathbf{x}_0+\mathbf{a}_0)+\int_{t_0}^{t_1-h_N}\prod_{j=1}^{1}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\\&\quad\times\mathbf{u}(s)\,\mathrm{d}s+\sum_{l=1}^{N-1}\int_{t_1-h_{l+1}}^{t_1-h_l}\Bigg\{\prod_{j=1}^{1}(1+\alpha_1)\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\\&\quad+\prod_{j=2}^{1}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\Bigg\}\mathbf{u}(s)\,\mathrm{d}s\\&\quad+\int_{t_1-h_1}^{t-h_N}\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s+\sum_{l=1}^{N-1}\int_{t-h_{l+1}}^{t-h_l}\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\\&\quad\times\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s.\end{aligned}\tag{9}$$

Again, as $\mathbf{x}(t_2^+)=\mathbf{x}(t_2)+\Delta\mathbf{x}(t_2)=(1+\alpha_2)\mathbf{x}(t_2)$, the solution to the system (3) on $(t_2,t_3]$ is given by

$$\begin{aligned}\mathbf{x}(t)&=\mathbf{\Phi}(t,t_2)\mathbf{x}(t_2^+)+\int_{t_2}^{t}\mathbf{\Phi}(t,s)\sum_{i=1}^{N}\mathbf{C}_i(s)\mathbf{u}(s-h_i)\,\mathrm{d}s\\&=(1+\alpha_2)\mathbf{\Phi}(t,t_2)\Bigg\{(1+\alpha_1)\mathbf{\Phi}(t_2,t_0)(\mathbf{x}_0+\mathbf{a}_0)+\int_{t_0}^{t_1-h_N}(1+\alpha_1)\sum_{i=1}^{N}\mathbf{\Phi}(t_2,s+h_i)\\&\quad\times\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s+\sum_{l=1}^{N-1}\int_{t_1-h_{l+1}}^{t_1-h_l}\Bigg[(1+\alpha_1)\sum_{i=1}^{l}\mathbf{\Phi}(t_2,s+h_i)\mathbf{C}_i(s+h_i)\\&\quad+\sum_{i=l+1}^{N}\mathbf{\Phi}(t_2,s+h_i)\mathbf{C}_i(s+h_i)\Bigg]\mathbf{u}(s)\,\mathrm{d}s+\int_{t_1-h_1}^{t_2-h_N}\sum_{i=1}^{N}\mathbf{\Phi}(t_2,s+h_i)\mathbf{C}_i(s+h_i)\\&\quad\times\mathbf{u}(s)\,\mathrm{d}s+\sum_{l=1}^{N-1}\int_{t_2-h_{l+1}}^{t_2-h_l}\sum_{i=1}^{l}\mathbf{\Phi}(t_2,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\Bigg\}\\&\quad+\sum_{i=1}^{N}\int_{t_2-h_i}^{t-h_i}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\\&=(1+\alpha_1)(1+\alpha_2)\mathbf{\Phi}(t,t_0)(\mathbf{x}_0+\mathbf{a}_0)\\&\quad+\int_{t_0}^{t_1-h_N}(1+\alpha_1)(1+\alpha_2)\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=1}^{N-1}\int_{t_1-h_{l+1}}^{t_1-h_l}\Bigg[(1+\alpha_1)(1+\alpha_2)\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\\
&+(1+\alpha_2)\sum_{i=l+1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\Bigg]\mathbf{u}(s)\,\mathrm{d}s\\
&+\int_{t_1-h_1}^{t_2-h_N}(1+\alpha_2)\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\\
&+\sum_{l=1}^{N-1}\int_{t_2-h_{l+1}}^{t_2-h_l}(1+\alpha_2)\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\\
&+\sum_{l=1}^{N-1}\int_{t_2-h_{l+1}}^{t_2-h_l}\sum_{i=l+1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\\
&+\int_{t_2-h_1}^{t-h_N}\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s+\sum_{l=1}^{N-1}\int_{t-h_{l+1}}^{t-h_l}\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\\
&\times\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbf{x}(t)={}&\prod_{j=1}^{2}(1+\alpha_j)\mathbf{\Phi}(t,t_0)(\mathbf{x}_0+\mathbf{a}_0)+\int_{t_0}^{t_1-h_N}\prod_{j=1}^{2}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\\
&\times\mathbf{u}(s)\,\mathrm{d}s+\sum_{l=1}^{N-1}\int_{t_1-h_{l+1}}^{t_1-h_l}\Bigg\{\prod_{j=1}^{2}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\\
&+\prod_{j=2}^{2}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\Bigg\}\mathbf{u}(s)\,\mathrm{d}s\\
&+\sum_{q=1}^{1}\Bigg\{\int_{t_q-h_1}^{t_{q+1}-h_N}\prod_{j=q+1}^{2}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\\
&+\sum_{l=1}^{N-1}\int_{t_{q+1}-h_{l+1}}^{t_{q+1}-h_l}\Bigg(\prod_{j=q+1}^{2}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\\
&+\prod_{j=q+2}^{2}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\Bigg)\mathbf{u}(s)\mathrm{d}s\Bigg\}\\
&+\int_{t_2-h_1}^{t-h_N}\sum_{i=1}^{N}\mathbf{\Phi}(t,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s+\sum_{l=1}^{N-1}\int_{t-h_{l+1}}^{t-h_l}\sum_{i=1}^{l}\mathbf{\Phi}(t,s+h_i)\\
&\times\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s.
\end{aligned}
\tag{10}
$$

Continuing this process on subintervals $(t_3,t_4],\dots,(t_M,T]$, in general the solution to system (3) on subinterval $(t_k,t_{k+1}]$ is given by (4). □

Let us now define the following matrices which are used throughout the paper:

$$
\begin{aligned}
\mathbf{W}_{1} &:= \int_{t_0}^{t_1-h_N} \left[\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right] \\
&\quad\times\left[\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right]^{*}\mathrm{d}s, \\
\mathbf{W}_{kN+1} &:= \int_{t_k-h_1}^{t_{k+1}-h_N} \left[\prod_{j=k+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right] \\
&\quad\times\left[\prod_{j=k+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right]^{*}\mathrm{d}s, \\
\mathbf{W}_{qN+1-l} &:= \int_{t_q-h_{l+1}}^{t_q-h_l} \left[\prod_{j=q}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right. \\
&\quad\left.+\prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right] \\
&\quad\times\left[\prod_{j=q}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right. \\
&\quad\left.+\prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right]^{*}\mathrm{d}s,
\end{aligned}
\tag{11}
$$

where it is assumed that $\prod_{j=k+1}^{k}(1+\alpha_j)=1$, $\prod_{j=k+2}^{k}(1+\alpha_j)=0$; $k=1,2,\dots,M$; $l=1,2,\dots,(N-1)$ and $q=1,2,\dots,(M+1)$.

Lemma 4.3. Each $\mathbf{W}_p$, $p=1,2,\dots,(M+1)N$, given in (11) is symmetric positive semidefinite $n^2\times n^2$ matrix and $rank\big(\mathbf{W}_1,\ \mathbf{W}_2,\dots,\mathbf{W}_{(M+1)N}\big) = rank\big(\mathbf{W}_1+\mathbf{W}_2+\dots+\mathbf{W}_{(M+1)N}\big)$.

Proof. Let $\mathbf{P}(s)$ be any $n^2\times mn$ matrix, with each of its entry to be a real valued continuous function of s, then for each fixed $s\in[t_0,T]$, for all $\mathbf{v}\in\mathbb{R}^{n^2}$ and under the usual inner product on $\mathbb{R}^{n^2}$, we have

$$
\big\langle \mathbf{v},\ \mathbf{P}(s)\mathbf{P}^*(s)\mathbf{v}\big\rangle_{\mathbb{R}^{n^2}} = \big\langle \mathbf{P}^*(s)\mathbf{v},\ \mathbf{P}^*(s)\mathbf{v}\big\rangle_{\mathbb{R}^{mn}} = \|\mathbf{P}^*(s)\mathbf{v}\|^2_{\mathbb{R}^{mn}} \ge 0,
$$

which shows that $\mathbf{P}(s)\mathbf{P}^*(s)$ is a symmetric positive semidefinite $n^2\times n^2$ matrix for each s. Now for $\alpha<\beta$, let us consider

$$
\Big\langle \int_\alpha^\beta \mathbf{P}(s)\mathbf{P}^*(s)\mathrm{d}s\ \mathbf{v},\ \mathbf{v}\Big\rangle_{\mathbb{R}^{n^2}} = \int_\alpha^\beta \big(\mathbf{P}^*(s)\mathbf{v}\big)^*\big(\mathbf{P}^*(s)\mathbf{v}\big)\,\mathrm{d}s = \int_\alpha^\beta \|\mathbf{P}^*(s)\mathbf{v}\|^2_{\mathbb{R}^{mn}}\,\mathrm{d}s \ge 0,
$$

which easily shows that $\int_\alpha^\beta \mathbf{P}(s)\mathbf{P}^*(s)\,\mathrm{d}s$ is symmetric positive semidefinite $n^2 \times n^2$ matrix. Therefore each $\mathbf{W}_p$ given in (11) is symmetric positive semidefinite $n^2 \times n^2$ matrix. Also, we know that

$$\begin{aligned}\Big\langle \big(\mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_{(M+1)N}\big)\mathbf{v},\ \mathbf{v}\Big\rangle_{\mathbb{R}^{n^2}} &= \big\langle \mathbf{W}_1\mathbf{v},\ \mathbf{v}\big\rangle_{\mathbb{R}^{n^2}} + \big\langle \mathbf{W}_2\mathbf{v},\ \mathbf{v}\big\rangle_{\mathbb{R}^{n^2}} + \cdots \\ &\quad + \big\langle \mathbf{W}_{(M+1)N}\mathbf{v},\ \mathbf{v}\big\rangle_{\mathbb{R}^{n^2}} \\ &\geq 0,\end{aligned}$$

for all $\mathbf{v} \in \mathbb{R}^{n^2}$, which shows that $\big(\mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_{(M+1)N}\big)$ is a symmetric positive semidefinite $n^2 \times n^2$ matrix.

Now, it remains to prove that $rank\big(\mathbf{W}_1+\mathbf{W}_2+\cdots+\mathbf{W}_{(M+1)N}\big) = rank\big(\mathbf{W}_1,\ \mathbf{W}_2, \ldots, \mathbf{W}_{(M+1)N}\big)$. This follows from the following estimation:

$$\begin{aligned}\mathbf{v} \in ker\Big[\big(\mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_{(M+1)N}\big)^*\Big] &\Longleftrightarrow \big(\mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_{(M+1)N}\big)^*(\mathbf{v}) = \mathbf{0} \\ &\Longleftrightarrow \mathbf{W}_p^*(\mathbf{v}) = \mathbf{0},\ \text{for all } p,\ \text{as each } \mathbf{W}_p \text{ is a} \\ &\qquad\qquad \text{positive semidefinite matrix} \\ &\Longleftrightarrow \mathbf{v} \in ker(\mathbf{W}_p^*),\ \text{for all } p \\ &\Longleftrightarrow \mathbf{v} \in ker\begin{bmatrix}\mathbf{W}_1^* \\ \mathbf{W}_2^* \\ s\vdots \\ \mathbf{W}_{(M+1)N}^*\end{bmatrix}_{(M+1)Nn^2\times n^2} \\ &\Longleftrightarrow \mathbf{v} \in ker\big(\mathbf{W}_1,\ \mathbf{W}_2, \ldots, \mathbf{W}_{(M+1)N}\big)^*.\end{aligned}$$

Therefore by using the rank-nullity theorem, we have

$$\begin{aligned}&ker\Big[\big(\mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_{(M+1)N}\big)^*\Big] = ker\Big[\big(\mathbf{W}_1,\ \mathbf{W}_2, \ldots, \mathbf{W}_{(M+1)N}\big)^*\Big] \\ &\Longrightarrow n^2 - rank\Big[\big(\mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_{(M+1)N}\big)^*\Big] = n^2 - rank\Big[\big(\mathbf{W}_1,\ \mathbf{W}_2, \ldots, \mathbf{W}_{(M+1)N}\big)^*\Big] \\ &\Longrightarrow rank\big(\mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_{(M+1)N}\big) = rank\big(\mathbf{W}_1,\ \mathbf{W}_2, \ldots, \mathbf{W}_{(M+1)N}\big),\end{aligned}$$

which completes the proof. □

The following classes of admissible control functions are considered in this paper, for which the controllability results of the system (1) are obtained:

(i) $\mathcal{U}_1 := \Big\{\mathbf{U}(\cdot)\,\Big|\,\mathbf{U}(\cdot) \in \mathcal{B}_2,\ \alpha_k := \sum_{r=1}^{m}\sum_{j=1}^{n} d_{rj}^k U_{rj}(t_k) \neq -1,\ \text{for all } k = 1, 2, \ldots, M\Big\}$

(ii) $\mathcal{U}_2 := \Big\{\mathbf{U}(\cdot)\,\Big|\,\mathbf{U}(\cdot) \in \mathcal{B}_2,\ \alpha_M := \sum_{r=1}^{m}\sum_{j=1}^{n} d_{rj}^M U_{rj}(t_M) = -1\Big\}$.

4.1. Controllability of the system (1) for the class $\mathcal{U}_1$ of control functions

In this subsection, the controllability results of the system (1) for the first class $\mathcal{U}_1$ of control functions is established.

Theorem 4.4. In system (1), if the control function $\mathbf{U}(\cdot) \in \mathcal{U}_1$, then a necessary and sufficient condition for the controllability of the system (1) on $[t_0, T]$ is

$$rank\big(\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_{(M+1)N}\big) = n^2.$$

Proof. To prove this Theorem, it is enough to show that the necessary and sufficient condition for the controllability of the system (3) is $rank\big(\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_{(M+1)N}\big) = n^2$ for this class of controls. Then the proof follows by Proposition 3.1.

First let us show that, the condition is sufficient. For this, assume

$$rank\big(\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_{(M+1)N}\big) = n^2,$$

so that $\mathbf{W} := \mathbf{W}_1 + \mathbf{W}_2 + \cdots + \mathbf{W}_{(M+1)N}$ is a positive definite matrix by Lemma 4.3, and hence $\mathbf{W}$ is invertible. Let us define a control function $\mathbf{u}(\cdot) = vec\,\mathbf{U}(\cdot) \in \mathcal{U}_1$ as follows:

$$\mathbf{u}(t) := \begin{cases} \Big[\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T, t+h_i)\mathbf{C}_i(t+h_i)\Big]^* \mathbf{W}^{-1}\Big[\mathbf{x}_T - \prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T, t_0) \\ \times (\mathbf{x}_0 + \mathbf{a}_0)\Big], \quad \text{for all } t \in [t_0, t_1 - h_N], \\ \Big[\prod_{j=k+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T, t+h_i)\mathbf{C}_i(t+h_i)\Big]^* \mathbf{W}^{-1}\Big[\mathbf{x}_T - \prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T, t_0) \\ \times (\mathbf{x}_0 + \mathbf{a}_0)\Big], \quad \text{for all } t \in (t_k - h_1, t_{k+1} - h_N] \setminus \{t_k\}, \\ \Big[\prod_{j=q}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T, t+h_i)\mathbf{C}_i(t+h_i) + \prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T, t+h_i) \\ \times \mathbf{C}_i(t+h_i)\Big]^* \mathbf{W}^{-1}\Big[\mathbf{x}_T - \prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T, t_0)(\mathbf{x}_0 + \mathbf{a}_0)\Big], \\ \qquad \text{for all } t \in (t_q - h_{l+1}, t_q - h_l], \\ \mathbf{v}_k, \qquad \text{at } t = t_k, \\ \mathbf{0}, \qquad \text{for all } t \in (T - h_1, T], \end{cases} \tag{12}$$

where $k = 1, \ldots, M;\ l = 1, \ldots, (N-1);\ q = 1, 2, \ldots, (M+1)$ and $\mathbf{v}_k = \big(v_{rj}^k\big) \in \mathbb{R}^{mn}$ is any vector such that $\sum_{r=1}^{m}\sum_{j=1}^{n} d_{rj}^k v_{rj}^k \neq -1$.
The state $\mathbf{x}(t)$ of the system (3) given in (4) at $t = T$ satisfies

$$\mathbf{x}(T) = \prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T, t_0)(\mathbf{x}_0 + \mathbf{a}_0) + \int_{t_0}^{t_1 - h_N}\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)$$

$$\times \mathbf{u}(s)\,\mathrm{d}s + \sum_{l=1}^{N-1} \int_{t_1-h_{l+1}}^{t_1-h_l} \left\{ \prod_{j=1}^{M}(1+\alpha_j) \sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i) \right.$$

$$\left. + \prod_{j=2}^{M}(1+\alpha_j) \sum_{i=l+1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i) \right\} \mathbf{u}(s)\,\mathrm{d}s$$

$$+ \sum_{q=1}^{M-1} \left\{ \int_{t_q-h_1}^{t_{q+1}-h_N} \prod_{j=q+1}^{M}(1+\alpha_j) \sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \right.$$

$$+ \sum_{l=1}^{N-1} \int_{t_{q+1}-h_{l+1}}^{t_{q+1}-h_l} \left(\prod_{j=q+1}^{M}(1+\alpha_j) \sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i) \right.$$

$$\left.\left. + \prod_{j=q+2}^{M}(1+\alpha_j) \sum_{i=l+1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i) \right) \mathbf{u}(s)\,\mathrm{d}s \right\}$$

$$+ \int_{t_M-h_1}^{T-h_N} \sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s + \sum_{l=1}^{N-1} \int_{T-h_{l+1}}^{T-h_l} \sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)$$

$$\times \mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s.$$

Substituting $\mathbf{u}(t)$ from (12) in the above expression we get,

$$\mathbf{x}(T) = \prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T, t_0)(\mathbf{x}_0 + \mathbf{a}_0) + \left[\int_{t_0}^{t_1-h_N} \left(\prod_{j=1}^{M}(1+\alpha_j) \sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i) \right) \right.$$

$$\times \left(\prod_{j=1}^{M}(1+\alpha_j) \sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i) \right)^* \mathrm{d}s$$

$$+ \sum_{l=1}^{N-1} \int_{t_1-h_{l+1}}^{t_1-h_l} \left\{ \prod_{j=1}^{M}(1+\alpha_j) \sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i) \right.$$

$$\left. + \prod_{j=2}^{M}(1+\alpha_j) \sum_{i=l+1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i) \right\}$$

$$\times \left\{ \prod_{j=1}^{M}(1+\alpha_j) \sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i) + \prod_{j=2}^{M}(1+\alpha_j) \sum_{i=l+1}^{N} \mathbf{\Phi}(T, s+h_i) \right.$$

$$\left. \times \mathbf{C}_i(s+h_i) \right\}^* \mathrm{d}s + \sum_{q=1}^{M-1} \left\{ \int_{t_q-h_1}^{t_{q+1}-h_N} \left(\prod_{j=q+1}^{M}(1+\alpha_j) \sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i) \right) \right.$$

$$\times \left(\prod_{j=q+1}^{M}(1+\alpha_j) \sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i) \right)^* \mathrm{d}s$$

$$+ \sum_{l=1}^{N-1} \int_{t_{q+1}-h_{l+1}}^{t_{q+1}-h_l} \left(\prod_{j=q+1}^{M}(1+\alpha_j) \sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i) \right.$$

$$
\begin{aligned}
&+ \prod_{j=q+2}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\bigg)\\
&\times\bigg(\prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)+\prod_{j=q+2}^{M}(1+\alpha_j)\\
&\times\sum_{i=l+1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\bigg)^{*}\mathrm{d}s\bigg\}+\int_{t_M-h_1}^{T-h_N}\bigg(\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\bigg)\\
&\times\bigg(\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\bigg)^{*}\mathrm{d}s+\sum_{l=1}^{N-1}\int_{T-h_{l+1}}^{T-h_l}\bigg(\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\bigg)\\
&\times\bigg(\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\bigg)^{*}\mathrm{d}s\bigg]\mathbf{W}^{-1}\bigg(\mathbf{x}_T-\prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0)\bigg)\\
=&\prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0)+\big(\mathbf{W}_1+\ldots+\mathbf{W}_{(M+1)N}\big)\mathbf{W}^{-1}\\
&\times\bigg(\mathbf{x}_T-\prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0)\bigg)\\
=&\prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0)+\mathbf{W}\mathbf{W}^{-1}\bigg(\mathbf{x}_T-\prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0)\bigg)=\mathbf{x}_T.
\end{aligned}
$$

Therefore the system (3) is controllable on $[t_0,T]$, and hence by Proposition 3.1, the system (1) is also controllable on $[t_0,T]$.

The necessary condition can be proved by contradiction. For this, let the system (1) be controllable on $[t_0,T]$, but assume that $0\le \mathit{rank}\big(\mathbf{W}_1,\,\mathbf{W}_2,\ldots,\mathbf{W}_{(M+1)N}\big)<n^2$. Then by Lemma 4.3, $\mathbf{W}=\mathbf{W}_1+\mathbf{W}_2+\cdots+\mathbf{W}_{(M+1)N}$ is singular matrix. Hence there exists at least one non-zero vector, say $\mathbf{v}\in\mathbb{R}^{n^2}$ such that $\mathbf{W}\mathbf{v}=\mathbf{0}$, i.e,

$$
\big(\mathbf{W}_1+\mathbf{W}_2+\cdots+\mathbf{W}_{(M+1)N}\big)\mathbf{v}=\mathbf{0}\Longrightarrow\mathbf{W}_1\mathbf{v}+\mathbf{W}_2\mathbf{v}+\cdots+\mathbf{W}_{(M+1)N}\mathbf{v}=\mathbf{0}.
$$

Hence $\mathbf{W}_p\mathbf{v}=\mathbf{0}$ for all p (since each $\mathbf{W}_p$ is positive semidefinite matrix). This shows that each $\mathbf{W}_p$ is a singular matrix and $\big\langle\mathbf{W}_p\mathbf{v},\mathbf{v}\big\rangle_{\mathbb{R}^{n^2}}=0$, for all p, i. e.,

$$
\left\{
\begin{aligned}
&\bigg\langle\int_{t_0}^{t_1-h_N}\bigg(\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\bigg)\\
&\times\bigg(\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\bigg)^{*}\mathrm{d}s\,\mathbf{v},\ \mathbf{v}\bigg\rangle_{\mathbb{R}^{n^2}}=0,
\end{aligned}
\right.
$$

$$
\begin{cases}
\Big\langle \displaystyle\int_{t_k-h_1}^{t_{k+1}-h_N} \Big(\prod_{j=k+1}^{M} (1+\alpha_j) \sum_{i=1}^{N} \mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \Big) \\
\times \Big(\displaystyle\prod_{j=k+1}^{M} (1+\alpha_j) \sum_{i=1}^{N} \mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \Big)^* \mathrm{d}s\, \mathbf{v},\ \mathbf{v} \Big\rangle_{\mathbb{R}^{n^2}} = 0, \\
\Big\langle \displaystyle\int_{t_q-h_{l+1}}^{t_q-h_l} \Big(\prod_{j=q}^{M} (1+\alpha_j) \sum_{i=1}^{l} \mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \\
+ \displaystyle\prod_{j=q+1}^{M} (1+\alpha_j) \sum_{i=l+1}^{N} \mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \Big) \\
\times \Big(\displaystyle\prod_{j=q}^{M} (1+\alpha_j) \sum_{i=1}^{l} \mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \\
+ \displaystyle\prod_{j=q+1}^{M} (1+\alpha_j) \sum_{i=l+1}^{N} \mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \Big)^* \mathrm{d}s\, \mathbf{v},\ \mathbf{v} \Big\rangle_{\mathbb{R}^{n^2}} = 0.
\end{cases}
$$

$$
\Longrightarrow
\begin{cases}
\displaystyle\int_{t_0}^{t_1-h_N} \Big\| \Big(\prod_{j=1}^{M} (1+\alpha_j) \sum_{i=1}^{N} \mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \Big)^* \mathbf{v} \Big\|^2_{\mathbb{R}^{mn}} \mathrm{d}s = 0, \\
\displaystyle\int_{t_k-h_1}^{t_{k+1}-h_N} \Big\| \Big(\prod_{j=k+1}^{M} (1+\alpha_j) \sum_{i=1}^{N} \mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \Big)^* \mathbf{v} \Big\|^2_{\mathbb{R}^{mn}} \mathrm{d}s = 0, \\
\displaystyle\int_{t_q-h_{l+1}}^{t_q-h_l} \Big\| \Big\{ \prod_{j=q}^{M} (1+\alpha_j) \sum_{i=1}^{l} \mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \\
\quad + \displaystyle\prod_{j=q+1}^{M} (1+\alpha_j) \sum_{i=l+1}^{N} \mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \Big\}^* \mathbf{v} \Big\|^2_{\mathbb{R}^{mn}} \mathrm{d}s = 0.
\end{cases}
$$

Since each $\mathbf{C}_i^*(\cdot)$ and $\mathbf{\Phi}^*(\cdot,\cdot)$ are continuous functions, so the above integrals implies that

$$
\begin{cases}
\mathbf{v}^* \Big(\displaystyle\prod_{j=1}^{M} (1+\alpha_j) \sum_{i=1}^{N} \mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \Big) = \mathbf{0}, \\
\mathbf{v}^* \Big(\displaystyle\prod_{j=k+1}^{M} (1+\alpha_j) \sum_{i=1}^{N} \mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \Big) = \mathbf{0}, \\
\mathbf{v}^* \Big(\displaystyle\prod_{j=q}^{M} (1+\alpha_j) \sum_{i=1}^{l} \mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \\
\quad + \displaystyle\prod_{j=q+1}^{M} (1+\alpha_j) \sum_{i=l+1}^{N} \mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i) \Big) = \mathbf{0},
\end{cases}
\tag{13}
$$

for all $k = 1, \ldots, M;\ l = 1, \ldots, (N-1);\ q = 1, \ldots, (M+1)$, and some $\mathbf{v} \neq 0 \in \mathbb{R}^{n^2}$.

We assumed that the system (1) is controllable on $[t_0, T]$, so the system (3). In particular, this system (3) is null controllable on $[t_0, T]$. Now, let us choose an initial state $\mathbf{x}_0 = -\mathbf{a}_0 + \Big(\prod_{j=1}^{M}(1+\alpha_j)\Big)^{-1}\mathbf{\Phi}^{-1}(T, t_0)\mathbf{v}$ and a final state $\mathbf{x}(T) = \mathbf{0}$. Then with some control $\mathbf{u}(\cdot)$, the state of the system (3) given in (4) satisfies $\mathbf{x}(T) = \mathbf{0}$. That is,

$$\begin{aligned}
\mathbf{0} = \mathbf{x}(T) = {} & \prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T, t_0)(\mathbf{x}_0 + \mathbf{a}_0) + \int_{t_0}^{t_1 - h_N} \prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T, s+h_i) \\
& \times \mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s + \sum_{l=1}^{N-1}\int_{t_1 - h_{l+1}}^{t_1 - h_l}\Big\{\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i) \\
& + \prod_{j=2}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\Big\}\mathbf{u}(s)\,\mathrm{d}s \\
& + \sum_{q=1}^{M-1}\Big\{\int_{t_q - h_1}^{t_{q+1} - h_N}\prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\
& + \sum_{l=1}^{N-1}\int_{t_{q+1} - h_{l+1}}^{t_{q+1} - h_l}\Big(\prod_{j=q+1}^{M}(1+\alpha_j)\sum_{i=1}^{l}\mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i) \\
& + \prod_{j=q+2}^{M}(1+\alpha_j)\sum_{i=l+1}^{N}\mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\Big)\mathbf{u}(s)\,\mathrm{d}s\Big\} \\
& + \int_{t_M - h_1}^{T - h_N}\sum_{i=1}^{N}\mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\
& + \sum_{l=1}^{N-1}\int_{T - h_{l+1}}^{T - h_l}\sum_{i=1}^{l}\mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s.
\end{aligned}$$

In the above expression, substitute $\mathbf{x}_0 = -\mathbf{a}_0 + \Big(\prod_{j=1}^{M}(1+\alpha_j)\Big)^{-1}\mathbf{\Phi}^{-1}(T, t_0)\mathbf{v}$, then premultiply with $\mathbf{v}^*$ and use (13) to get, $0 = \mathbf{v}^*\mathbf{v}$. Thus $\mathbf{v} = \mathbf{0}$. This is contradiction. Hence our assumption that $0 \leq rank\big(\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_{(M+1)N}\big) < n^2$ is wrong. Thus, finally we have $rank\big(\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_{(M+1)N}\big) = n^2$. □

Remark 4.5. In system (1), if delays are absent in the control function $\mathbf{U}(\cdot) \in \mathcal{U}_1$ i.e $h_i = 0$, $\forall\, i$, then the necessary and sufficient condition of controllability of the system (1) given in Theorem 4.4 reduces to

$$rank\big(\mathbf{W}_1, \mathbf{W}_{N+1}, \mathbf{W}_{2N+1}, \ldots, \mathbf{W}_{MN+1}\big) = n^2,$$

where $\mathbf{W}_1, \mathbf{W}_{N+1}, \mathbf{W}_{2N+1}, \ldots, \mathbf{W}_{MN+1}$ are obtained from (11) by taking $h_i = 0$, for all $i = 1, 2, \ldots, N$, i.e

$$\mathbf{W}_{kN+1} := \int_{t_k}^{t_{k+1}}\Big[\prod_{j=k+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T, s)\mathbf{C}_i(s)\Big]\Big[\prod_{j=k+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T, s)\mathbf{C}_i(s)\Big]^*\mathrm{d}s,$$

$\forall\, k = 0, 1, \dots, M$. Note that in this case, the other $\mathbf{W}_p$'s are zero matrices. Further, the steering controller defined in (12) reduces to

$$\mathbf{u}(t) := \begin{cases} \left[\prod_{j=1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,t)\mathbf{C}_i(t)\right]^*\mathbf{W}^{-1}\left[\mathbf{x}_T - \prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0)\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } t \in [t_0, t_1], \\ \left[\prod_{j=k+1}^{M}(1+\alpha_j)\sum_{i=1}^{N}\mathbf{\Phi}(T,t)\mathbf{C}_i(t)\right]^*\mathbf{W}^{-1}\left[\mathbf{x}_T - \prod_{j=1}^{M}(1+\alpha_j)\mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0)\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } t \in (t_k, t_{k+1}], \end{cases}$$

where $\mathbf{W} := \mathbf{W}_1 + \mathbf{W}_{N+1} + \mathbf{W}_{2N+1} + \dots + \mathbf{W}_{MN+1}$ and $k = 1, \dots, M$.

Corollary 4.6. Suppose in the system (1), the control function $\mathbf{U}(\cdot) \in \mathcal{U}_1$ such that $\alpha_k := \sum_{r=1}^{m}\sum_{j=1}^{n} d_{rj}^k U_{rj}(t_k) = 0$, for all $k = 1, 2, \dots, M$, then the necessary and sufficient condition for the controllability of the system (1) obtained in Theorem 4.4 reduces to

$$\mathit{rank}\big(\mathbf{V},\ \mathbf{W}_{MN+2}, \dots, \mathbf{W}_{(M+1)N}\big) = n^2,$$

where

$$\mathbf{V} := \int_{t_0}^{T-h_N}\left[\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right]\left[\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right]^*\mathrm{d}s. \qquad (14)$$

and $\mathbf{W}_{MN+2}, \dots, \mathbf{W}_{(M+1)N}$ are obtained from (11) by taking $h_i = 0$, $\forall\, i = 1, \dots, N$, that is,

$$\mathbf{W}_{(M+1)N+1-l} := \int_{T-h_{l+1}}^{T-h_l}\left[\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right]\left[\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right]^*\mathrm{d}s,$$

where $l = 1, \dots, (N-1)$.

Proof. In this scenario, there are no impulses in the system (1) as $\sum_{r=1}^{m}\sum_{j=1}^{n} d_{rj}^k U_{rj}(t_k) = \alpha_k = 0$, so that $D^k\mathbf{U}(t_k) = \mathbf{O}$ and hence $\Delta\mathbf{X}(t_k) = \mathbf{O}$, for all $k = 1, \dots, M$. As a result, the matrices $\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_{MN+1}$ can be combined to form a single matrix $\mathbf{V}$ i.e. $\mathbf{V} = \mathbf{W}_1 + \mathbf{W}_2 + \dots + \mathbf{W}_{MN+1}$.

The proof of this corollary is similar to the proof of the Theorem 4.4. The steering control function given in (12) reduces to

$$\mathbf{u}(t) := \begin{cases} \left[\sum_{i=1}^{N}\mathbf{\Phi}(T,t+h_i)\mathbf{C}_i(t+h_i)\right]^*\mathbf{W}^{-1}\big[\mathbf{x}_T - \mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0)\big], \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for all } t \in [t_0, T-h_N], \\ \left[\sum_{i=1}^{l}\mathbf{\Phi}(T,t+h_i)\mathbf{C}_i(t+h_i)\right]^*\mathbf{W}^{-1}\big[\mathbf{x}_T - \mathbf{\Phi}(T,t_0)(\mathbf{x}_0+\mathbf{a}_0)\big], \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for all } t \in (T-h_{l+1}, T-h_l], \\ \mathbf{0}, \qquad\qquad\qquad\qquad\qquad \text{for all } t \in (T-h_1, T], \end{cases} \qquad (15)$$

where $\mathbf{W} := \mathbf{V} + \mathbf{W}_{MN+2} + \cdots + \mathbf{W}_{(M+1)N}$ and $l = 1, 2, \ldots, (N-1)$. □

Remark 4.7. The control function defined in (15) is in fact the minimum energy controller among all steering controllers, which steers the state of the linear delay system (3) without impulses from $\mathbf{x}_0$ to $\mathbf{x}_T$ and hence corresponding $\mathbf{U}(\cdot) \in \mathcal{U}_1$ (where $\mathbf{u}(t) = vec\,\mathbf{U}(t)$) is the minimum energy control function of the system (1) without impulses and with delays.

Remark 4.8. In system (1), if delays are absent in the control function $\mathbf{U}(\cdot) \in \mathcal{U}_1$ and $\alpha_k := \sum_{r=1}^{m} \sum_{j=1}^{n} d_{rj}^k U_{rj}(t_k) = 0$, for all $k = 1, 2, \ldots, M$ then the necessary and sufficient condition for the controllability of the system (1) obtained in Corollary 4.6 reduces to the controllability condition given in remark 3.4, p.3 of Dubey and George [8] and is given by

$$rank(\mathbf{V}) = n^2,$$

where $\mathbf{V}$ is obtained from (14) by substituting $h_i = 0$, for all $i = 1, 2, \ldots, N$ i. e.

$$\mathbf{V} = \int_{t_0}^{T} \left[\mathbf{\Phi}(T, s) \sum_{i=1}^{N} \mathbf{C}_i(s) \right] \left[\mathbf{\Phi}(T, s) \sum_{i=1}^{N} \mathbf{C}_i(s) \right]^* \mathrm{d}s.$$

The matrix $\mathbf{V}$ given above is called the controllability Grammian of the system (1) having no impulses and no delays. Further the steering controller defined in (15) reduces to

$$\mathbf{u}(t) := \left[\mathbf{\Phi}(T, t) \sum_{i=1}^{N} \mathbf{C}_i(t) \right]^* \mathbf{V}^{-1} \big[\mathbf{x}_T - \mathbf{\Phi}(T, t_0)(\mathbf{x}_0 + \mathbf{a}_0) \big], \text{ for all } t \in [t_0, T].$$

This is the minimum energy controller among all steering controllers which steers the state of the system (3) having no impulses and no delays, from $\mathbf{x}_0$ to $\mathbf{x}_T$. Hence corresponding $\mathbf{U}(\cdot) \in \mathcal{U}_1$ (where $\mathbf{u}(t) = vec\,\mathbf{U}(t)$) is the minimum energy control function of the system (1) without impulses and without delays.

4.2. Controllability of the system (1) for the class $\mathcal{U}_2$ of control functions

In this subsection, a necessary and sufficient condition for the controllability of the system (1) for the class $\mathcal{U}_2$ of control functions is derived. Further, if the control function $\mathbf{U}(\cdot) \in \mathcal{U}_2$, then $\mathbf{W}_p = \mathbf{O}$, for $p = 1, 2, \ldots, (M-1)N$, $(M-1)N + 1$.

Theorem 4.9. In system (1), if the control function belongs to the class $\mathcal{U}_2$, then a necessary and sufficient condition for the controllability of the system (1) on $[t_0, T]$ is given by

$$rank\big(\mathbf{W}_{(M-1)N+2}, \mathbf{W}_{(M-1)N+3}, \ldots, \mathbf{W}_{MN}, \mathbf{W}_{MN+1}, \ldots, \mathbf{W}_{(M+1)N}\big) = n^2.$$

Here, $\mathbf{W}_{(M-1)N+2}, \mathbf{W}_{(M-1)N+3}, \ldots, \mathbf{W}_{MN}, \mathbf{W}_{MN+1}, \ldots, \mathbf{W}_{(M+1)N}$ are obtained from (11) and are given by

$$\mathbf{W}_{MN+1-l} := \int_{t_M - h_{l+1}}^{t_M - h_l} \left[\sum_{i=l+1}^{N} \mathbf{\Phi}(T, s + h_i) \mathbf{C}_i(s + h_i) \right] \left[\sum_{i=l+1}^{N} \mathbf{\Phi}(T, s + h_i) \mathbf{C}_i(s + h_i) \right]^* \mathrm{d}s,$$

$$\mathbf{W}_{MN+1} := \int_{t_M-h_1}^{T-h_N} \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right]\left[\sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right]^* \mathrm{d}s,$$

$$\mathbf{W}_{(M+1)N+1-l} := \int_{T-h_{l+1}}^{T-h_l} \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right]\left[\sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right]^* \mathrm{d}s,$$

where $l = 1, \ldots, (N-1)$.

Proof. To prove the sufficiency, it is enough to show that, the sufficient condition for the controllability of the system (3) is

$$rank\big(\mathbf{W}_{(M-1)N+2}, \mathbf{W}_{(M-1)N+3}, \ldots, \mathbf{W}_{MN}, \mathbf{W}_{MN+1}, \ldots, \mathbf{W}_{(M+1)N}\big) = n^2$$

for the class $\mathcal{U}_2$ of controls. Then the proof follows by Proposition 3.1.

Let us begin by considering $rank\big(\mathbf{W}_{(M-1)N+2}, \mathbf{W}_{(M-1)N+3}, \ldots, \mathbf{W}_{MN}, \mathbf{W}_{MN+1}, \ldots, \mathbf{W}_{(M+1)N}\big) = n^2$. Then $\mathbf{W} := \mathbf{W}_{(M-1)N+2} + \mathbf{W}_{(M-1)N+3} + \cdots + \mathbf{W}_{MN} + \mathbf{W}_{MN+1} + \cdots + \mathbf{W}_{(M+1)N}$ is a positive definite matrix. Now, define a control function $\mathbf{u}(\cdot) = vec\,\mathbf{U}(\cdot) \in \mathcal{U}_2$ as follows:

$$\mathbf{u}(t) := \begin{cases} \left[\sum_{i=l+1}^{N} \mathbf{\Phi}(T, t+h_i)\mathbf{C}_i(t+h_i)\right]^* \mathbf{W}^{-1}\mathbf{x}_T, \text{ for all } t \in (t_M - h_{l+1}, t_M - h_l], \\ \left[\sum_{i=1}^{N} \mathbf{\Phi}(T, t+h_i)\mathbf{C}_i(t+h_i)\right]^* \mathbf{W}^{-1}\mathbf{x}_T, \text{ for all } t \in (t_M - h_1, T - h_N] \setminus \{t_M\}, \\ \left[\sum_{i=1}^{l} \mathbf{\Phi}(T, t+h_i)\mathbf{C}_i(t+h_i)\right]^* \mathbf{W}^{-1}\mathbf{x}_T, \text{ for all } t \in (T - h_{l+1}, T - h_l], \\ \mathbf{v}_M, \quad \text{at } t = t_M, \\ \mathbf{0}, \quad \text{for all } t \in [t_0, t_M - h_N] \cup (T - h_1, T], \end{cases} \tag{16}$$

where $l = 1, \ldots, (N-1)$ and $\mathbf{v}_M = \big(v_{rj}^M\big) \in \mathbb{R}^{mn}$ is any vector such that $\sum_{r=1}^{m}\sum_{j=1}^{n} d_{rj}^M v_{rj}^M = -1$. Now the state $\mathbf{x}(t)$ of the system (3) given in (4) at $t = T$ satisfies,

$$\begin{aligned} \mathbf{x}(T) = &\sum_{l=1}^{N-1} \int_{t_M-h_{l+1}}^{t_M-h_l} \sum_{i=l+1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\ &+ \int_{t_M-h_1}^{T-h_N} \sum_{i=1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s \\ &+ \sum_{l=1}^{N-1} \int_{T-h_{l+1}}^{T-h_l} \sum_{i=1}^{l} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s. \end{aligned}$$

Substitute $\mathbf{u}(t)$ from (16) in the above expression to get,

$$\mathbf{x}(T) = \Bigg\{ \sum_{l=1}^{N-1} \int_{t_M-h_{l+1}}^{t_M-h_l} \left[\sum_{i=l+1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right]\left[\sum_{i=l+1}^{N} \mathbf{\Phi}(T, s+h_i)\mathbf{C}_i(s+h_i)\right]^* \mathrm{d}s$$

$$
\begin{aligned}
&+\int_{t_M-h_1}^{T-h_N}\left[\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right]\left[\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right]^*\mathrm{d}s\\
&+\sum_{l=1}^{N-1}\int_{T-h_{l+1}}^{T-h_l}\left[\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right]\left[\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\right]^*\mathrm{d}s\Bigg\}\mathbf{W}^{-1}\mathbf{x}_T\\
&=\Big\{\mathbf{W}_{(M-1)N+2}+\mathbf{W}_{(M-1)N+3}+\cdots+\mathbf{W}_{MN}+\mathbf{W}_{MN+1}+\cdots+\mathbf{W}_{(M+1)N}\Big\}\mathbf{W}^{-1}\mathbf{x}_T\\
&=\mathbf{W}\mathbf{W}^{-1}\mathbf{x}_T=\mathbf{x}_T.
\end{aligned}
$$

Hence the system (3) is controllable over $\mathbb{R}^{n^2}$ on $[t_0,T]$. Then by Proposition 3.1, the system (1) is also controllable over $\mathbb{R}^{n\times n}$ on $[t_0,T]$.

Now the necessity of the condition can be proved by contradiction. Let the system (1) be controllable on $[t_0,T]$ for the class $\mathcal{U}_2$ of control functions, but assume that

$$0\le rank\big(\mathbf{W}_{(M-1)N+2},\,\mathbf{W}_{(M-1)N+3},\dots,\mathbf{W}_{MN},\,\mathbf{W}_{MN+1},\dots,\mathbf{W}_{(M+1)N}\big)<n^2.$$

Then the matrix $\mathbf{W}=\mathbf{W}_{(M-1)N+2}+\mathbf{W}_{(M-1)N+3}+\cdots+\mathbf{W}_{MN}+\mathbf{W}_{MN+1}+\cdots+\mathbf{W}_{(M+1)N}$ is singular and hence there exists a non-zero vector, say $\mathbf{v}\in\mathbb{R}^{n^2}$ such that $\mathbf{W}\mathbf{v}=\mathbf{0}$, i.e

$$\big(\mathbf{W}_{(M-1)N+2}+\mathbf{W}_{(M-1)N+3}+\cdots+\mathbf{W}_{MN}+\mathbf{W}_{MN+1}+\cdots+\mathbf{W}_{(M+1)N}\big)\mathbf{v}=\mathbf{0},$$

and hence each $\mathbf{W}_{(M-1)N+2}$, $\mathbf{W}_{(M-1)N+3}$, $\mathbf{W}_{MN}$, $\mathbf{W}_{MN+1}$, $\mathbf{W}_{(M+1)N}$ is a singular matrix and $\langle\mathbf{W}_p\mathbf{v},\mathbf{v}\rangle_{\mathbb{R}^{n^2}}=0$ for $p=(M-1)N+2,\ (M-1)N+3,\dots,MN,\ (MN+1),\dots,(M+1)N$. Proceeding similar to the Theorem 4.4, we get

$$
\begin{cases}
\mathbf{v}^*\Big(\displaystyle\sum_{i=l+1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\Big)=\mathbf{0},\\
\mathbf{v}^*\Big(\displaystyle\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\Big)=\mathbf{0},\\
\mathbf{v}^*\Big(\displaystyle\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\Big)=\mathbf{0},
\end{cases}
\tag{17}
$$

for all $l=1,2,\dots,(N-1)$.

Since the system (1) is controllable, so the system (3) on $[t_0,T]$, and hence any initial state $\mathbf{x}_0$ can be steered to the final state $\mathbf{x}(T)=\mathbf{v}$ with certain control function $\mathbf{u}(\cdot)=vec\,\mathbf{U}(\cdot)$, where $\mathbf{U}(\cdot)\in\mathcal{U}_2$. That is,

$$
\begin{aligned}
\mathbf{v}=\mathbf{x}(T)=&\sum_{l=1}^{N-1}\int_{t_M-h_{l+1}}^{t_M-h_l}\sum_{i=l+1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\\
&+\int_{t_M-h_1}^{T-h_N}\sum_{i=1}^{N}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s\\
&+\sum_{l=1}^{N-1}\int_{T-h_{l+1}}^{T-h_l}\sum_{i=1}^{l}\mathbf{\Phi}(T,s+h_i)\mathbf{C}_i(s+h_i)\mathbf{u}(s)\,\mathrm{d}s.
\end{aligned}
$$

Premultiply the above expression with $\mathbf{v}^*$ and use (17), to get $\mathbf{v}^*\mathbf{v} = 0$, and hence $\mathbf{v} = \mathbf{0}$, a contradiction. Hence $rank\big(\mathbf{W}_{(M-1)N+2}, \mathbf{W}_{(M-1)N+3}, \ldots, \mathbf{W}_{MN}, \mathbf{W}_{MN+1}, \ldots, \mathbf{W}_{(M+1)N}\big) = n^2$. □

Remark 4.10. The control function given in (16) is independent of an initial state of the system (1) and depends only on the final state $\mathbf{X}(T)$ (where $\mathbf{x}_T = vec\,\mathbf{X}(T)$). Therefore, the control function given in (16) steers any initial state of the system (1) to $\mathbf{X}(T)$.

Remark 4.11. Suppose the control function $\mathbf{U}(\cdot) \in \mathcal{U}_2$ in the system (1) does not have delays, then the necessary and sufficient condition of this system obtained in the Theorem 4.9 reduces to the controllability condition given in Theorem 3.1, p.330 of Dubey and George [9] and is given by

$$rank(\mathbf{W}_{MN+1}) = n^2,$$

where $\mathbf{W}_{MN+1}$ is obtained from (11) by taking $h_i = 0$, for all $i = 1, 2, \ldots, N$, that is,

$$\mathbf{W}_{MN+1} := \int_{t_M}^{T} \left[\mathbf{\Phi}(T,s)\sum_{i=1}^{N}\mathbf{C}_i(s)\right]\left[\mathbf{\Phi}(T,s)\sum_{i=1}^{N}\mathbf{C}_i(s)\right]^* \mathrm{d}s.$$

Further the sterring control function given in (16) reduces to

$$\mathbf{u}(t) := \begin{cases} \mathbf{0}, & \text{for all } t \in [t_0, t_M], \\ \left[\mathbf{\Phi}(T,t)\sum_{i=1}^{N}\mathbf{C}_i(t)\right]^* \mathbf{W}_{MN+1}^{-1}\mathbf{x}_T, & \text{for all } t \in (t_M, T]. \end{cases}$$

5. NUMERICAL EXAMPLES

Example 5.1. Consider the following $2 \times 2-$dimensional linear impulsive matrix Lyapunov autonomous ordinary differential system with one impulse and two delays in the control function:

$$\left.\begin{aligned} \begin{bmatrix} \dot{x}_{11}(t) & \dot{x}_{12}(t) \\ \dot{x}_{21}(t) & \dot{x}_{22}(t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{bmatrix} + \begin{bmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} U_{11}(t-0.2) & U_{12}(t-0.2) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} U_{11}(t-0.4) & U_{12}(t-0.4) \end{bmatrix},\ t \in [0,1] \setminus \{0.5\}, \\ \begin{bmatrix} x_{11}(0) & x_{12}(0) \\ x_{21}(0) & x_{22}(0) \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ \begin{bmatrix} \Delta x_{11}(0.5) & \Delta x_{12}(0.5) \\ \Delta x_{21}(0.5) & \Delta x_{22}(0.5) \end{bmatrix} &= \big(U_{11}(0.5) + U_{12}(0.5)\big)\begin{bmatrix} x_{11}(0.5) & x_{12}(0.5) \\ x_{21}(0.5) & x_{22}(0.5) \end{bmatrix}, \\ \begin{bmatrix} U_{11}(t) & U_{12}(t) \end{bmatrix} &= \begin{bmatrix} 1 & t \end{bmatrix}, \quad t \in [-0.4, 0). \end{aligned}\right\} \quad (18)$$

After applying the vector operator, the system (18) becomes

$$\begin{bmatrix}\dot{x}_{11}(t)\\ \dot{x}_{21}(t)\\ \dot{x}_{12}(t)\\ \dot{x}_{22}(t)\end{bmatrix} = \begin{bmatrix}2&0&0&0\\0&4&0&0\\0&0&1&0\\0&0&0&3\end{bmatrix}\begin{bmatrix}x_{11}(t)\\ x_{21}(t)\\ x_{12}(t)\\ x_{22}(t)\end{bmatrix} + \begin{bmatrix}1&0\\0&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}U_{11}(t-0.2)\\ U_{12}(t-0.2)\end{bmatrix} + \begin{bmatrix}0&0\\1&0\\0&0\\0&1\end{bmatrix}\begin{bmatrix}U_{11}(t-0.4)\\ U_{12}(t-0.4)\end{bmatrix}, \quad t\in[0,1]\setminus\{0.5\},$$

$$\begin{bmatrix}x_{11}(0)\\ x_{21}(0)\\ x_{12}(0)\\ x_{22}(0)\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix},$$

$$\begin{bmatrix}\Delta x_{11}(0.5)\\ \Delta x_{21}(0.5)\\ \Delta x_{12}(0.5)\\ \Delta x_{22}(0.5)\end{bmatrix} = \big(U_{11}(0.5)+U_{12}(0.5)\big)\begin{bmatrix}x_{11}(0.5)\\ x_{21}(0.5)\\ x_{12}(0.5)\\ x_{22}(0.5)\end{bmatrix},$$

$$\begin{bmatrix}U_{11}(t)\\ U_{12}(t)\end{bmatrix} = \begin{bmatrix}1\\ t\end{bmatrix}, \quad t\in[-0.4,0).$$

On comparing the above system with (3), we get

$$\mathbf{A} = \begin{bmatrix}2&0&0&0\\0&4&0&0\\0&0&1&0\\0&0&0&3\end{bmatrix}, \mathbf{C}_1 = \begin{bmatrix}1&0\\0&0\\0&1\\0&0\end{bmatrix}, \mathbf{C}_2 = \begin{bmatrix}0&0\\1&0\\0&0\\0&1\end{bmatrix}, t_0=0, h_1=0.2, h_2=0.4, t_1=0.5,$$
$$T=1,\ \alpha = U_{11}(0.5)+U_{12}(0.5).$$

By calculation, we get

$$\mathbf{\Phi}(t,s) = \begin{bmatrix}e^{2(t-s)}&0&0&0\\0&e^{4(t-s)}&0&0\\0&0&e^{(t-s)}&0\\0&0&0&e^{3(t-s)}\end{bmatrix}, \quad \mathbf{a}_0 = \begin{bmatrix}0.1648\\0.1995\\-0.0187\\-0.0557\end{bmatrix},$$

$$\mathbf{W}_1 = (1+\alpha)^2\begin{bmatrix}2.0217&4.1054&0&0\\4.1054&8.3637&0&0\\0&0&0.4487&1.1094\\0&0&1.1094&2.7521\end{bmatrix},$$

$$\mathbf{W}_2 = \begin{bmatrix}2.2636(1+\alpha)^2&3.4896(1+\alpha)&0&0\\3.4896(1+\alpha)&5.4467&0&0\\0&0&0.6684(1+\alpha)^2&1.2424(1+\alpha)\\0&0&1.2424(1+\alpha)&2.3393\end{bmatrix},$$

$$\mathbf{W}_3 = \begin{bmatrix}1.2907&1.2555&0&0\\1.2555&1.2529&0&0\\0&0&0.6131&0.7084\\0&0&0.7084&0.8416\end{bmatrix} \text{ and } \mathbf{W}_4 = \begin{bmatrix}0.3064&0&0&0\\0&0&0&0\\0&0&0.2459&0\\0&0&0&0\end{bmatrix}.$$

Let the desired final state of the system (18) be $\mathbf{X}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Now we compute the steering controller and controlled trajectory in different cases.

Case (i): If we choose the control function from the class $\mathcal{U}_1$ such that $\alpha = U_{11}(0.5) + U_{12}(0.5) = 1$, then $rank\big(\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \mathbf{W}_4\big) = 4$, hence the system (18) is controllable on $[0,1]$ by Theorem 4.4.

Further, $\mathbf{W}^{-1} = \begin{bmatrix} 0.2779 & -0.1706 & 0 & 0 \\ -0.1706 & 0.1297 & 0 & 0 \\ 0 & 0 & 0.8172 & -0.4395 \\ 0 & 0 & -0.4395 & 0.3068 \end{bmatrix}$ and one of the control function that steers the state from $\mathbf{X}_0$ to $\mathbf{X}_1$ is given by

$$\mathbf{U}(t) = \begin{cases} \big[(31.2e^{-2t} - 54.0563e^{-4t}) \quad (-2.326e^{-t} + 6.1598e^{-3t})\big], & \text{for all } t \in [0, 0.1], \\ \big[(31.2e^{-2t} - 27.0283e^{-4t}) \quad (-2.326e^{-t} + 3.08e^{-3t})\big], & \text{for all } t \in (0.1, 0.3], \\ \big[(15.6e^{-2t} - 27.028e^{-4t}) \quad (-1.163e^{-t} + 3.08e^{-3t})\big], & \text{for all } t \in (0.3, 0.6] \setminus \{0.5\}, \\ \big[15.6e^{-2t} \quad -1.163e^{-t}\big], & \text{for all } t \in (0.6, 0.8], \\ \big[0 \quad 0\big], & \text{for all } t \in (0.8, 1], \end{cases}$$

and the controlled trajectory is given by

$$\mathbf{X}(t) = \begin{cases} \begin{bmatrix} 0.5(e^{2t} - 1) & 0.8(e^{t} - 1.25t - 1) \\ 0.25(e^{4t} - 1) & -0.0222(e^{3t} + 15.0135t - 1) \end{bmatrix}, & \text{for all } t \in [0, 0.2], \\ \begin{bmatrix} (20.05e^{-4t} - 11.629e^{-2t} - 0.6456e^{2t}) & (-2.806e^{-3t} + 1.4198e^{-t} + 0.289e^{t}) \\ 0.25(e^{4t} - 1) & -0.0222(e^{3t} + 15.136t - 1) \end{bmatrix}, & \text{for all } t \in (0.2, 0.3], \\ \begin{bmatrix} (10.025e^{-4t} - 11.634e^{-2t} + 1.0125e^{2t}) & (-1.403e^{-3t} + 1.4198e^{-t} - 1.3258e^{t}) \\ 0.25(e^{4t} - 1) & -0.0222(e^{3t} + 15.136t - 1) \end{bmatrix}, & \text{for all } t \in (0.3, 0.4], \\ \begin{bmatrix} (10.025e^{-4t} - 11.689e^{-2t} + 1.0406e^{2t}) & (-1.403e^{-3t} + 1.4198e^{-t} - 0.1326e^{t}) \\ (33.469e^{-4t} - 11.4464e^{-2t} - 0.1272e^{4t}) & (-3.409e^{-3t} + 0.8693e^{-t} + 0.0784e^{3t}) \end{bmatrix}, & \text{for all } t \in (0.4, 0.5], \\ \begin{bmatrix} (10.025e^{-4t} - 5.8175e^{-2t} + 0.1584e^{2t}) & (-1.403e^{-3t} + 0.7102e^{-t} + 0.3286e^{t}) \\ (16.734e^{-4t} - 11.5716e^{-2t} + 0.1138e^{4t}) & (-1.7042e^{-3t} + 0.8674e^{-t} + 0.0194e^{3t}) \end{bmatrix}, & \text{for all } t \in (0.5, 0.7], \\ \begin{bmatrix} (10.025e^{-4t} - 5.8175e^{-2t} + 0.1584e^{2t}) & (-1.403e^{-3t} + 0.7102e^{-t} + 0.3286e^{t}) \\ (16.734e^{-4t} - 5.785e^{-2t} + 0.0268e^{4t}) & (-1.7042e^{-3t} + 0.4334e^{-t} + 0.046e^{3t}) \end{bmatrix}, & \text{for all } t \in (0.7, 0.8], \\ \begin{bmatrix} -5.8178e^{-2t} + 0.2415e^{2t} & 0.7102e^{-t} + 0.2715e^{t} \\ (16.734e^{-4t} - 5.785e^{-2t} + 0.0268e^{4t}) & (-1.7042e^{-3t} + 0.4334e^{-t} + 0.046e^{3t}) \end{bmatrix}, & \text{for all } t \in (0.8, 1], \end{cases}$$

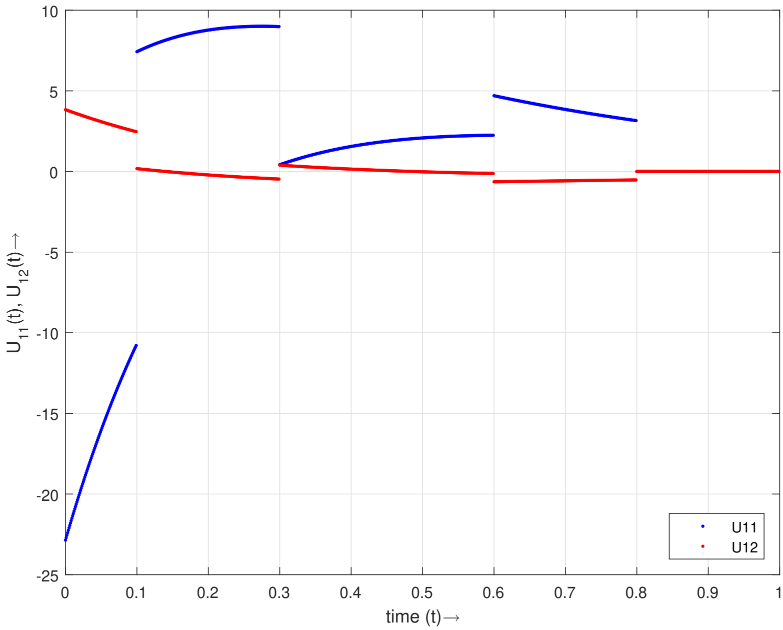

Fig. 1. Plot of control function in case (i) of example 5.1.

and these are shown in the Figures1 and 2 respectively.

Case (ii): If we choose the control function from the class $\mathcal{U}_1$ such that $\alpha = U_{11}(0.5) + U_{12}(0.5) = 0$, then there are no impulses in the system (18) and hence the matrices $\mathbf{W}_1$, $\mathbf{W}_2$ and $\mathbf{W}_3$ can be combined to get a matrix $\mathbf{V} = \mathbf{W}_1 + \mathbf{W}_2 + \mathbf{W}_3 = \begin{bmatrix} 5.576 & 8.8505 & 0 & 0 \\ 8.8505 & 15.0633 & 0 & 0 \\ 0 & 0 & 1.7302 & 3.0602 \\ 0 & 0 & 3.0602 & 5.933 \end{bmatrix}$. Then we see that $rank(\mathbf{V}, \mathbf{W}_4) = 4$, and hence by Corollary 4.6, the system (18) is controllable on $[0, 1]$. Further,
$\mathbf{W}^{-1} = \begin{bmatrix} 1.4606 & -0.8578 & 0 & 0 \\ -0.8578 & 0.5701 & 0 & 0 \\ 0 & 0 & 2.5147 & -1.2970 \\ 0 & 0 & -1.2970 & 0.8375 \end{bmatrix}$ and one of the control function that steers the state from $\mathbf{X}_0$ to $\mathbf{X}_1$ is given by

$$\mathbf{U}(t) = \begin{cases} \left[(-60.11e^{-4t} + 40.4515e^{-2t}) \quad (2.4901e^{-3t} - 0.2351e^{-t})\right], \ \forall t \in [0, 0.6] \setminus \{0.5\}, \\ \left[40.4515e^{-2t} \quad -0.2351e^{-t}\right], \ \forall t \in (0.6, 0.8], \\ \left[0 \quad 0\right], \ \forall t \in (0.8, 1], \end{cases}$$

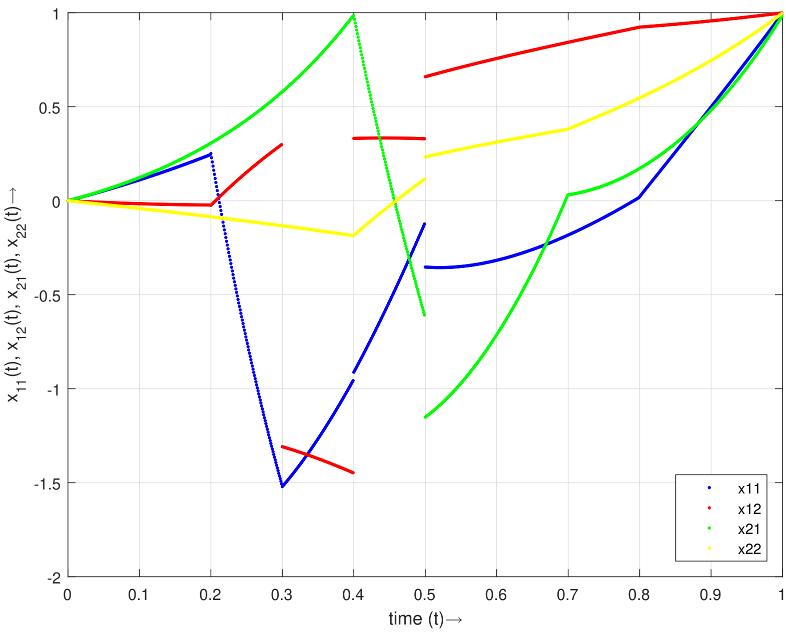

Fig. 2. Plot of controlled trajectory in case (i) of example 5.1.

and the controlled trajectory is given by

$$\mathbf{X}(t) = \begin{cases} \begin{bmatrix} 0.5(e^{2t}-1) & 0.8(e^{t}-1.25t-1) \\ 0.25(e^{4t}-1) & -0.0222(e^{3t}+15.0135t-1) \end{bmatrix}, & \text{for all } t \in [0, 0.2], \\ \begin{bmatrix} (22.2946e^{-4t}-15.086e^{-2t}+0.2285e^{2t}) & (-1.1342e^{-3t}+0.1436e^{-t}+0.3947e^{t}) \\ 0.25(e^{4t}-1) & -0.0222(e^{3t}+15.0135t-1) \end{bmatrix}, \\ \qquad \text{for all } t \in (0.2, 0.4], \\ \begin{bmatrix} (22.2954e^{-4t}-15.086e^{-2t}+0.22825e^{2t}) & (-1.1343e^{-3t}+0.1436e^{-t}+0.3947e^{t}) \\ (37.216e^{-4t}-15.005e^{-2t}+0.0437e^{4t}) & (-1.378e^{-3t}+0.0877e^{-t}+0.0516e^{3t}) \end{bmatrix}, \\ \qquad \text{for all } t \in (0.4, 0.8], \\ \begin{bmatrix} -15.086e^{-2t}+0.4117e^{2t} & 0.1436e^{-t}+0.3484e^{t} \\ (37.216e^{-4t}-15.005e^{-2t}+0.0437e^{4t}) & (-1.378e^{-3t}+0.0877e^{-t}+0.0516e^{3t}) \end{bmatrix}, \\ \qquad \text{for all } t \in (0.8, 1], \end{cases}$$

and these are shown in Figures 3 and 4 respectively.

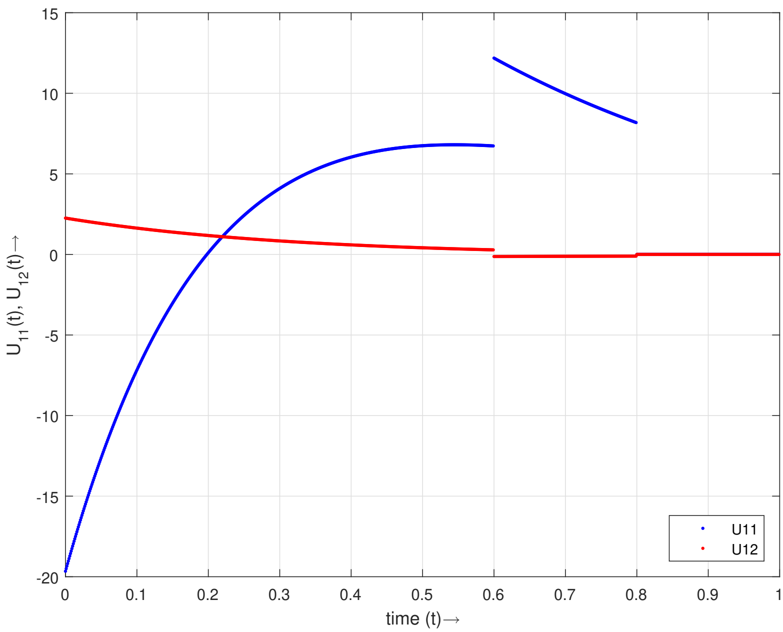

Fig. 3. Plot of control function in case (ii) of example 5.1.

Case (iii): If we choose the control function from the class $\mathcal{U}_2$ such that $\alpha = U_{11}(0.5) + U_{12}(0.5) = -1$, then $\mathbf{W}_1 = \mathbf{O}$ and $rank\big(\mathbf{W}_2, \mathbf{W}_3, \mathbf{W}_4\big) = 4$, hence by Theorem 4.9, the system (18) is controllable on $[0, 1]$. Now,

$$\mathbf{W}^{-1} = \begin{bmatrix} 0.7342 & -0.1374 & 0 & 0 \\ -0.1374 & 0.1748 & 0 & 0 \\ 0 & 0 & 1.4260 & -0.3176 \\ 0 & 0 & -0.3176 & 0.3851 \end{bmatrix}$$

and one of the control function that steers the state from $\mathbf{X}_0$ to $\mathbf{X}_1$ is given by

$$\mathbf{U}(t) = \begin{cases} \begin{bmatrix} 0.4122e^{-4t} & 0.40846e^{-3t} \end{bmatrix}, \ \forall\, t \in (0.1, 0.3], \\ \begin{bmatrix} (2.9556e^{-2t} + 0.4122e^{-4t}) & (2.4668e^{-t} + 0.40846e^{-3t}) \end{bmatrix}, \ \forall\, t \in (0.3, 0.6] \setminus \{0.5\}, \\ \begin{bmatrix} 2.9556e^{-2t} & 2.4668e^{-t} \end{bmatrix}, \ \forall\, t \in (0.6, 0.8], \\ \begin{bmatrix} 0 & 0 \end{bmatrix}, \ \forall\, t \in [0, 0.1] \cup (0.8, 1], \end{cases}$$

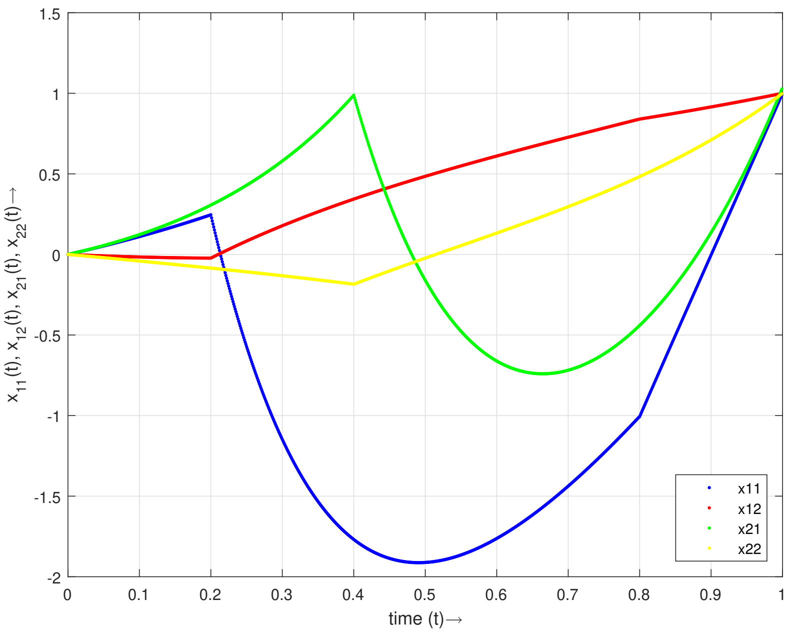

Fig. 4. Plot of controlled trajectory in case (ii) of example 5.1.

and the controlled trajectory is given by

$$\mathbf{X}(t)=\begin{cases}\begin{bmatrix}0.5(e^{2t}-1) & 0.8(e^{t}-1.25t-1)\\ 0.25(e^{4t}-1) & -0.0222(e^{3t}+15.0135t-1)\end{bmatrix}, & \text{for all } t\in[0,0.2],\\ \begin{bmatrix}0.1648e^{2t} & -0.0187e^{t}\\ 0.25(e^{4t}-1) & -0.0222(e^{3t}+15.136t-1)\end{bmatrix}, & \text{for all } t\in(0.2,0.3],\\ \begin{bmatrix}0.19e^{2t}-0.153e^{4t} & -0.186e^{-3t}+0.0373e^{t}\\ 0.25(e^{4t}-1) & -0.0222(e^{3t}+15.136t-1)\end{bmatrix}, & \text{for all } t\in(0.3,0.4],\\ \begin{bmatrix}0.19e^{2t}-0.153e^{4t} & -0.186e^{-3t}+0.0373e^{t}\\ 0.1995e^{4t} & -0.0557e^{3t}\end{bmatrix}, & \text{for all } t\in(0.4,0.5],\\ \begin{bmatrix}(-0.153e^{-4t}-1.103e^{-2t}+0.1568e^{2t}) & (-0.186e^{-3t}-1.5064e^{-t}+0.5794e^{t})\\ -0.255e^{-4t}+0.0047e^{4t} & -0.226e^{-3t}+0.0113e^{3t}\end{bmatrix}, & \\ \qquad \text{for all } t\in(0.5,0.7], & \\ \begin{bmatrix}(-0.153e^{-4t}-1.103e^{-2t}+0.1568e^{2t}) & (-0.186e^{-3t}-1.5064e^{-t}+0.5794e^{t})\\ (-0.255e^{-4t}-1.096e^{-2t}+0.0211e^{4t}) & (-0.226e^{-3t}-0.9204e^{-t}+0.067e^{3t})\end{bmatrix}, & \\ \qquad \text{for all } t\in(0.7,0.8], & \\ \begin{bmatrix}-1.1023e^{-2t}+0.1555e^{2t} & -1.5064e^{-t}+0.5718e^{t}\\ (-0.255e^{-4t}-1.096e^{-2t}+0.0211e^{4t}) & (-0.226e^{-3t}-0.9204e^{-t}+0.067e^{3t})\end{bmatrix}, & \\ \qquad \text{for all } t\in(0.8,1], & \end{cases}$$

and these are shown in Figures 5 and 6 respectively.

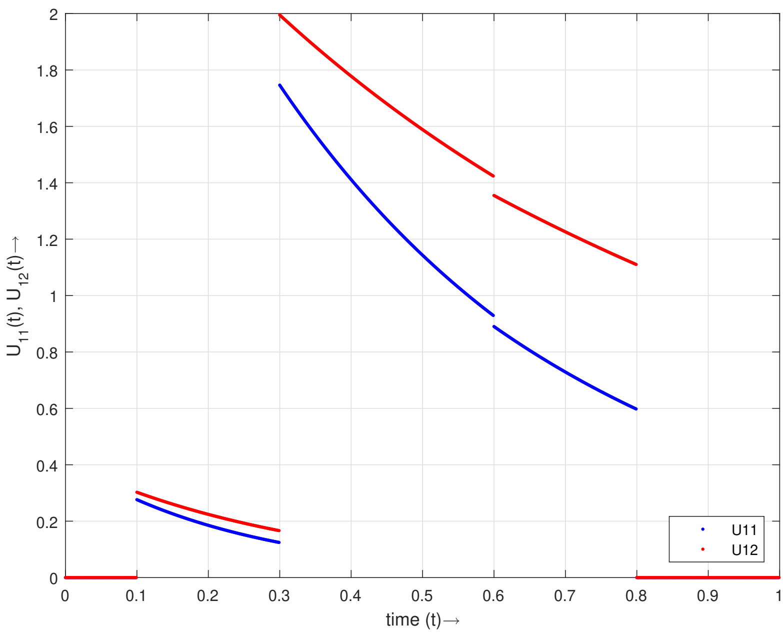

Fig. 5. Plot of control function in case (iii) of example 5.1.

Example 5.2. Consider another $2 \times 2-$dimensional linear impulsive matrix Lyapunov autonomous ordinary differential system with one impulse and two delays in the control function:

$$\left.\begin{aligned}
\begin{bmatrix}\dot{x}_{11}(t) & \dot{x}_{12}(t)\\ \dot{x}_{21}(t) & \dot{x}_{22}(t)\end{bmatrix} &= \begin{bmatrix}0 & 2\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_{11}(t) & x_{12}(t)\\ x_{21}(t) & x_{22}(t)\end{bmatrix} + \begin{bmatrix}x_{11}(t) & x_{12}(t)\\ x_{21}(t) & x_{22}(t)\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\\
&\quad + \begin{bmatrix}1\\ 1\end{bmatrix}\begin{bmatrix}U_{11}(t-0.2) & U_{12}(t-0.2)\end{bmatrix}\\
&\quad + \begin{bmatrix}0\\ 2\end{bmatrix}\begin{bmatrix}U_{11}(t-0.4) & U_{12}(t-0.4)\end{bmatrix},\ t \in [0,1]\setminus\{0.5\},\\
\begin{bmatrix}x_{11}(0) & x_{12}(0)\\ x_{21}(0) & x_{22}(0)\end{bmatrix} &= \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},\\
\begin{bmatrix}\Delta x_{11}(0.5) & \Delta x_{12}(0.5)\\ \Delta x_{21}(0.5) & \Delta x_{22}(0.5)\end{bmatrix} &= \big(U_{11}(0.5) + U_{12}(0.5)\big)\begin{bmatrix}x_{11}(0.5) & x_{12}(0.5)\\ x_{21}(0.5) & x_{22}(0.5)\end{bmatrix},\\
\begin{bmatrix}U_{11}(t) & U_{12}(t)\end{bmatrix} &= \begin{bmatrix}1 & t\end{bmatrix}, \qquad t \in [-0.4, 0).
\end{aligned}\right\} \quad (19)$$

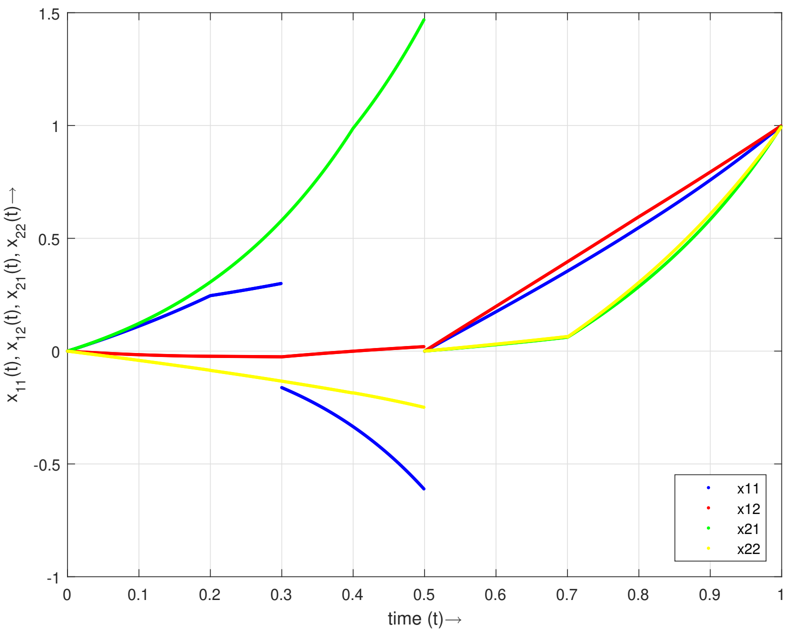

Fig. 6. Plot of controlled trajectory in case (iii) of example 5.1.

After applying the vector operator, the system (19) becomes

$$\begin{bmatrix}\dot{x}_{11}(t)\\ \dot{x}_{21}(t)\\ \dot{x}_{12}(t)\\ \dot{x}_{22}(t)\end{bmatrix} = \begin{bmatrix}1&2&1&0\\1&0&0&1\\1&0&1&2\\0&1&1&0\end{bmatrix}\begin{bmatrix}x_{11}(t)\\ x_{21}(t)\\ x_{12}(t)\\ x_{22}(t)\end{bmatrix} + \begin{bmatrix}1&0\\1&0\\0&1\\0&1\end{bmatrix}\begin{bmatrix}U_{11}(t-0.2)\\ U_{12}(t-0.2)\end{bmatrix} + \begin{bmatrix}0&0\\2&0\\0&0\\0&2\end{bmatrix}$$

$$\times\begin{bmatrix}U_{11}(t-0.4)\\ U_{12}(t-0.4)\end{bmatrix}, \qquad t\in[0,1]\setminus\{0.5\},$$

$$\begin{bmatrix}x_{11}(0)\\ x_{21}(0)\\ x_{12}(0)\\ x_{22}(0)\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix},$$

$$\begin{bmatrix}\Delta x_{11}(0.5)\\ \Delta x_{21}(0.5)\\ \Delta x_{12}(0.5)\\ \Delta x_{22}(0.5)\end{bmatrix} = \big(U_{11}(0.5)+U_{12}(0.5)\big)\begin{bmatrix}x_{11}(0.5)\\ x_{21}(0.5)\\ x_{12}(0.5)\\ x_{22}(0.5)\end{bmatrix},$$

$$\begin{bmatrix}U_{11}(t)\\ U_{12}(t)\end{bmatrix} = \begin{bmatrix}1\\ t\end{bmatrix}, \qquad t\in[-0.4,0).$$

On comparing the above system with (3), we get

$$\begin{aligned}
\mathbf{A} &= \begin{bmatrix} 1 & 2 & 1 & 0\\ 1 & 0 & 0 & 1\\ 1 & 0 & 1 & 2\\ 0 & 1 & 1 & 0 \end{bmatrix},\\
\mathbf{C}_1 &= \begin{bmatrix} 1 & 0\\ 1 & 0\\ 0 & 1\\ 0 & 1 \end{bmatrix},\\
\mathbf{C}_2 &= \begin{bmatrix} 0 & 0\\ 2 & 0\\ 0 & 0\\ 0 & 2 \end{bmatrix},\\
& t_0 = 0, h_1 = 0.2, h_2 = 0.4, t_1 = 0.5, \quad T = 1, \quad \alpha = U_{11}(0.5) + U_{12}(0.5).
\end{aligned}$$

By calculation, we get

$$\Phi(t,s) = \frac{1}{6}\left[\begin{matrix}
\begin{pmatrix} e^{-2(t-s)}+1\\ +2e^{t-s}+2e^{3(t-s)} \end{pmatrix} & \begin{pmatrix} -2e^{-2(t-s)}-2\\ +2e^{t-s}+2e^{3(t-s)} \end{pmatrix}\\
\begin{pmatrix} -e^{-2(t-s)}-1\\ +e^{t-s}+e^{3(t-s)} \end{pmatrix} & \begin{pmatrix} 2e^{-2(t-s)}+2\\ +e^{t-s}+e^{3(t-s)} \end{pmatrix}\\
\begin{pmatrix} -e^{-2(t-s)}+1\\ -2e^{t-s}+2e^{3(t-s)} \end{pmatrix} & \begin{pmatrix} 2e^{-2(t-s)}-2\\ -2e^{t-s}+2e^{3(t-s)} \end{pmatrix}\\
\begin{pmatrix} e^{-2(t-s)}-1\\ -e^{t-s}+e^{3(t-s)} \end{pmatrix} & \begin{pmatrix} -2e^{-2(t-s)}+2\\ -e^{t-s}+e^{3(t-s)} \end{pmatrix}
\end{matrix}\right.$$

$$\left.\begin{matrix}
\begin{pmatrix} -e^{-2(t-s)}+1\\ -2e^{t-s}+2e^{3(t-s)} \end{pmatrix} & \begin{pmatrix} 2e^{-2(t-s)}-2\\ -2e^{t-s}+2e^{3(t-s)} \end{pmatrix}\\
\begin{pmatrix} e^{-2(t-s)}-1\\ -e^{t-s}+e^{3(t-s)} \end{pmatrix} & \begin{pmatrix} -2e^{-2(t-s)}+2\\ -e^{t-s}+e^{3(t-s)} \end{pmatrix}\\
\begin{pmatrix} e^{-2(t-s)}+1\\ +2e^{t-s}+2e^{3(t-s)} \end{pmatrix} & \begin{pmatrix} -2e^{-2(t-s)}-2\\ +2e^{t-s}+2e^{3(t-s)} \end{pmatrix}\\
\begin{pmatrix} -e^{-2(t-s)}-1\\ +e^{t-s}+e^{3(t-s)} \end{pmatrix} & \begin{pmatrix} 2e^{-2(t-s)}+2\\ +e^{t-s}+e^{3(t-s)} \end{pmatrix}
\end{matrix}\right],$$

$$\mathbf{a}_0 = \begin{bmatrix} -0.1602\\ 1.0713\\ 0.0883\\ 0.5817 \end{bmatrix}, \quad \mathbf{W}_1 = (1+\alpha)^2 \begin{bmatrix} 17.4115 & 11.135 & 15.277 & 9.709\\ 11.1350 & 7.125 & 9.709 & 6.172\\ 15.2770 & 9.709 & 17.4115 & 11.1349\\ 9.709 & 6.172 & 11.1349 & 7.125 \end{bmatrix},$$

$$\mathbf{W}_{2}=\begin{bmatrix} \begin{matrix}6.915(1+\alpha)^2\\+5.8373(1+\alpha)\\+1.2553\end{matrix} & \begin{matrix}3.8854(1+\alpha)^2\\+4.965(1+\alpha)\\+1.3835\end{matrix} & \begin{matrix}5.8225(1+\alpha)^2\\+4.5299(1+\alpha)\\+0.8625\end{matrix} & \begin{matrix}3.2689(1+\alpha)^2\\+3.7637(1+\alpha)\\+0.9396\end{matrix} \\ \\ \begin{matrix}3.8854(1+\alpha)^2\\+4.965(1+\alpha)\\+0.8625\end{matrix} & \begin{matrix}2.1839(1+\alpha)^2\\+3.646(1+\alpha)\\+1.5526\end{matrix} & \begin{matrix}3.269(1+\alpha)^2\\+3.7636(1+\alpha)\\+0.9395\end{matrix} & \begin{matrix}1.8359(1+\alpha)^2\\2.8035(1+\alpha)\\1.041\end{matrix} \\ \\ \begin{matrix}5.8225(1+\alpha)^2\\+4.5299(1+\alpha)\\+0.8625\end{matrix} & \begin{matrix}3.269(1+\alpha)^2\\+3.7636(1+\alpha)\\+0.9395\end{matrix} & \begin{matrix}6.9152(1+\alpha)^2\\+5.8373(1+\alpha)\\+1.2553\end{matrix} & \begin{matrix}3.8854(1+\alpha)^2\\+4.8776(1+\alpha)\\+1.3834\end{matrix} \\ \\ \begin{matrix}3.2689(1+\alpha)^2\\+3.7637(1+\alpha)\\+0.9396\end{matrix} & \begin{matrix}1.8359(1+\alpha)^2\\+2.8035(1+\alpha)\\+1.041\end{matrix} & \begin{matrix}3.8854(1+\alpha)^2\\+4.8776(1+\alpha)\\+1.3834\end{matrix} & \begin{matrix}2.1839(1+\alpha)^2\\+3.646(1+\alpha)\\+1.5525\end{matrix} \end{bmatrix},$$

$$\mathbf{W}_3=\begin{bmatrix}4.143 & 4.3136 & 2.5038 & 2.3366\\ 4.3136 & 4.6915 & 2.3366 & 2.1719\\ 2.5038 & 2.3366 & 4.143 & 4.3136\\ 2.3366 & 2.1719 & 4.3136 & 4.6915\end{bmatrix},\ \mathbf{W}_4=\begin{bmatrix}0.3787 & 0.313 & 0.0874 & 0.0694\\ 0.313 & 0.2605 & 0.0694 & 0.0554\\ 0.0874 & 0.0694 & 0.3787 & 0.313\\ 0.0694 & 0.0554 & 0.313 & 0.2605\end{bmatrix}.$$

Let the desired final state of the system (19) be $\mathbf{X}_1=\begin{bmatrix}1&1\\1&1\end{bmatrix}$. Now we discuss the controllability of the system (19) in different cases.

Case (i): If we choose the control function $\mathbf{U}(\cdot)\in\mathcal{U}_1$ such that $\alpha=U_{11}(0.5)+U_{12}(0.5)=1$, then $rank\big(\mathbf{W}_1,\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_4\big)=4$, and hence by Theorem 4.4, the system (19) is controllable on $[0,1]$.

Case (ii): If the control function $\mathbf{U}(\cdot)\in\mathcal{U}_1$ such that $\alpha=U_{11}(0.5)+U_{12}(0.5)=0$, then there are no impulses in the system (19) and hence the matrices $\mathbf{W}_1$, $\mathbf{W}_2$ and $\mathbf{W}_3$ can be combined to get a matrix $\mathbf{V}=\mathbf{W}_1+\mathbf{W}_2+\mathbf{W}_3=\begin{bmatrix}35.5621 & 25.6825 & 28.9957 & 20.0177\\ 25.6825 & 19.199 & 20.0177 & 14.0243\\ 28.9957 & 20.0177 & 35.5623 & 25.5949\\ 20.0177 & 14.0243 & 25.5949 & 19.1989\end{bmatrix}$.

Then we see that $rank(\mathbf{V},\mathbf{W}_4)=4$, and hence by Corollary 4.6, the system (19) is controllable on $[0,1]$.

Case (iii): If we choose the control function $\mathbf{U}(\cdot)\in\mathcal{U}_2$ such that $\alpha=U_{11}(0.5)+U_{12}(0.5)=-1$, then $\mathbf{W}_1=\mathbf{O}$ and $rank\big(\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_4\big)=4$. Therefore by Theorem 4.9, the system (19) is controllable on $[0,1]$.

In all the above three cases, the computation of the control function and corresponding controlled trajectory are similar to that of Example 5.1.

Concluding remarks: In this paper, a dynamical control system modelled by an $n\times n-$dimensional linear impulsive matrix Lyapunov ordinary differential equations having multiple constant time-delays in its control function is considered. The controllability conditions of this system for certain classes of admissible control functions are derived.

Further, these controllability conditions are reduced to the corresponding system without impulses and with delays; with impulses and without delays; and without impulses and without delays. In each of such case, the obtained controllability results coincides with the results available in the existing works of the literature. Numerical examples are given to show the effectiveness of obtained results.

As we know that, the control function in the matrix Lyapunov systems may possesses variable time delays. In such scenario, the ideas of this work can be extended to study the controllability of impulsive matrix Lyapunov systems with variable time delays in control function. Further, it should be mentioned that, for the constrained controllability (e.g, in a given closed convex cone with vertex at zero) of impulsive matrix Lyapunov differential systems in finite or infinite-dimensional space, one has to consider the different methods discussed in [5, 21] etc, where the authors studied the controllability of infinite-dimensional nonlinear systems with constrained controls in a given closed convex cone with vertex at zero.

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