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SOME GLOBALLY DETERMINED CLASSES OF GRAPHS

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Abstract. For a class of graphs we say that it is globally determined if any two nonisomorphic graphs from that class have nonisomorphic globals. We will prove that the class of so called CCB graphs and the class of finite forests are globally determined.

Keywords: globals of graphs; global determination; isomorphism

MSC 2010: 05C76, 05C60, 05C25

1. Introduction and preliminaries

In general case, if $\mathcal{A}$ is some (operational-relational) structure with a carrier set A, the global of $\mathcal{A}$ is the structure induced in a natural way on the powerset of A. For specific types of structures several other names have been used in the literature instead of global, such as power structure, complex algebra, power algebra. For a general overview on globals see [3], [4]. In the present paper we consider the question of global determinism of certain classes of graphs. We say that the class K of structures is globally determined if every time when two structures from the class have isomorphic globals, these two structures are isomorphic too. This problem is extensively studied for semigroups, see [8], [10], [16], [18]. Besides semigroups, some other classes of algebraic structures were studied in this context, which includes unary algebras, see [6], [9], [14]. In [6] Drápal showed that the class of finite partial monounary algebras is globally determined. Since monounary algebras can be viewed as graphs, this result can be interpreted as one of the first results on global determination of classes of graphs, which is the topic of our paper.

Different definitions of powering of relations can be found in the literature, especially in theoretical computer science, but the most general definition of the power

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of an n-ary relation can be found in [19]. Globals of graphs are investigated in [2], while in [11], [12] it is demonstrated that power graphs could be useful in the theory of concurrent systems.

In the present paper we will prove that the classes of finite forests and CCB graphs are globally determined. For a finite (undirected) graph G we say that it is a *CCB graph* if every component of G is a complete graph with loops or a complete bipartite graph. This class of graphs plays an important role in problems concerning the finiteness of equational bases of universal algebras. Namely, Shallon [17] in her PhD thesis (see also [15]) proposed a method for constructing algebras from graphs, which in many cases gives examples of nonfinitely based finite algebras. It turned out that many later discovered nonfinitely based finite algebras can be obtained as graph algebras of some special graphs. The class of CCB graphs is precisely the class of finite graphs whose corresponding graph algebras have finite equational bases.

In this paper we consider finite undirected graphs possibly with loops. In other words, a graph is a structure $G = (V, E)$, where V is a finite nonempty set and E is a symmetric binary relation on V. The complete graph with n vertices, all of them having a loop, will be denoted by K_n^s. The complete bipartite graph with partition classes having p and q vertices respectively, will be denoted by $K_{p,q}$. By $G_1 + G_2$ we will denote the disjoint union of graphs G_1 and G_2. We call graphs G_1 and G_2 isomorphic, and write $G_1 \simeq G_2$, if they are isomorphic as relational structures. For basic notions and terminology of graph theory we refer the reader to [5].

The *global* of $G = (V, E)$, denoted by $\mathcal{P}(G)$, is the graph with the set of vertices $\mathcal{P}(V)$ (the powerset of V), whose edges are determined as: for all $X, Y \in \mathcal{P}(G)$, $(X, Y) \in E^+ \Leftrightarrow (\forall x \in X)(\exists y \in Y)(x, y) \in E$ and $(\forall y \in Y)(\exists x \in X)(x, y) \in E$. Note that $(X, \emptyset) \in E^+$ if and only if $X = \emptyset$. Therefore at least one of the components of $\mathcal{P}(G)$ is equal to K_1^s. Some authors exclude the empty set from the vertex set of the global of a graph G. We will call the graph obtained in this way the *positive global* of G, and denote it by $\mathcal{P}^+(G)$.

For a class K of graphs we say that it is *globally determined* if for all graphs G_1 and G_2 from K, $\mathcal{P}(G_1) \simeq \mathcal{P}(G_2)$ implies $G_1 \simeq G_2$.

The structure of this paper is the following: In Section 2 we describe globals of CCB graphs. In Section 3 we prove that the class of CCB graphs is globally determined. An algorithm for reconstructing CCB graphs from its globals is described in Section 4. Finally, in the last section we prove that the class of finite forests is globally determined.

2. Globals of CCB graphs

Definition 1. A finite undirected graph G is a *CCB graph* if all its connected components are complete graphs with a loop at every vertex, or complete bipartite graphs. We say that the dimension of G is (n, m), and write $\dim(G) = (n, m)$, if G has n complete components and m complete bipartite components.

Definition 2. Let $G = (V, E)$ be a CCB graph with complete components $A_1, A_2, \ldots, A_n$ and bipartite components $(B_1, C_1), (B_2, C_2), \ldots, (B_m, C_m)$ such that $|B_i| \leqslant |C_i|$.

(1) We define the *type* of an arbitrary nonempty subset H of V in the following way: $\text{type}(H) = (\alpha, \delta)$, where
$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i \in \{0, 1\}$, $\delta = ((\beta_1, \gamma_1), (\beta_2, \gamma_2), \ldots, (\beta_m, \gamma_m))$, $\beta_i, \gamma_i \in \{0, 1\}$ so that
$\alpha_i = 1$ if and only if $A_i \cap H \neq \emptyset$,
$\beta_i = 1$ if and only if $B_i \cap H \neq \emptyset$,
$\gamma_i = 1$ if and only if $C_i \cap H \neq \emptyset$.

(2) If each component of G is bipartite, then $\alpha = \emptyset$, and if each component of G is complete, then $\delta = \emptyset$.

(3) By $\text{Type}(G)$ we will denote the set of all types of subsets of V.

(4) For a type $\tau = (\alpha, \delta) \in \text{Type}(G)$, its *dual type* τ^{-1} is defined in the following way: if $\delta = ((\beta_1, \gamma_1), (\beta_2, \gamma_2), \ldots, (\beta_m, \gamma_m))$, then $\tau^{-1} = (\alpha, \delta^{-1})$, where $\delta^{-1} = ((\gamma_1, \beta_1), (\gamma_2, \beta_2), \ldots, (\gamma_m, \beta_m))$. If $\delta = \emptyset$, then $\tau^{-1} = \tau$. A type $\tau = (\alpha, \delta)$ is *even* if $\tau = \tau^{-1}$. Otherwise, it is *odd*.

The following well known theorem will help us to describe globals of CCB graphs.

Theorem 1 ([1])**.** *A graph G is a CCB graph if and only if it does not contain any of the following graphs as an induced subgraph: M, K_3, L_3, P_4.*

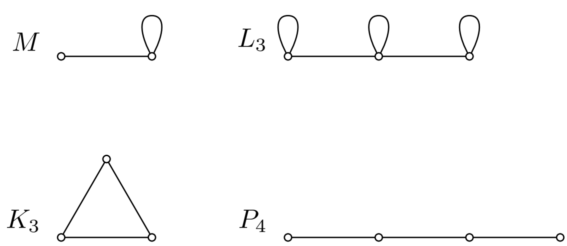

Figure 1.

Proposition 1. *If the global of a graph G is a CCB graph, then G is a CCB graph.*

P r o o f. Suppose G is not a CCB graph. Then it contains one of four forbidden subgraphs from Theorem 1. Since $\mathcal{P}(G)$ contains G as an induced subgraph, it must also contain one of the forbidden subgraphs and can not be a CCB graph. $\square$

For CCB graphs with exactly one component it is easy to see that $\mathcal{P}(K_r^s) = K_1^s + K_{2^r-1}^s$ and $\mathcal{P}(K_{s,t}) = K_1^s + K_{(2^s-1)\cdot(2^t-1)}^s + K_{2^s-1,2^t-1}$. In general, the following statement holds:

Proposition 2. *Let G be a CCB graph of dimension (n, m).*

(1) *If $X, Y \in \mathcal{P}(V)$, then $(X, Y) \in E^+$ if and only if* $\operatorname{type}(X) = (\operatorname{type}(Y))^{-1}$.

(2) *$\mathcal{P}(G)$ is a CCB graph and*

$$\dim(\mathcal{P}(G)) = \Big(2^{n+m}, \frac{2^{n+2m} - 2^{n+m}}{2}\Big).$$

P r o o f. Part (1) follows directly from the definition of relation E^+. Let $G = (V, E)$, $X, Y \in \mathcal{P}(V)$. According to (1), if X is of even type, it will (in the power graph) be adjacent precisely to the sets of the same type. Thus, $\{Y \in \mathcal{P}(V)\colon \operatorname{type}(X) = \operatorname{type}(Y)\}$ is a complete component of $\mathcal{P}(G)$. Consequently, the number of complete components of $\mathcal{P}(G)$ is equal to the number of even types, which is precisely 2^{n+m}.

Let X be of odd type and $(X, Y) \in E^+$. Then X and Y have different dual types. This means that every pair of different dual types determines a complete bipartite component of $\mathcal{P}(G)$ (which consists of all subsets of V that belong to one of those types). Consequently, the number of complete bipartite components of $\mathcal{P}(G)$ is equal to a half of the number of odd types, which is precisely $\frac{1}{2}(2^{n+2m} - 2^{n+m})$. $\square$

From now, instead of saying that a graph is isomorphic to a global of another graph, or that a graph is (up to isomorphism) a global of another graph, we will shortly say that a graph is a global of another graph.

Of course, not every CCB graph G is a global of another CCB graph.

Proposition 3.

(1) *There exists a CCB graph of dimension (a, b) which is a global of some CCB graph if and only if there exist positive integers k and l such that $k \leqslant l \leqslant 2k$ and $a = 2^k$, $a + 2b = 2^l$.*

(2) *If G is a CCB graph such that $G \simeq \mathcal{P}(G_0)$ for some graph G_0, then the dimension of G_0 is uniquely determined by the dimension of G.*

P r o o f. (1) Let G be a CCB graph of dimension (a, b) which is a global of some graph G_0. Then G_0 is a CCB graph with dimension, say, (n, m). From Proposition 2 we obtain $a = 2^{n+m}$ and $a + 2b = 2^{n+2m}$. Therefore $k = n + m$, $l = n + 2m$ and obviously $k \leqslant l \leqslant 2k$.

Suppose now that $a = 2^k$, $a+2b = 2^l$, and $k \leqslant l \leqslant 2k$. Let $n = 2k-l$ and $m = l-k$. Then n and m are nonnegative numbers such that $k = n+m$ and $l = n+2m$, which gives $a = 2^{n+m}$ and $b = \frac{1}{2}(2^{n+2m} - 2^{n+m})$. So, according to Proposition 2, if G_0 is any CCB graph of dimension (n, m), then $\dim(\mathcal{P}(G_0)) = (a, b)$.

(2) Let G be of dimension (a, b) and G_0 be of dimension (n, m). As we have shown in the proof of (1), there exist (uniquely determined) positive integers k and l satisfying $k \leqslant l \leqslant 2k$, such that $k = n + m$ and $l = n + 2m$. This implies $n = 2k - l$ and $m = l - k$. □

Let a CCB graph G of dimension (a, b) be given. Suppose that G is a global of some CCB graph G_0. The proof of the previous theorem gives a simple algorithm for determining the dimension of G_0.

Example 1. Let $\dim(G) = (8, 12)$. Then $a = 2^3$ and $a + 2b = 2^5$. This gives $k = 3$, $l = 5$, and $n = 6 - 5 = 1$, $m = 5 - 3 = 2$. So if $G = \mathcal{P}(G_0)$, then $\dim(G_0) = (1, 2)$.

Naturally, if G is a CCB graph of dimension (a, b) such that a and b fulfill the conditions from Proposition 3, this still does not guarantee that G is a global of some graph. Another obvious necessary condition is that the number of vertices of G is 2^t for some positive integer t. Even this will usually not be sufficient.

Example 2. Let G consist of 6 copies of K_1^s, 2 copies of K_3^s, 7 copies of $K_{1,1}$, 4 copies of $K_{1,7}$ and 1 copy of $K_{3,3}$. Necessary conditions for a and b from Proposition 3 are satisfied, number of vertices is 2^6, but G can not be the global of any CCB graph. The reason for that is the structure of trivial components of G, as we will see in Proposition 4.

A component of a CCB graph will be called *trivial* if it is isomorphic to K_1^s or $K_{1,1}$. Knowing the number of trivial components of $\mathcal{P}(G_0)$, we can determine the number of trivial components in G_0.

Proposition 4. *Let G be a CCB graph with j trivial complete components and k trivial bipartite components. Then $\mathcal{P}(G)$ has 2^{j+k} trivial complete components and $2^{j+k-1}(2^k - 1)$ trivial bipartite components.*

P r o o f. Trivial components of $\mathcal{P}(G)$ will be obtained from those types (α, δ) for which $\alpha_i = 0$ for all nontrivial components A_i of G and $(\beta_i, \gamma_i) = (0, 0)$ for all nontrivial components (B_i, C_i). There are exactly 2^{j+2k} such types, and 2^{j+k} of them are even. Therefore the number of complete trivial components of $\mathcal{P}(G)$ is 2^{j+k}, and the number of bipartite trivial components of $\mathcal{P}(G)$ is $\frac{1}{2}(2^{j+2k} - 2^{j+k}) = 2^{j+k-1}(2^k - 1)$ (a half of the number of odd types of this kind). □

It is now clear that when the global of a CCB graph G is given, we can easily reconstruct the number of copies of K_1^s and $K_{1,1}$ among the components of G. Also, we can now verify that the graph from Example 2 is not a global of any CCB graph. It is sufficient to notice that it has 6 copies of K_1^s, and 6 is different from 2^{j+k} for any positive integers j, k.

3. CCB graphs are globally determined

Lemma 1. *Let G be a CCB graph. If G is the global of some CCB graph G_0, then we can determine at least one component of G_0.*

Proof. Let G_0 have complete components $A_1, A_2, \ldots, A_n$ and bipartite components $(B_1, C_1), (B_2, C_2), \ldots, (B_m, C_m)$. Note that according to Proposition 3 (2) we can determine n and m, but the cardinalities of components are unknown. Put $|A_i| = a_i$, $|B_i| = b_i$, $|C_i| = c_i$. Let G have at least one complete bipartite component. Pick a complete bipartite component (M, N) of G with $|M| + |N|$ minimal. According to Proposition 2 (1), if $X \in M$ and $Y \in N$, then $\text{type}(X) = (\text{type}(Y))^{-1}$. Let $\text{type}(X) = (\alpha^M, \delta^M)$, $\text{type}(Y) = (\alpha^N, \delta^N)$, $I = \{i\colon\ \alpha_i^M = 1\}$, $J = \{j\colon\ \beta_j^M = 1\}$, $K = \{k\colon\ \gamma_k^M = 1\}$.

Now the exact number of elements in M is

$$|M| = \prod_{i\in I}(2^{a_i} - 1) \cdot \prod_{j\in J}(2^{b_j} - 1) \cdot \prod_{k\in K}(2^{c_k} - 1).$$

Of course, $|N|$ can be calculated in a similar way. Since the types of X and Y are mutually dual, we know that $\alpha^M = \alpha^N$ and there exists j such that $(\beta_j^M, \gamma_j^M) = (1, 0)$ and $(\beta_j^N, \gamma_j^N) = (0, 1)$. Consider the type $\tau_0 = (\alpha, \delta)$ such that $(\beta_j, \gamma_j) = (1, 0)$, $(\beta_i, \gamma_i) = (0, 0)$ for $i \neq j$ and $\alpha_i = 0$ for all $i \in \{1, \ldots, n\}$. This type and its dual type determine a bipartite component in G isomorphic to $K_{2^{b_j}-1, 2^{c_j}-1}$. Since $2^{b_j} - 1$ divides $|M|$ and $2^{c_j} - 1$ divides $|N|$, the minimality of (M, N) implies $|M| = 2^{b_j} - 1$, $|N| = 2^{c_j} - 1$. This way we have determined the cardinality of a bipartite component of G_0, i.e. $|B_j| = b_j$, $|C_j| = c_j$.

Suppose now that all components of G are complete (which means that all components of G_0 are complete, too). We know that one trivial complete component of G corresponds to the empty subset of G_0. Let G' be the graph obtained from G

by removing a trivial complete component. Pick a component K of G' with minimal cardinality. We know that all X from K have the same even type, say $(\alpha^K, \emptyset)$. Since $|K| = \prod_{\{i\colon \alpha_i^K=1\}} (2^{a_i} - 1)$ and there exists j such that $\alpha_j = 1$, we conclude that $2^{a_j} - 1$ divides $|K|$ and $a_j \geqslant 1$. Let K_0 be a complete component of G determined by the type $(\alpha, \emptyset)$ such that $\alpha_j = 1$ and $\alpha_t = 0$ for $t \neq j$. Then $|K_0| = 2^{a_j} - 1$, and the minimality of K implies $|K| = 2^{a_j} - 1$. In this way we have determined the cardinality of a complete component of G_0: $|A_j| = a_j$. □

To complete our proof of global determination of the class of CCB graphs, we need two additional statements.

Lemma 2 ([7]). *Let $\Sigma_{i\in I}\mathcal{A}_i$ be the disjoint union of a family $\langle \mathcal{A}_i\colon i \in I\rangle$ of relational structures. Then $\mathcal{P}\Big(\sum_{i\in I} \mathcal{A}_i\Big) \simeq \prod_{i\in I} \mathcal{P}(\mathcal{A}_i)$.*

Lemma 3 ([13]). *Let G_1, G_2 and H be graphs. If $G_1 \times H \simeq G_2 \times H$ and H has a loop, then $G_1 \simeq G_2$.*

Theorem 2. *The class of CCB graphs is globally determined.*

P r o o f. Let G_1 and G_2 be CCB graphs such that $\mathcal{P}(G_1) \simeq \mathcal{P}(G_2)$. We are going to prove that G_1 is isomorphic to G_2. Notice that G_1 and G_2 have the same dimension and consequently the same number of components. Therefore the proof will be done by induction on the number of components of G_1 (G_2). Let G_1 have exactly one component. If $G_1 = K_r$, then $\mathcal{P}(G_1) = K_1^s + K_{2^r-1}^s$, which means that $\mathcal{P}(G_1)$ uniquely determines G_1. If $G_1 = K_{s,t}$, then $\mathcal{P}(G_1) = K_1^s + K^s_{(2^s-1)\cdot(2^t-1)} + K_{2^s-1,2^t-1}$ and s, t are uniquely determined by $\mathcal{P}(G_1)$.

Let G_1 have $k > 1$ components. Suppose that the statement holds for all graphs with less than k components. According to Lemma 1, we can determine a component H of both G_1 and G_2. Then $G_1 = H + G_1'$ and $G_2 = H + G_2'$ for some graphs G_1' and G_2'. Using Lemma 2 we obtain $\mathcal{P}(H) \times \mathcal{P}(G_1') \simeq \mathcal{P}(G_1) \simeq \mathcal{P}(G_2) \simeq \mathcal{P}(H) \times \mathcal{P}(G_2')$. Since $\mathcal{P}(H)$ always has a loop, we can apply Lemma 3 to obtain $P(G_1') \simeq P(G_2')$. By the induction hypothesis this gives $G_1' \simeq G_2'$, and finally $G_1 \simeq G_2$. □

4. Reconstructing a CCB graph from its global

In this section we are going to present an algorithm for reconstructing a CCB graph from its power graph. The algorithm is based on the following simple properties of graphs:

Lemma 4. *Let G, H_1 and H_2 be graphs. Then*

$$G \times (H_1 + H_2) = G \times H_1 + G \times H_2.$$

Lemma 5.

$$K_n^s \times K_m^s = K_{n\cdot m}^s,$$
$$K_n^s \times K_{s,t} = K_{n\cdot s, n\cdot t},$$
$$K_{p,q} \times K_{s,t} = K_{p\cdot s, q\cdot t} + K_{p\cdot t, q\cdot s}.$$

Let a graph G' be given and $G' = \mathcal{P}(G_0)$. The algorithm for determining G_0 is inductive. It consists of two subroutines (A) and (B) used repeatedly until all components of G_0 are identified. Suppose that a graph $\mathcal{P}(G_i)$ has been obtained at some stage of the algorithm. Then the inductive step is the following:

(A) Determine a component H_{i+1} of G_i (which is also a component of the graph G_0).

(B) Determine the graph $\mathcal{P}(G_{i+1})$, where G_{i+1} is obtained by removing the component H_{i+1} from G_i.

According to Lemma 2, in part (A) we first determine complete bipartite components of G_i, and then complete components. So, the algorithm has two phases: phase 1, when there exist complete bipartite components of G_i, and phase 2, when G_i consists of complete components only.

Phase 1 (G_i has complete bipartite components)

(A) Determine a component $H_{i+1} = K_{a,b}$ of G_i.

(B) Determine $\mathcal{P}(G_{i+1}) = \mathcal{P}(G_i - H_{i+1})$ in the following way: Put $r = 2^a - 1$, $t = 2^b - 1$. Using Lemma 2 and Lemma 4 we obtain

$$\begin{aligned}\mathcal{P}(G_i) &= \mathcal{P}(G_{i+1} + K_{a,b}) = \mathcal{P}(G_{i+1}) \times \mathcal{P}(K_{a,b})\\ &= \mathcal{P}(G_{i+1}) \times (K_1^s + K_{r\cdot t}^s + K_{r,t})\\ &= \mathcal{P}(G_{i+1}) + \mathcal{P}(G_{i+1}) \times K_{r\cdot t}^s + \mathcal{P}(G_{i+1}) \times K_{r,t}.\end{aligned}$$

This means that $\mathcal{P}(G_i)$ contains all components of $\mathcal{P}(G_{i+1})$. Therefore it is necessary to decide what components of $\mathcal{P}(G_i)$ belong to $\mathcal{P}(G_{i+1})$, and remove those that do not. Distributivity of the direct product implies that for every component K of $\mathcal{P}(G_{i+1})$, two associated graphs, $K \times K_{r\cdot t}^s$ and $K \times K_{r,t}$, are also in $\mathcal{P}(G_i)$. Let us start with complete components. Take a minimal complete component $K = K_v^s$ of $\mathcal{P}(G_i)$. This component (or some of its isomorphic copies) obviously belongs to $\mathcal{P}(G_{i+1})$, so it needs to be moved to the list of components of $\mathcal{P}(G_{i+1})$. Then we remove one copy of $K_v^s \times K_{r\cdot t}^s = K_{v\cdot r\cdot t}^s$ and one copy of $K_v^s \times K_{r,t} = K_{v\cdot r, v\cdot t}$ from $\mathcal{P}(G_i)$

and repeat the described procedure for another minimal complete component in the new graph. In the end of this process, the list of all complete components of $\mathcal{P}(G_{i+1})$ is obtained. In that moment, all remaining components of the graph $\mathcal{P}(G_i)$ are complete bipartite and it is necessary to distinguish those which belong to $\mathcal{P}(G_{i+1})$. In order to do that, pick a minimal bipartite component $K = K_{p,q}$, move it to the list of components of $\mathcal{P}(G_{i+1})$, and remove one copy of $K_{p,q} \times K^s_{r\cdot t} = K_{p\cdot r\cdot t, q\cdot r\cdot t}$ and one copy of $K_{p,q} \times K_{r,t} = K_{p\cdot r, q\cdot t} + K_{p\cdot tr, q\cdot r}$ from what remained of $\mathcal{P}(G_i)$. Repeating the procedure as long as it is necessary, we eventually obtain $\mathcal{P}(G_{i+1})$.

Phase 2 (G_i is a disjoint union of complete components)

(A) Determine a component $H = K^s_a$ of G_i.

(B) Determine $\mathcal{P}(G_{i+1}) = \mathcal{P}(G_i - H_{i+1})$ in the following way: Put $r = 2^a - 1$. Using Lemma 2 and Lemma 4 we obtain

$$\begin{aligned}\mathcal{P}(G_i) &= \mathcal{P}(G_{i+1} + K^s_a) = \mathcal{P}(G_{i+1}) \times \mathcal{P}(K^s_a)\\ &= \mathcal{P}(G_{i+1}) \times (K^s_1 + K^s_r) = \mathcal{P}(G_{i+1}) + \mathcal{P}(G_{i+1}) \times K^s_r.\end{aligned}$$

Take a minimal complete component $K = K^s_v$ of $\mathcal{P}(G_i)$ and move it to the list of components of $\mathcal{P}(G_{i+1})$. Then remove one copy of $K^s_v \times K^s_r = K^s_{v\cdot r}$ from $\mathcal{P}(G_i)$ and repeat the described procedure for another minimal component in the new graph. In the end of this process, we obtain the list of all components of $\mathcal{P}(G_{i+1})$.

The pseudo code of the algorithm described above is given bellow.

function DEGLOBALIZE($\mathcal{P}(G)$)
  $U \leftarrow \mathcal{P}(G)$, $X \leftarrow \emptyset$, $G \leftarrow \emptyset$
  while there is a bipartite component in U **do**
    choose a minimal bipartite component $K_{r,t}$ from U
    $a \leftarrow \log_2(r+1)$, $b \leftarrow \log_2(t+1)$
    $G \leftarrow G + K_{a,b}$
    while there is a complete component in U **do**
      choose a minimal complete component K^s_v from U
      $X \leftarrow X + K^s_v$
      remove K^s_{rtv}, $K_{rv,tv}$, K^s_v from U
    end while
    while there is a bipartite component in U **do**
      choose a minimal bipartite component $K_{p,q}$ from U
      $X \leftarrow X + K_{p,q}$
      remove $K_{rtp,rtq}$, $K_{rp,tq}$, $K_{rq,tp}$, $K_{p,q}$ from U
    end while
    $U \leftarrow X$, $X \leftarrow \emptyset$
  end while

while there is a complete component in $U - K_1^s$ **do**
choose a minimal complete component K_r^s from $U - K_1^s$
$a \leftarrow \log_2(r+1)$
$G \leftarrow G + K_a^s$
while there is a complete component in U **do**
choose a minimal complete component K_v^s of U
$X \leftarrow X + K_v^s$
remove K_{rv}^s, K_v^s from U
end while
$U \leftarrow X$, $X \leftarrow \emptyset$
end while
return G
end function

5. Global determination of finite forests

A *tree* is a connected graph without cycles. A disjoint union of trees is called a *forest*. In this section we prove that the class of finite forests is globally determined.

Let G be a graph. By $N_G(u)$, or simply $N(u)$, we denote the set of all neighbours of a vertex u of the graph G. If U is a subset of the vertex set of G, by $N[U]$ we will denote the set of all neighbours of vertices from U (while $N(U)$ denotes the set of all neighbours of vertex U in $\mathcal{P}(G)$). The degree of vertex v is the number $|N(v)|$ (the number of neighbours of vertex v). We will denote it by $d(v)$.

It is well known that every tree has at least two leaves (vertices of degree 1). By $T(G)$ we will denote the set of all vertices of graph G that are adjacent to at least one leaf of G.

Lemma 6. *Let $G = (V, E)$ be an undirected graph without isolated vertices, X a leaf in the graph $\mathcal{P}(G)$, and Y a neighbour of X. Then $N[X] = Y$ and every vertex from Y is adjacent to some leaf from X.*

Proof. Since X does not contain any isolated vertex, $N[X]$ is a neighbour of X in the global of G. This gives $N[X] = Y$. Suppose $y \in Y$ and y does not have neighbours among the leaves from X. Then X would be adjacent to $Y \setminus \{y\}$, which is clearly a contradiction. □

Lemma 7. *Let $G = (V, E)$ be an undirected graph, $Y \in T(\mathcal{P}(G))$, and $y \in Y$. Then $\{y\} \in T(\mathcal{P}(G))$.*

P r o o f. According to Lemma 6 there exists a leaf x of graph G which is adjacent to y. This means that $\{y\}$ is a neighbour of $\{x\}$, which is a leaf in $\mathcal{P}(G)$. □

Lemma 8. *Let $G = (V, E)$ be a finite undirected connected graph and $Y = \{y_1, \ldots, y_r\} \in T(\mathcal{P}(G))$, $r \geqslant 2$. Then*

$$d(Y) > \max_{y_i \in Y} d(\{y_i\}).$$

P r o o f. Let y_s be an arbitrary vertex from Y and $d(y_s) = p$. Then $d(\{y_s\}) = 2^p - 1$. Let X be the set of leaves of G which are neighbours of vertices from $Y \setminus \{y_s\}$ and $Z = N(y_s)$. For every nonempty subset Z' of Z, $X \cup Z'$ is a neighbour of Y. This gives $d(Y) \geqslant 2^p - 1$. Let $x_i \in X$ be a leaf adjacent to y_i, $i \neq s$. According to Lemma 6, there are no leaves in Y. Consequently, Y is a neighbour of $(X \cup N(y_i)) \setminus \{x_i\}$, which means that at least one neighbour of Y does not contain all vertices from X. This implies $d(Y) \geqslant 2^p > d(\{y_s\})$. □

Let G be a tree. In the further text we want to describe $\mathcal{P}(G)$. It is clear that the global of every graph has one trivial component, which corresponds to $\emptyset$. The remaining components will be referred to as nontrivial. It is well known that every tree is a bipartite graph.

Lemma 9. *Let X and Y be some nonempty sets of vertices of a graph G such that for all $x \in X$ there exists a walk of odd length from x to some $y \in Y$, and for all $y \in Y$ there exists a walk of odd length from some $x \in X$ to y. Then X and Y are in the same connected component of $\mathcal{P}(G)$.*

P r o o f. Let $X = \{x_1, \ldots, x_k\}$, $Y = \{y_1, \ldots, y_l\}$. Every vertex $x_i \in X$ is a starting point of a walk W_i of odd length ending in Y. Also, for every $y_j \in Y$ there is a walk W_{k+j} of odd length from some $x \in X$ to y_j. Every walk W_s, $s \in \{1, \ldots, k + l\}$, could be extended to a walk W'_s, whose length is equal to the maximal length of walks $W_1, \ldots, W_{j+k}$ (by traversing the last edge backward and forward as many times as necessary). Thus, we obtain $k + l$ walks of the same length d:

$$W'_1\colon\ x_1 = z_{01}, z_{11}, \ldots, z_{d1};$$

$$\vdots$$

$$W'_k\colon\ x_k = z_{0k}, z_{1k}, \ldots, z_{dk};$$

$$W'_{k+1}\colon\ z_{0k+1}, z_{1k+1}, \ldots, z_{dk+1} = y_1;$$

$$\vdots$$

$$W'_k + l\colon\ z_{0k+l}, z_{1k+l}, \ldots, z_{dk+l} = y_l.$$

Put $Z_m = \{z_{m1}, \ldots, z_{mk+l}\}$, $m = 0, 1, \ldots d$. Then $X = Z_0, Z_1, \ldots, Z_d = Y$ is a walk from X to Y in $\mathcal{P}(G)$. □

In [2] it is proved that the positive global of a connected graph G is connected if and only if G is not bipartite. Here we will describe globals of bipartite graphs (and trees in particular) in more details.

Proposition 5. *Let G be a connected bipartite graph. Then $\mathcal{P}(G)$ has two nontrivial connected components and one of them is bipartite.*

P r o o f. Let us denote partition classes of G by A and B. By A' and B' we will denote the set of all nonempty subsets of A and B, respectively. By C' we will denote the set of all subsets of $A \cup B$ having nonempty intersection with both A and B. It is clear that in $\mathcal{P}(G)$ a vertex from $A' \cup B'$ and a vertex from C' can not be neighbours. The subgraph of $\mathcal{P}(G)$ induced by $A' \cup B'$ is bipartite, with A' and B' as biparite classes, and it is connected, according to Lemma 9. The subgraph induced by C' is connected too, by the same reason. □

Lemma 10. *Let $G = (V, E)$ be a finite tree and $u \in V$. If $X \in N(\{u\})$ and $d(X) = 2^k - 1$ for $k \geqslant 2$, then X is a singleton. If $d(X) = 1$, then there is at least one singleton among the leaves of $\mathcal{P}(G)$ which are neighbours of $\{u\}$.*

P r o o f. Suppose X is not a singleton. Let $X = \{x_1, \ldots, x_r\}$, $r \geqslant 2$, and $d(x_i) = k_i + 1$, for $i = 1, \ldots, r$. Since G is a tree, every two different vertices x_i and x_j from X have at most one common neighbour, and that must be u. If $Y \in N(X)$ and $u \in Y$, then $Y \setminus \{u\}$ can be any subset of $\bigcup\limits_{i \in 1,\ldots,r} \{N(x_i) \setminus \{u\} \colon x_i \in X\}$, and there are exactly $2^{k_1+\ldots+k_r}$ such subsets. If $Y \in N(X)$ and $u \notin Y$, then $Y = U_1 \cup \ldots \cup U_r$, where U_i is a nonempty subset of $N(x_i) \setminus \{u\}$. There are $\prod\limits_{i \in 1,\ldots,r} (2^{k_i} - 1)$ such subsets, which gives

$$d(X) = 2^{k_1+\ldots+k_r} + (2^{k_1} - 1)(2^{k_2} - 1) \ldots (2^{k_r} - 1).$$

For $d(X) > 1$ at least one of the numbers k_i is different from 0. It is now a routine exercise to show that $2^{k_1+\ldots+k_r} - 1 < d(X) < 2^{k_1+\ldots+k_r+1} - 1$, which means that $d(X)$ can not be equal to $2^k - 1$ for a positive integer k. If $d(X) = 1$, then the leaves $\{x_1\}, \ldots, \{x_r\}$ are neighbours of $\{u\}$. □

Theorem 3. *The class of finite trees is globally determined.*

P r o o f. Let $G = (V, E)$ be a finite tree. According to Proposition 5 and its proof, the global of G has two nontrivial components. One of them is bipartite and it contains all singletons. Let $m = \min\limits_{Y \in T(\mathcal{P}(G))} d(Y)$ and $M = \{Y \in T(\mathcal{P}(G)) \colon d(Y) = m\}$

(note that $T(\mathcal{P}(G))$ can contain vertices from both nontrivial components, but M must be contained in the bipartite component). Suppose that $X \in M$ and X is not a singleton. If $x \in X$, according to Lemma 8 we get $d(\{x\}) < d(X) = m$. However, this is impossible, since $\{x\} \in T(\mathcal{P}(G))$ according to Lemma 7. Thus, we conclude that all vertices from M are singletons of degree greater than one. The subgraph U of $\mathcal{P}(G)$ induced by all singletons which are not leaves is connected. Therefore, according to Lemma 10, a vertex Y of a graph $\mathcal{P}(G)$ belongs to U if and only if $d(Y) = 2^k - 1$ for $k \geqslant 2$, and there exists a path from Y to some vertex from M consisting of vertices of degree $2^m - 1$ for some $m \geqslant 2$. This means that it is possible to reconstruct U from $\mathcal{P}(G)$.

A vertex $\{u\}$ from U is adjacent to a leaf Y from $\mathcal{P}(G)$ if and only if Y is a set of some leaves of G which are adjacent to u. So, if $\{u\}$ has $2^k - 1$ leaves of $\mathcal{P}(G)$ among its neighbours, then exactly k of its neighbours are singletons. Therefore, G is uniquely determined (up to an isomorphism). □

Theorem 4. *The class of finite forests is globally determined.*

P r o o f. Let G be a finite forest with connected components $G_1, \ldots, G_k$ (which are, of course, trees). Pick one of the bipartite components of $\mathcal{P}(G)$ with minimal number of vertices. Similarly as in the proof of Lemma 1, we can conclude that the chosen component is isomorphic to the bipartite component of $\mathcal{P}(G_i)$ for some $i \in \{1, \ldots, k\}$. According to Theorem 3, we can reconstruct the component G_i. Using Lemmas 2 and 3, in a similar way as in the proof of Theorem 2, we can now prove that the class of all finite forests is globally determined. □

6. Concluding remarks

In this paper we proved that two classes of finite graphs are globally determined. The same ideas are present in both proofs, including cancelation properties of finite relational structures, discovered by László Lovász. To apply this method to prove that some class of finite graphs is globally determined, two conditions are necessary. The first one is that the subclass of connected graphs from the given class is globally determined. The second one is that, given the global of a graph from the class, we are able to reconstruct a component of that graph.

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