Zhanmin Zhu

Coherence relative to a weak torsion class

Czechoslovak Mathematical Journal, Vol. 68 (2018), No. 2, 455–474

Persistent URL: `http://dml.cz/dmlcz/147229`

Czechoslovak Mathematical Journal, 68 (143) (2018), 455–474

COHERENCE RELATIVE TO A WEAK TORSION CLASS

Zhanmin Zhu, Jiaxing

Received September 22, 2016. First published February 10, 2018.

Abstract. Let R be a ring. A subclass $\mathcal{T}$ of left R-modules is called a weak torsion class if it is closed under homomorphic images and extensions. Let $\mathcal{T}$ be a weak torsion class of left R-modules and n a positive integer. Then a left R-module M is called $\mathcal{T}$-finitely generated if there exists a finitely generated submodule N such that $M/N \in \mathcal{T}$; a left R-module A is called $(\mathcal{T}, n)$-presented if there exists an exact sequence of left R-modules

$$0 \longrightarrow K_{n-1} \longrightarrow F_{n-1} \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

such that $F_0, \ldots, F_{n-1}$ are finitely generated free and K_{n-1} is $\mathcal{T}$-finitely generated; a left R-module M is called $(\mathcal{T}, n)$-injective, if $\mathrm{Ext}_R^n(A, M) = 0$ for each $(\mathcal{T}, n+1)$-presented left R-module A; a right R-module M is called $(\mathcal{T}, n)$-flat, if $\mathrm{Tor}_n^R(M, A) = 0$ for each $(\mathcal{T}, n+1)$-presented left R-module A. A ring R is called $(\mathcal{T}, n)$-coherent, if every $(\mathcal{T}, n+1)$-presented module is $(n+1)$-presented. Some characterizations and properties of these modules and rings are given.

Keywords: $(\mathcal{T}, n)$-presented module; $(\mathcal{T}, n)$-injective module; $(\mathcal{T}, n)$-flat module; $(\mathcal{T}, n)$-coherent ring

MSC 2010: 16D40, 16D50, 16P70

1. Introduction

Recall that a *torsion theory*, see [14], $\tau = (\mathcal{T}, \mathcal{F})$ for the category of all left R-modules consists of two subclasses $\mathcal{T}$ and $\mathcal{F}$ such that:

(1) $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

(2) If $\mathrm{Hom}(T, F) = 0$ for all $F \in \mathcal{F}$, then $T \in \mathcal{T}$.

(3) If $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$, then $F \in \mathcal{F}$.

In this case, $\mathcal{T}$ is called a torsion class.

This research was supported by the Natural Science Foundation of Zhejiang Province, China (LY18A010018).

DOI: 10.21136/CMJ.2018.0494-16

A torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ is called hereditary if $\mathcal{T}$ is closed under submodules. By [14], page 139, Proposition 2.1, a class $\mathcal{T}$ of left R-modules is a torsion class for some torsion theory if and only if $\mathcal{T}$ is closed under quotient modules, direct sums and extensions. Inspired by this result, in this paper we will call a nonempty subclass $\mathcal{T}$ of left R-modules a weak torsion class if $\mathcal{T}$ is closed under homomorphic images and extensions.

Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory for the category of all left R-modules. Then according to [8], a left R-module M is called *τ-finitely generated* (or τ-FG for short) if there exists a finitely generated submodule N such that $M/N \in \mathcal{T}$; a left R-module A is called *τ-finitely presented* (or τ-FP for short) if there exists an exact sequence of left R-modules $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$ with F finitely generated free and K τ-finitely generated. In Section 2, we will give the concepts of $\mathcal{T}$-finitely generated modules and $\mathcal{T}$-finitely presented modules by taking $\mathcal{T}$ to be a weak torsion class of left R-modules, which extends the two concepts of Jones's τ-finitely generated modules and τ-finitely presented modules respectively. And then we will establish some properties of $\mathcal{T}$-finitely generated modules and $\mathcal{T}$-finitely presented modules.

Let n be a nonnegative integer. Then according to [4], a left R-module A is called n-presented in case there exists an exact sequence of left R-modules $F_n \longrightarrow F_{n-1} \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ in which every F_i is finitely generated free. Motivated by the concepts of n-presented modules and $\mathcal{T}$-finitely presented modules, in Section 3 we will define and investigate $(\mathcal{T}, n)$-presented modules.

Recall that a left R-module M is called *FP-injective*, see [13], or *absolutely pure*, see [11], if $\mathrm{Ext}^1_R(A, M) = 0$ for any finitely presented left R-module A; a right R-module M is flat if and only if $\mathrm{Tor}^R_1(M, A) = 0$ for any finitely presented left R-module A; a ring R is *left coherent*, see [1], if every finitely generated left ideal of R is finitely presented, or equivalently, if every finitely generated submodule of a projective left R-module is finitely presented. The FP-injective modules, flat modules, coherent rings and their generalizations have been studied extensively by many authors (see, for example, [1], [3], [4], [8], [10], [13], [18], [17]).

In 1994, Costa introduced the concept of *left n-coherent rings* in [4]. According to [4], a ring R is called left n-coherent in case every n-presented left R-module is $(n+1)$-presented. In 1996, Chen and Ding introduced the concepts of *n-FP-injective modules* and *n-flat modules*, see [3]. According to [3], a left R-module M is called n-FP-injective in case $\mathrm{Ext}^n_R(A, M) = 0$ for any n-presented left R-module A, a right R-module M is called n-flat in case $\mathrm{Tor}^R_n(M, A) = 0$ for any n-presented left R-module A. By using the concepts of n-FP-injective and n-flat modules, they characterized n-coherent rings. In 2012, Mao and Ding introduced the concepts of *τ-f-injective modules*, *τ-flat modules* and *τ-coherent rings*, see [10]. According to [10], a left R-module M is called τ-f-injective in case $\mathrm{Ext}^1_R(R/I, M) = 0$ for any τ-finitely presented left ideal I; a right R-module M is called τ-flat in case $\mathrm{Tor}_1^R(M, R/I) = 0$ for any τ-finitely presented left ideal I; a ring R is called τ-coherent in case every τ-finitely presented left ideal is finitely presented. By using the concepts of τ-f-injective and τ-flat modules, they characterized τ-coherent rings.

Motivated by the characterization of n-coherent rings and τ-coherent rings (where τ is a hereditary torsion theory), in Section 5 we extend the concept of n-coherent rings and introduce the concept of $(\mathcal{T}, n)$-coherent rings (where $\mathcal{T}$ is a weak torsion class). To characterize $(\mathcal{T}, n)$-coherent rings, $(\mathcal{T}, n)$-injective modules and $(\mathcal{T}, n)$-flat modules are introduced and studied in Section 4; some elementary properties of $(\mathcal{T}, n)$-injective modules and $(\mathcal{T}, n)$-flat modules are obtained in that section.

In Section 5, a series of characterizations and properties of $(\mathcal{T}, n)$-coherent rings are given. For instance, we prove: (1) A ring R is $(\mathcal{T}, n)$-coherent $\Leftrightarrow$ any direct product of $(\mathcal{T}, n)$-flat right R-modules is $(\mathcal{T}, n)$-flat $\Leftrightarrow$ any direct limit of $(\mathcal{T}, n)$-injective left R-modules is $(\mathcal{T}, n)$-injective $\Leftrightarrow$ every right R-module has a $(\mathcal{T}, n)$-flat preenvelope $\Leftrightarrow$ if N is a $(\mathcal{T}, n)$-injective left R-module, N_1 is an FP-injective submodule of N, then N/N_1 is $(\mathcal{T}, n)$-injective. (2) If R is a $(\mathcal{T}, n)$-coherent ring, then every left R-module has a $(\mathcal{T}, n)$-injective cover. (3) Every right R-module has a monic $(\mathcal{T}, n)$-flat preenvelope $\Leftrightarrow$ R is $(\mathcal{T}, n)$-coherent and ${}_RR$ is $(\mathcal{T}, n)$-injective $\Leftrightarrow$ R is $(\mathcal{T}, n)$-coherent and every left R-module has an epic $(\mathcal{T}, n)$-injective cover $\Leftrightarrow$ R is $(\mathcal{T}, n)$-coherent and every injective right R-module is $(\mathcal{T}, n)$-flat $\Leftrightarrow$ R is $(\mathcal{T}, n)$-coherent and every flat left R-module is $(\mathcal{T}, n)$-injective. As corollaries, some interesting results on n-coherent rings are obtained.

Throughout this paper, R is an associative ring with identity and all modules considered are unitary, n is a positive integer, $\mathcal{T}$ is a weak torsion class of left R-modules. R-Mod denotes the class of all left R-modules. For any R-module M, $M^+ = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ will be the character module of M. Given a class $\mathcal{L}$ of R-modules, we denote by $\mathcal{L}^\perp = \{M\colon \mathrm{Ext}_R^1(L, M) = 0,\ L \in \mathcal{L}\}$ the right orthogonal class of $\mathcal{L}$, and by ${}^\perp\mathcal{L} = \{M\colon \mathrm{Ext}_R^1(M, L) = 0,\ L \in \mathcal{L}\}$ the left orthogonal class of $\mathcal{L}$.

2. $\mathcal{T}$-finitely generated and $\mathcal{T}$-finitely presented modules

We begin with the following definition.

Definition 2.1. A nonempty subclass $\mathcal{T}$ of left R-modules is called a *weak torsion class* if $\mathcal{T}$ is closed under homomorphic images and extensions. If a class $\mathcal{T}$ of left R-modules is a weak torsion class, then a left R-module M is called *$\mathcal{T}$-finitely generated* (or $\mathcal{T}$-FG for short) if there exists a finitely generated submodule N such that $M/N \in \mathcal{T}$. A left R-module A is called *$\mathcal{T}$-finitely presented* (or $\mathcal{T}$-FP for short) if there exists an exact sequence of left R-modules $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$ with F finitely generated free and K $\mathcal{T}$-finitely generated.

Example 2.2.

(1) Let R be a non-left noetherian left hereditary ring and $\mathcal{T}$ the class of all injective left R-modules. Then by [16], Section 39.16, $\mathcal{T}$ is a weak torsion class. But $\mathcal{T}$ is not a torsion class.

(2) Let $\mathcal{T}$ be the class of all finitely generated left R-modules. Then by [16], Section 13.9 (1), $\mathcal{T}$ is a weak torsion class. But $\mathcal{T}$ is not a torsion class.

(3) Let $\mathcal{T}$ be the class of all finitely generated semisimple left R-modules. Then $\mathcal{T}$ is a weak torsion class but not a torsion class.

(4) Let $\mathcal{T}$ be the class of all finitely generated left R-modules. Then a left R-module A is $\mathcal{T}$-finitely generated if and only if it is finitely generated.

(5) Let $\mathcal{T} = R$-Mod. Then a left R-module A is $\mathcal{T}$-finitely presented if and only if it is finitely generated.

(6) Let $\mathcal{T} = 0$. Then a left R-module A is $\mathcal{T}$-finitely presented if and only if it is finitely presented.

Theorem 2.3. (1) *Any homomorphic image of a $\mathcal{T}$-FG module is $\mathcal{T}$-FG.*

(2) *Any finite direct sum of $\mathcal{T}$-FG modules is $\mathcal{T}$-FG.*

(3) *Any sum of a finite number of $\mathcal{T}$-FG submodules of a module M is $\mathcal{T}$-FG.*

(4) *A direct summand of a $\mathcal{T}$-FP module is $\mathcal{T}$-FP.*

Proof. (1) Let M be a $\mathcal{T}$-FG module and N a submodule of N. Since M is $\mathcal{T}$-FG, there exists a finitely generated submodule K of M such that $M/K \in \mathcal{T}$. Since $\mathcal{T}$ is closed under homomorphic images, we have $(M/K)/[(K+N)/K] \in \mathcal{T}$, so $M/(K+N) \in \mathcal{T}$, and thus $(M/N)/(K+N)/N \in \mathcal{T}$. Observing that $(K+N)/N$ is finitely generated, we have that M/N is $\mathcal{T}$-FG.

(2) Let N_1, N_2 be two $\mathcal{T}$-FG modules. Then there exists a finitely generated submodule K_i of N_i such that $N_i/K_i \in \mathcal{T}$, $i = 1, 2$. So, $K_1 \oplus K_2$ is finitely generated and $(N_1 \oplus N_2)/(K_1 \oplus K_2) \cong N_1/K_1 \oplus N_2/K_2 \in \mathcal{T}$ because $\mathcal{T}$ is closed under extensions. And thus $N_1 \oplus N_2$ is $\mathcal{T}$-FG.

(3) Let M_1, M_2 be two $\mathcal{T}$-FG submodules of M. Then by (2), $M_1 \oplus M_2$ is $\mathcal{T}$-FG. Note that $M_1 + M_2$ is a homomorphic image of $M_1 \oplus M_2$; by (1), $M_1 + M_2$ is $\mathcal{T}$-FG.

(4) Suppose that $M \cong F/K$ where F is finitely generated free and K is $\mathcal{T}$-FG. If $F/K = (A+K)/K \oplus (B+K)/K$, where A, B are finitely generated, then by (3), $B+K$ is $\mathcal{T}$-FG . But $(A+K)/K \cong F/(B+K)$, so $(A+K)/K$ is $\mathcal{T}$-FP. $\square$

Corollary 2.4. *A direct summand of a $\mathcal{T}$-FG module is $\mathcal{T}$-FG.*

Theorem 2.5. *Let $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$ be an exact sequence of left R-modules.*

(1) *If both A and C are $\mathcal{T}$-FG, then B is $\mathcal{T}$-FG.*

(2) *If both A and C are $\mathcal{T}$-FP, then B is $\mathcal{T}$-FP.*

(3) *If B is FG and C is $\mathcal{T}$-FP, then A is $\mathcal{T}$-FG.*

(4) *If B is $\mathcal{T}$-FP and A is $\mathcal{T}$-FG, then C is $\mathcal{T}$-FP.*

P r o o f. (1) Suppose that A and C are $\mathcal{T}$-FG. Then there exist a finitely generated submodule A' of A and a finitely generated submodule C' of C such that $A/A' \in \mathcal{T}$ and $C/C' \in \mathcal{T}$. Choose a finitely generated submodule B' of B such that $p(B') = C'$, let $A'' = A \cap (A' + B') = A' + (A \cap B')$, and define

$$\alpha\colon\ A/A'' \longrightarrow B/(A' + B'); \qquad a + A'' \mapsto a + (A' + B')$$

and

$$\overline{p}\colon\ B/(A' + B') \longrightarrow C/C'; \qquad b + (A' + B') \mapsto p(b) + C'.$$

Then we get an exact sequence $0 \longrightarrow A/A'' \xrightarrow{\alpha} B/(A' + B') \xrightarrow{\overline{p}} C/C' \longrightarrow 0$. Thus $A/A'' \cong (A/A')/(A''/A') \in \mathcal{T}$ and $C/C' \in \mathcal{T}$, so $B/(A' + B') \in \mathcal{T}$, and hence B is $\mathcal{T}$-FG.

(2) Since A and C are $\mathcal{T}$-FP, we have two exact sequences $0 \longrightarrow K' \xrightarrow{\iota_1} F' \xrightarrow{f} A \longrightarrow 0$ and $0 \longrightarrow K'' \xrightarrow{\iota_2} F'' \xrightarrow{g} C \longrightarrow 0$, where F', F'' are finitely generated free, K', K'' are $\mathcal{T}$-FG, ι_1, ι_2 are inclusion maps. Since F'' is projective, there exists a homomorphism $\sigma\colon F'' \to B$ such that $g = p\sigma$. And so we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K' \arrow[r, "\lambda\iota_1"] \arrow[d, "\iota_1"] & \mathrm{Ker}(h) \arrow[r, "\pi\iota"] \arrow[d, "\iota"] & K'' \arrow[r] \arrow[d, "\iota_2"] & 0 \\
0 \arrow[r] & F' \arrow[r, "\lambda"] \arrow[d, "f"] & F' \oplus F'' \arrow[r, "\pi"] \arrow[d, "h"] & F'' \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] \arrow[d] & B \arrow[r, "p"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where λ is the natural injection, ι is the inclusion map, π is the natural projection, and

$$h\colon\ F'\oplus F''\to B;\quad (x',x'')\mapsto if(x')+\sigma(x'').$$

By (1), $\mathrm{Ker}(h)$ is $\mathcal{T}$-FG, and hence B is $\mathcal{T}$-FP.

(3) Suppose that B is FG and C is $\mathcal{T}$-FP. Let $F\xrightarrow{\varphi}B\longrightarrow 0$ be exact with F FG free, let $K=\mathrm{Ker}(p\varphi)$. Then $0\longrightarrow K\longrightarrow F\longrightarrow C\longrightarrow 0$ is exact. Since C is $\mathcal{T}$-FP, there exists an exact sequence $0\longrightarrow K'\longrightarrow F'\longrightarrow C\longrightarrow 0$ with F' FG free and K' $\mathcal{T}$-FG. By Schanuel's lemma, we have $K'\oplus F\cong K\oplus F'$, and thus K is $\mathcal{T}$-FG because a finite direct sum and a direct summand of $\mathcal{T}$-FG modules are $\mathcal{T}$-FG. Now let $\psi=\varphi\big|_K$. Observing that φ is epic, it is easy to see that ψ is an epimorphism from K to A. Hence, by Theorem 2.3 (1), A is $\mathcal{T}$-FG.

(4) Since B is $\mathcal{T}$-FP, there exists an exact sequence of left R-modules $0\longrightarrow K\longrightarrow F\longrightarrow B\longrightarrow 0$ such that F is finitely generated free and K is $\mathcal{T}$-FG. Therefore, we can now from the pullback of $A\longrightarrow B$ and $F\longrightarrow B$ get the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & P \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

with exact rows and columns. Since both K and A are $\mathcal{T}$-FG, by (1), P is also $\mathcal{T}$-FG, and so C is $\mathcal{T}$-FP. $\square$

3. $(\mathcal{T},n)$-PRESENTED MODULES

Definition 3.1. Let $\mathcal{T}$ be a weak torsion class and n a positive integer. Then a left R-module A is said to be $(\mathcal{T},n)$-presented if there exists an exact sequence of left R-modules

$$0\longrightarrow K_{n-1}\longrightarrow F_{n-1}\longrightarrow\ldots\longrightarrow F_1\longrightarrow F_0\longrightarrow M\longrightarrow 0$$

such that $F_0,\ldots,F_{n-1}$ are finitely generated free and K_{n-1} is $\mathcal{T}$-finitely generated.

Clearly, a left R-module A is $\mathcal{T}$-finitely presented if and only if it is $(\mathcal{T},1)$-presented. It is easy to see that every $(\mathcal{T},n)$-presented module is $(\mathcal{T},n-1)$-presented. We also call $\mathcal{T}$-finitely generated modules $(\mathcal{T},0)$-presented.

Example 3.2. (1) Let $\mathcal{T}=R$-Mod. Then a left R-module A is $(\mathcal{T},n)$-presented if and only if it is $(n-1)$-presented.

(2) Let $\mathcal{T}=0$. Then a left R-module A is $(\mathcal{T},n)$-presented if and only if it is n-presented.

Lemma 3.3. *Let A,B be two left R-modules and n a positive integer. If both A and B are $(\mathcal{T},n)$-presented, then $A\oplus B$ is also $(\mathcal{T},n)$-presented.*

Proof. It is a consequence of Theorem 2.3 (2). $\square$

Proposition 3.4. *The following statements are equivalent for a left R-module A:*

(1) *A is $(\mathcal{T},n)$-presented.*

(2) *A is $(n-1)$-presented, and if there exists an exact sequence of left R-modules*

$$0\longrightarrow K_{n-1}\longrightarrow F_{n-1}\longrightarrow\ldots\longrightarrow F_1\longrightarrow F_0\longrightarrow A\longrightarrow 0$$

such that $F_0,\ldots,F_{n-1}$ are finitely generated free, then K_{n-1} is $\mathcal{T}$-finitely generated.

(3) *There exists an exact sequence of left R-modules*

$$0\longrightarrow K\longrightarrow F\longrightarrow A\longrightarrow 0$$

such that F is finitely generated free and K is $(\mathcal{T},n-1)$-presented.

If $n\geqslant 2$, then the above conditions are also equivalent to:

(4) *A is $(n-2)$-presented, and if there exists an exact sequence of left R-modules*

$$0\longrightarrow K_{n-2}\longrightarrow F_{n-2}\longrightarrow\ldots\longrightarrow F_1\longrightarrow F_0\longrightarrow A\longrightarrow 0$$

such that $F_0,\ldots,F_{n-2}$ are finitely generated free, then K_{n-2} is $\mathcal{T}$-finitely presented.

Proof. (1) $\Rightarrow$ (2) Since A is $(\mathcal{T},n)$-presented, there exists an exact sequence of left R-modules

$$0\longrightarrow L_{n-1}\longrightarrow F'_{n-1}\longrightarrow\ldots\longrightarrow F'_1\longrightarrow F'_0\longrightarrow A\longrightarrow 0$$

such that $F'_0,\ldots,F'_{n-1}$ are finitely generated free and L_{n-1} is $\mathcal{T}$-finitely generated, so A is $(n-1)$-presented. Now if there exists an exact sequence of left R-modules

$$0\longrightarrow K_{n-1}\longrightarrow F_{n-1}\longrightarrow\ldots\longrightarrow F_1\longrightarrow F_0\longrightarrow A\longrightarrow 0$$

such that $F_0, \dots, F_{n-1}$ are finitely generated free, then by the generalization of Schanuel's lemma [12], Exercise 3.37, and by Theorem 2.3 (2) and Corollary 2.4, K_{n-1} is $\mathcal{T}$-finitely generated.

(2) $\Rightarrow$ (1); (1) $\Leftrightarrow$ (3); and (2) $\Leftrightarrow$ (4) are obvious. $\square$

Proposition 3.5. *Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of left R-modules. Then:*

(1) *If both A and C are $(\mathcal{T}, n)$-presented, then so is B.*

(2) *If B is $(\mathcal{T}, n)$-presented and A is $(\mathcal{T}, n-1)$-presented, then C is $(\mathcal{T}, n)$-presented.*

Proof. (1) Use induction on n. If $n = 1$, then (1) holds by Theorem 2.5 (2). Suppose that (1) holds for $n - 1$. Let A and C be $(\mathcal{T}, n)$-presented. Then by Proposition 3.4, we have two exact sequences $0 \longrightarrow K' \xrightarrow{\iota_1} F' \xrightarrow{f} A \longrightarrow 0$ and $0 \longrightarrow K'' \xrightarrow{\iota_2} F'' \xrightarrow{g} C \longrightarrow 0$, where F', F'' are finitely generated free, K', K'' are $(\mathcal{T}, n-1)$-presented, ι_1, ι_2 are inclusion maps. Using a method similar to the proof of Theorem 2.5 (2), by induction hypothesis and Proposition 3.4 we can get that B is also $(\mathcal{T}, n)$-presented.

(2) Since B is $(\mathcal{T}, n)$-presented, by Proposition 3.4 there exists an exact sequence of left R-modules $0 \longrightarrow K \longrightarrow F \longrightarrow B \longrightarrow 0$ such that F is finitely generated free and K is $(\mathcal{T}, n-1)$-presented. Now, using a method similar to the proof of Theorem 2.5 (4), by (1) and Proposition 3.4, we can get that C is $(\mathcal{T}, n)$-presented. $\square$

Corollary 3.6. *A direct summand of a $(\mathcal{T}, n)$-presented module is $(\mathcal{T}, n)$-presented.*

Proof. Use induction on n. If $n = 1$, then the conclusion holds by Theorem 2.3 (4). Suppose that the conclusion holds for $n - 1$. Let B be $(\mathcal{T}, n)$-presented and $B = A \oplus C$. Then by hypothesis, A is $(\mathcal{T}, n-1)$-presented , and so C $(\mathcal{T}, n)$-presented by Proposition 3.5 (2), as required. $\square$

Corollary 3.7. *The following statements are equivalent for a left R-module M:*

(1) *M is $(\mathcal{T}, n)$-presented.*

(2) *M is finitely generated and, if the sequence of left R-modules $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ is exact with F finitely generated free, then K is $(\mathcal{T}, n-1)$-presented.*

Proof. (1) $\Rightarrow$ (2). Since M is $(\mathcal{T}, n)$-presented, by Proposition 3.4 (3) there exists an exact sequence of left R-modules $0 \longrightarrow K' \longrightarrow F' \longrightarrow M \longrightarrow 0$ such that F' is finitely generated free and K' is $(\mathcal{T}, n-1)$-presented. So, by Schanuel's lemma, we have $K' \oplus F \cong K \oplus F'$, and thus K is $(\mathcal{T}, n-1)$-presented because finite direct sums and direct summands of $(\mathcal{T}, n-1)$-presented modules are $(\mathcal{T}, n-1)$-presented by Lemma 3.3 and Corollary 3.6.

(2) $\Rightarrow$ (1). It follows from Proposition 3.4 (3). $\square$

Corollary 3.8. *Let $n > 1$ and let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of left R-modules. If C is $(\mathcal{T}, n)$-presented and B is $(\mathcal{T}, n-1)$-presented, then A is $(\mathcal{T}, n-1)$-presented.*

P r o o f. Since $n > 1$ and B is $(\mathcal{T}, n-1)$-presented, we have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & P \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

with exact rows and columns, where F is finitely generated free. Moreover, by Corollary 3.7, K is $(\mathcal{T}, n-2)$-presented. Since C is $(\mathcal{T}, n)$-presented, by Corollary 3.7, P is $(\mathcal{T}, n-1)$-presented, and so A is $(\mathcal{T}, n-1)$-presented by Proposition 3.5 (2). $\square$

4. $(\mathcal{T}, n)$-injective and $(\mathcal{T}, n)$-flat modules

Definition 4.1. A left R-module M is called $(\mathcal{T}, n)$-*injective*, if $\operatorname{Ext}_R^n(A, M) = 0$ for each $(\mathcal{T}, n+1)$-presented left R-module A. A right R-module M is called $(\mathcal{T}, n)$-*flat*, if $\operatorname{Tor}_n^R(M, A) = 0$ for each $(\mathcal{T}, n+1)$-presented left R-module A.

Clearly, n-FP-injective left R-modules are $(\mathcal{T}, n)$-injective, n-flat right R-modules are $(\mathcal{T}, n)$-flat. By Proposition 3.4 (3), it is easy to see that a $(\mathcal{T}, n)$-injective module is $(\mathcal{T}, n+1)$-injective, a $(\mathcal{T}, n)$-flat module is $(\mathcal{T}, n+1)$-flat. We denote by $\mathcal{T}_n\mathcal{I}$ the class of all $(\mathcal{T}, n)$-injective left R-modules, and denote by $\mathcal{T}_n\mathcal{F}$ the class of all $(\mathcal{T}, n)$-flat right R-modules. We recall that if n, d are nonnegative integers, then according to [18], a right R-module M is called (n, d)-injective if $\operatorname{Ext}_R^{d+1}(A, M)=0$ for every n-presented right R-module A; a left R-module M is called (n, d)-flat if $\operatorname{Tor}_{d+1}^R(A, M)=0$ for every n-presented right R-module A.

Example 4.2. (1) Let $\mathcal{T} = R$-Mod. Then a left R-module M is $(\mathcal{T}, n)$-injective if and only if M is n-FP-injective, a right R-module M is $(\mathcal{T}, n)$-flat if and only if M is n-flat. In particular, a left R-module M is $(\mathcal{T}, 1)$-injective if and only if M is FP-injective, a right R-module M is $(\mathcal{T}, 1)$-flat if and only if M is flat.

(2) Let $\mathcal{T} = \{0\}$. Then a left R-module M is $(\mathcal{T}, n)$-injective if and only if M is $(n+1, n-1)$-injective, a right R-module M is $(\mathcal{T}, n)$-flat if and only if M is $(n+1, n-1)$-flat. In particular, a left R-module M is $(\mathcal{T}, 1)$-injective if and only if M is $(2, 0)$-injective, a right R-module M is $(\mathcal{T}, 1)$-flat if and only if M is $(2, 0)$-flat.

Recall that an exact sequence of left R-modules $0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0$ is said to be pure if every finitely presented left R-module is projective with respect to this exact sequence.

Definition 4.3. Let $0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0$ be an exact sequence of left R-modules. Then it is said to be $\mathcal{T}$-*pure* if every $(\mathcal{T}, 2)$-presented left R-module is projective with respect to it.

Example 4.4. (1) Let $\mathcal{T} = R$-Mod. Then it is easy to see that an exact sequence of left R-modules $0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0$ is pure if and only if it is $\mathcal{T}$-pure.

(2) Let $\mathcal{T} = \{0\}$. Then it is easy to see that an exact sequence of left R-modules $0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0$ is $\mathcal{T}$-pure if and only if every 2-presented left R-module is projective with respect to it.

Let $\ldots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \longrightarrow 0$ be a projective resolution of a module A. As usual, we will denote $\mathrm{Ker}(d_i)$ by K_i, and we will call K_i an i-syzygy of A. If $n \geqslant 2$, then it is easy to see that a left R-module A is $(\mathcal{T}, n+1)$-presented if and only if it is $(n-2)$-presented; and if the sequence of right R-modules $0 \longrightarrow K_{n-2} \longrightarrow F_{n-2} \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$ is exact, where $F_0, \ldots, F_{n-2}$ are finitely generated free, then K_{n-2} is $(\mathcal{T}, 2)$-presented.

Theorem 4.5. *Let M be a left R-module and $n \geqslant 2$. Then the following statements are equivalent:*

(1) *M is $(\mathcal{T}, n)$-injective.*
(2) *If the sequence $0 \longrightarrow K_{n-2} \longrightarrow F_{n-2} \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$ is exact, where $F_0, \ldots, F_{n-2}$ are finitely generated free and K_{n-2} is $(\mathcal{T}, 2)$-presented, then $\mathrm{Ext}^1_R(K_{n-2}, M) = 0$.*
(3) *For every $(n-1)$-presentation $F_{n-1} \longrightarrow \ldots \longrightarrow F_0 \longrightarrow A \longrightarrow 0$ of a $(\mathcal{T}, n+1)$-presented module A with $F_0, \ldots, F_{n-2}, F_{n-1}$ finitely generated free, every homomorphism from the $(n-1)$-syzygy K_{n-1} to M can be extended to a homomorphism from F_{n-1} to M.*
(4) *There exists a $\mathcal{T}$-pure exact sequence $0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0$ of left R-modules with M' $(\mathcal{T}, n)$-injective.*

P r o o f. (1) $\Leftrightarrow$ (2). By the isomorphism $\operatorname{Ext}^n_R(A,M)\cong \operatorname{Ext}^1_R(K_{n-2},M)$.

(2) $\Leftrightarrow$ (3). By the exact sequence

$$\operatorname{Hom}(F_{n-1},M)\longrightarrow \operatorname{Hom}(K_{n-1},M)\longrightarrow \operatorname{Ext}^1_R(K_{n-2},M)\longrightarrow \operatorname{Ext}^1_R(F_{n-1},M)=0.$$

(1) $\Rightarrow$ (4). It is obvious.

(4) $\Rightarrow$ (2). Since $0\longrightarrow M\longrightarrow M'\longrightarrow M''\longrightarrow 0$ is $\mathcal{T}$-pure and K_{n-2} is $(\mathcal{T},2)$-presented, we have that the map $\operatorname{Hom}(K_{n-2},M')\longrightarrow \operatorname{Hom}(K_{n-2},M'')$ is epic. So from the exact sequence

$$\operatorname{Hom}(K_{n-2},M')\longrightarrow \operatorname{Hom}(K_{n-2},M'')\longrightarrow \operatorname{Ext}^1_R(K_{n-2},M)\longrightarrow 0$$

we have $\operatorname{Ext}^1_R(K_{n-2},M)=0$. □

Proposition 4.6. *Let $\{M_i\colon i\in I\}$ be a family of left R-modules. Then the following statements are equivalent:*

(1) *Each M_i is $(\mathcal{T},n)$-injective.*

(2) $\prod\limits_{i\in I} M_i$ *is $(\mathcal{T},n)$-injective.*

(3) $\bigoplus\limits_{i\in I} M_i$ *is $(\mathcal{T},n)$-injective.*

P r o o f. (1) $\Leftrightarrow$ (2). By the isomorphism $\operatorname{Ext}^n_R\Big(A,\prod\limits_{i\in I} M_i\Big)\cong \prod\limits_{i\in I}\operatorname{Ext}^n_R(A,M_i)$.

(2) $\Rightarrow$ (3). For every $(n-1)$-presentation $F_{n-1}\longrightarrow\ldots\longrightarrow F_0\longrightarrow A\longrightarrow 0$ of a $(\mathcal{T},n+1)$-presented module A with $F_0,\ldots,F_{n-2},F_{n-1}$ finitely generated free, by Proposition 3.4 (4), the $(n-1)$-syzygy K_{n-1} is $\mathcal{T}$-finitely presented and hence finitely generated. Let f be any homomorphism from K_{n-1} to $\bigoplus\limits_{i\in I} M_i$. Then there exists a finite subset I_0 of I such that $\operatorname{Im}(f)\subseteq \bigoplus\limits_{i\in I_0} M_i$. By (2), $\bigoplus\limits_{i\in I_0} M_i$ is $(\mathcal{T},n)$-injective. So, by Theorem 4.5 (3), f can be extended to a homomorphism from F_{n-1} to $\bigoplus\limits_{i\in I_0} M_i$, and then f can be extended to a homomorphism from F_{n-1} to $\bigoplus\limits_{i\in I} M_i$. Therefore $\bigoplus\limits_{i\in I} M_i$ is $(\mathcal{T},n)$-injective by Theorem 4.5 (3) again.

(3) $\Rightarrow$ (1). It is trivial. □

Proposition 4.7. *Let $\{M_i\colon i\in I\}$ be a family of right R-modules. Then the following conditions are equivalent:*

(1) *Every M_i is $(\mathcal{T},n)$-flat.*

(2) $\bigoplus\limits_{i\in I} M_i$ *is $(\mathcal{T},n)$-flat.*

P r o o f. By the isomorphism $\operatorname{Tor}^R_n\Big(\bigoplus\limits_{i\in I} M_i,A\Big)\cong \bigoplus\limits_{i\in I}\operatorname{Tor}^R_n(M_i,A)$. □

Theorem 4.8. *Let M be a right R-module. Then M is $(\mathcal{T}, n)$-flat if and only if M^+ is $(\mathcal{T}, n)$-injective.*

Proof. It follows from the isomorphism $\mathrm{Tor}_n^R(M, A)^+ \cong \mathrm{Ext}_R^n(A, M^+)$. $\square$

Proposition 4.9.

(1) *Pure submodules of $(\mathcal{T}, n)$-injective modules are $(\mathcal{T}, n)$-injective.*

(2) *Pure submodules of $(\mathcal{T}, n)$-flat modules are $(\mathcal{T}, n)$-flat.*

Proof. (1) Let N be a pure submodule of a $(\mathcal{T}, n)$-injective module M. Then N is $\mathcal{T}$-pure in M, and so, by Theorem 4.5 (4), N is $(\mathcal{T}, n)$-injective.

(2) Let M be a $(\mathcal{T}, n)$-flat module and N a pure submodule of M. Then the pure exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ induces a split exact sequence $0 \longrightarrow (M/N)^+ \longrightarrow M^+ \longrightarrow N^+ \longrightarrow 0$. By Theorem 4.8, M^+ is $(\mathcal{T}, n)$-injective, so N^+ is $(\mathcal{T}, n)$-injective by Proposition 4.6, and hence N is $(\mathcal{T}, n)$-flat by Theorem 4.8 again. $\square$

Remark 4.10. From Theorem 4.8, the $(\mathcal{T}, n)$-flatness of M_R can be characterized by the $(\mathcal{T}, n)$-injectivity of M^+. On the other hand, by [3], Lemma 2.7 (1), the sequence $\mathrm{Tor}_n^R(M^+, A) \longrightarrow \mathrm{Ext}_R^n(A, M)^+ \longrightarrow 0$ is exact for any n-presented left R-module A and any left R-module M. So, for any left R-module M, if M^+ is $(\mathcal{T}, n)$-flat, then M is $(\mathcal{T}, n)$-injective.

Let $\mathcal{F}$ be a class of R-modules and M an R-module. Following [6], we say that a homomorphism $\varphi\colon M \longrightarrow F$ where $F \in \mathcal{F}$ is an $\mathcal{F}$-preenvelope of M if for any morphism $f\colon M \longrightarrow F'$ with $F' \in \mathcal{F}$ there is a $g\colon F \longrightarrow F'$ such that $g\varphi = f$. An $\mathcal{F}$-preenvelope $\varphi\colon M \longrightarrow F$ is said to be an $\mathcal{F}$-envelope if every endomorphism $g\colon F \longrightarrow F$ such that $g\varphi = \varphi$ is an isomorphism. Dually, we have the definitions of an $\mathcal{F}$-precover and an $\mathcal{F}$-cover. The $\mathcal{F}$-envelopes ($\mathcal{F}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

A pair $(\mathcal{A}, \mathcal{B})$ of classes of R-modules is called a cotorsion theory, see [6], if $\mathcal{A}^\perp = \mathcal{B}$ and ${}^\perp\mathcal{B} = \mathcal{A}$. A cotorsion theory $(\mathcal{A}, \mathcal{B})$ is called perfect, see [7], if every R-module has a $\mathcal{B}$-envelope and an $\mathcal{A}$-cover. A cotorsion theory $(\mathcal{A}, \mathcal{B})$ is called complete (see [6], Definition 7.1.6, and [15], Lemma 1.13) if for any R-module M there are exact sequences $0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and $0 \longrightarrow B' \longrightarrow A' \longrightarrow M \longrightarrow 0$ with $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$.

For a class $\mathcal{F}$ of R-modules, we put $\mathcal{F}^+ = \{F^+\colon F \in \mathcal{F}\}$. We recall that a left R-module M is said to be *pure injective* if it is injective with respect to all pure exact sequences of left R-modules. Following [15], we denote by $\mathcal{PI}$ the class of pure injective left R-modules.

Theorem 4.11. *Let R be a ring. Then:*

(1) $({}^{\perp}(\mathcal{T}_n\mathcal{I}), \mathcal{T}_n\mathcal{I})$ *is a complete cotorsion theory.*
(2) $(\mathcal{T}_n\mathcal{F}, (\mathcal{T}_n\mathcal{F})^{\perp})$ *is a perfect cotorsion theory.*

P r o o f. (1) Let X be the set of representatives of all K_{n-2}'s in Theorem 4.5 (2). Then by Theorem 4.5, $\mathcal{T}_n\mathcal{I} = X^{\perp}$, and so $({}^{\perp}(\mathcal{T}_n\mathcal{I}), \mathcal{T}_n\mathcal{I}) = ({}^{\perp}(X^{\perp}), X^{\perp})$ is a complete cotorsion theory by [15], Theorem 2.2 (2).

(2) Write $\mathcal{A} = \mathcal{T}_n\mathcal{F}$ and let $\mathcal{X}$ be the class of all K_{n-2}'s in Theorem 4.5 (2). Then by dimension shifting one shows that $A \in \mathcal{T}_n\mathcal{F}$ if and only if $\mathrm{Tor}_1^R(A, X) = 0$ for each $X \in \mathcal{X}$. Thus, by the isomorphism $\mathrm{Tor}_1^R(A, B)^+ \cong \mathrm{Ext}_R^1(A, B^+)$, we have $\mathcal{A} = {}^{\perp}(\mathcal{X}^+)$, and so $(\mathcal{T}_n\mathcal{F}, (\mathcal{T}_n\mathcal{F})^{\perp}) = ({}^{\perp}(\mathcal{X}^+), ({}^{\perp}(\mathcal{X}^+))^{\perp})$ is a cotorsion theory generated by $\mathcal{X}^+$. Since every character module is pure injective by [6], Proposition 5.3.7, we have $\mathcal{X}^+ \subseteq \mathcal{PI}$, and so it is a perfect cotorsion theory by [15], Theorem 2.8. $\square$

Following [6], Definition 5.3.22, a right R-module M is said to be *cotorsion* if $\mathrm{Ext}_R^1(F, M) = 0$ for all flat right R-modules F. We call a right R-module M $(\mathcal{T}, n)$-*cotorsion* if $\mathrm{Ext}_R^1(F, M) = 0$ for all $(\mathcal{T}, n)$-flat right R-modules F. By Theorem 4.11, we have the following results.

Corollary 4.12. *Let R be a ring. Then:*
(1) *Every right R-module has a $(\mathcal{T}, n)$-flat cover.*
(2) *Every right R-module has a $(\mathcal{T}, n)$-cotorsion envelope.*

5. $(\mathcal{T}, n)$-coherent rings

We begin this section with the concepts of $(\mathcal{T}, n)$-coherent rings and $\mathcal{T}$-coherent rings.

Definition 5.1. A ring R is called $(\mathcal{T}, n)$-*coherent*, if every $(\mathcal{T}, n+1)$-presented module is $(n+1)$-presented. A ring R is called $\mathcal{T}$-coherent if it is $(\mathcal{T}, 1)$-coherent.

It is easy to see that a ring R is $(\mathcal{T}, n)$-coherent if and only if every $(\mathcal{T}, n)$-presented submodule of a finitely generated free left R-module is n-presented, and a ring R is $\mathcal{T}$-coherent if and only if every $\mathcal{T}$-finite presented submodule of a finitely generated free left R-module is finitely presented.

Example 5.2. (1) Let $\mathcal{T} = R$-Mod. Then R is $(\mathcal{T}, n)$-coherent if and only if R is left n-coherent. In particular, R is $(\mathcal{T}, 1)$-coherent if and only if R is left coherent.
(2) Let $\mathcal{T} = \{0\}$. Then R is $(\mathcal{T}, n)$-coherent for any positive integer n.

Next we will characterize $(\mathcal{T}, n)$-coherent rings in terms of, among others, $(\mathcal{T}, n)$-injective modules and $(\mathcal{T}, n)$-flat modules. These results extend the theory of coherence of rings.

Theorem 5.3. *The following statements are equivalent for the ring R:*

(1) *R is $(\mathcal{T}, n)$-coherent.*

(2) *$\varinjlim \operatorname{Ext}_R^n(A, M_i) \cong \operatorname{Ext}_R^n(A, \varinjlim M_i)$ for any $(\mathcal{T}, n+1)$-presented module A and direct system $(M_i)_{i\in I}$ of left R-modules.*

(3) *$\operatorname{Tor}_n^R(\prod N_i, A) \cong \prod \operatorname{Tor}_n^R(N_i, A)$ for any family $\{N_i\}$ of right R-modules and any $(\mathcal{T}, n+1)$-presented module A.*

(4) *Any direct product of copies of R_R is $(\mathcal{T}, n)$-flat.*

(5) *Any direct product of $(\mathcal{T}, n)$-flat right R-modules is $(\mathcal{T}, n)$-flat.*

(6) *Any direct limit of $(\mathcal{T}, n)$-injective left R-modules is $(\mathcal{T}, n)$-injective.*

(7) *Any direct limit of injective left R-modules is $(\mathcal{T}, n)$-injective.*

(8) *A left R-module M is $(\mathcal{T}, n)$-injective if and only if M^+ is $(\mathcal{T}, n)$-flat.*

(9) *A left R-module M is $(\mathcal{T}, n)$-injective if and only if M^{++} is $(\mathcal{T}, n)$-injective.*

(10) *A right R-module M is $(\mathcal{T}, n)$-flat if and only if M^{++} is $(\mathcal{T}, n)$-flat.*

(11) *For any ring S, $\operatorname{Tor}_n^R(\operatorname{Hom}_S(B, E), A) \cong \operatorname{Hom}_S(\operatorname{Ext}_R^n(A, B), E)$ for the situation $({}_RA, {}_RB_S, E_S)$ with A $(\mathcal{T}, n+1)$-presented and E_S injective.*

(12) *Every right R-module has a $(\mathcal{T}, n)$-flat preenvelope.*

Proof. (1) $\Rightarrow$ (2). follows from [3], Lemma 2.9 (2).

(1) $\Rightarrow$ (3). follows from [3], Lemma 2.10 (2).

(2) $\Rightarrow$ (6) $\Rightarrow$ (7) and (3) $\Rightarrow$ (5) $\Rightarrow$ (4) are trivial.

(7) $\Rightarrow$ (1). Let A be $(\mathcal{T}, n+1)$-presented with a finite n-presentation $F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \ldots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} A \longrightarrow 0$. Write $K_{n-1} = \operatorname{Ker}(d_{n-1})$ and $K_{n-2} = \operatorname{Ker}(d_{n-2})$. Then K_{n-1} is finitely generated, and we get an exact sequence of left R-modules $0 \longrightarrow K_{n-1} \longrightarrow F_{n-1} \longrightarrow K_{n-2} \longrightarrow 0$. Let $(E_i)_{i\in I}$ be any direct system of injective left R-modules (with I directed). Then $\varinjlim E_i$ is $(\mathcal{T}, n)$-injective by (7), so $\operatorname{Ext}_R^n(A, \varinjlim E_i) = 0$ and then $\operatorname{Ext}_R^1(K_{n-2}, \varinjlim E_i) = 0$. Thus, we have a commutative diagram

$$\begin{array}{ccccccc}
\varinjlim \operatorname{Hom}(K_{n-2}, E_i) & \longrightarrow & \varinjlim \operatorname{Hom}(F_{n-1}, E_i) & \longrightarrow & \varinjlim \operatorname{Hom}(K_{n-1}, E_i) & \longrightarrow & 0 \\
\Big\downarrow f & & \Big\downarrow g & & \Big\downarrow h & & \\
\operatorname{Hom}(K_{n-2}, \varinjlim E_i) & \longrightarrow & \operatorname{Hom}(F_{n-1}, \varinjlim E_i) & \longrightarrow & \operatorname{Hom}(K_{n-1}, \varinjlim E_i) & \longrightarrow & 0
\end{array}$$

with exact rows. Since f and g are isomorphisms by [16], 25.4(d), h is an isomorphism by the Five lemma. Now, let $(M_i)_{i\in I}$ be any direct system of left R-modules (with

I directed). Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & \varinjlim \mathrm{Hom}(K_{n-1}, M_i) & \longrightarrow & \varinjlim \mathrm{Hom}(K_{n-1}, E(M_i)) & \longrightarrow & \varinjlim \mathrm{Hom}(K_{n-1}, E(M_i)/M_i) \\
& \Big\downarrow{\scriptstyle \varphi_1} & & \Big\downarrow{\scriptstyle \varphi_2} & & \Big\downarrow{\scriptstyle \varphi_3} \\
0 \longrightarrow & \mathrm{Hom}(K_{n-1}, \varinjlim M_i) & \longrightarrow & \mathrm{Hom}(K_{n-1}, \varinjlim E(M_i)) & \longrightarrow & \mathrm{Hom}(K_{n-1}, \varinjlim E(M_i)/M_i)
\end{array}$$

where $E(M_i)$ is the injective hull of M_i. Since K_{n-1} is finitely generated, by [16], Section 24.9, the maps φ_1, φ_2 and φ_3 are monic. By the above proof, φ_2 is an isomorphism. Hence φ_1 is also an isomorphism by the Five lemma again, so K_{n-1} is finitely presented by [16], Section 25.4 (d), again, and thus A is $(n+1)$-presented. Therefore R is $(\mathcal{T}, n)$-coherent.

(4) $\Rightarrow$ (1). It follows similarly to (7) $\Rightarrow$ (1).

(5) $\Rightarrow$ (12). Let N be any left R-module. By [6], Lemma 5.3.12, there is a cardinal number $\aleph_\alpha$ dependent on $\mathrm{Card}(N)$ and $\mathrm{Card}(R)$ such that for any homomorphism $f\colon N \longrightarrow F$ with F $(\mathcal{T}, n)$-flat, there is a pure submodule S of F such that $f(N) \subseteq S$ and $\mathrm{Card}\ S \leqslant \aleph_\alpha$. Thus f has a factorization $N \longrightarrow S \longrightarrow F$ with S $(\mathcal{T}, n)$-flat by Proposition 4.9 (2). Now let $(\varphi_\beta)_{\beta \in B}$ be all such homomorphisms $\varphi_\beta\colon N \longrightarrow S_\beta$ with $\mathrm{Card}\ S_\beta \leqslant \aleph_\alpha$ and S_β $(\mathcal{T}, n)$-flat. Then any homomorphism $N \longrightarrow F$ with F $(\mathcal{T}, n)$-flat has a factorization $N \longrightarrow S_i \longrightarrow F$ for some $i \in B$. Thus the homomorphism $N \longrightarrow \prod\limits_{\beta \in B} S_\beta$ induced by all φ_β is a $(\mathcal{T}, n)$-flat preenvelope since $\prod\limits_{\beta \in B} S_\beta$ is $(\mathcal{T}, n)$-flat by (5).

(12) $\Rightarrow$ (5). For any family $\{F_i\}_{i \in I}$ of $(\mathcal{T}, n)$-flat left R-modules, by hypothesis, $\prod\limits_{i \in I} F_i$ has a $(\mathcal{T}, n)$-flat preenvelope $\varphi\colon \prod\limits_{i \in I} F_i \longrightarrow F$. Let $p_i\colon \prod\limits_{i \in I} F_i \longrightarrow F_i$ be the projection. Then there exists $f_i\colon F \longrightarrow F_i$ such that $p_i = f_i \varphi$. Define $\psi\colon F \longrightarrow \prod\limits_{i \in I} F_i$ by $\psi(x) = (f_i(x))$ for every $x \in F$, then it is easy to check that $\psi\varphi = 1$. Hence $\prod\limits_{i \in I} F_i$ is isomorphic to a direct summand of F, and so $\prod\limits_{i \in I} F_i$ is $(\mathcal{T}, n)$-flat.

(1) $\Rightarrow$ (11). For any $(\mathcal{T}, n+1)$-presented module A, since R is $(\mathcal{T}, n)$-coherent, A is $(n+1)$-presented. And so (11) follows from [3], Lemma 2.7 (2).

(11) $\Rightarrow$ (8). Let $S = \mathbb{Z}$, $E = \mathbb{Q}/\mathbb{Z}$ and $B = M$. Then $\mathrm{Tor}_n^R(M^+, A) \cong \mathrm{Ext}_R^n(A, M)^+$ for any $(\mathcal{T}, n+1)$-presented module A by (11), and hence (8) holds.

(8) $\Rightarrow$ (9). Let M be a left R-module. If M is $(\mathcal{T}, n)$-injective, then M^+ is $(\mathcal{T}, n)$-flat by (8), and so M^{++} is $(\mathcal{T}, n)$-injective by Theorem 4.8. Conversely, if M^{++} is $(\mathcal{T}, n)$-injective, then M, being a pure submodule of M^{++} (see [14], Exercise 41, page 48), is $(\mathcal{T}, n)$-injective by Proposition 4.9 (1).

(9) $\Rightarrow$ (10). If M is a $(\mathcal{T}, n)$-flat right R-module, then M^+ is a $(\mathcal{T}, n)$-injective left R-module by Theorem 4.8, and so M^{+++} is $(\mathcal{T}, n)$-injective by (9). Thus M^{++}

is $(\mathcal{T}, n)$-flat by Theorem 4.8 again. Conversely, if M^{++} is $(\mathcal{T}, n)$-flat, then M is $(\mathcal{T}, n)$-flat by Proposition 4.9 (2) as M is a pure submodule of M^{++}.

(10) $\Rightarrow$ (5). Let $\{N_i\}_{i\in I}$ be a family of $(\mathcal{T}, n)$-flat right R-modules. Then by Proposition 4.7, $\bigoplus\limits_{i\in I} N_i$ is $(\mathcal{T}, n)$-flat, and so $\Big(\prod\limits_{i\in I} N_i^+\Big)^+ \cong \Big(\bigoplus\limits_{i\in I} N_i\Big)^{++}$ is $(\mathcal{T}, n)$-flat by (10). Since $\bigoplus\limits_{i\in I} N_i^+$ is a pure submodule of $\prod\limits_{i\in I} N_i^+$ by [2], Lemma 1 (1), $\Big(\prod\limits_{i\in I} N_i^+\Big)^+ \longrightarrow \Big(\bigoplus\limits_{i\in I} N_i^+\Big)^+ \longrightarrow 0$ splits, and hence $\Big(\bigoplus\limits_{i\in I} N_i^+\Big)^+$ is $(\mathcal{T}, n)$-flat. Thus $\prod\limits_{i\in I} N_i^{++} \cong \Big(\bigoplus\limits_{i\in I} N_i^+\Big)^+$ is $(\mathcal{T}, n)$-flat. Since $\prod\limits_{i\in I} N_i$ is a pure submodule of $\prod\limits_{i\in I} N_i^{++}$ by [2], Lemma 1 (2), $\prod\limits_{i\in I} N_i$ is $(\mathcal{T}, n)$-flat by Proposition 4.9 (2). $\square$

Corollary 5.4. *The following statements are equivalent for a ring R:*

(1) *R is left n-coherent.*

(2) $\varinjlim \operatorname{Ext}_R^n(C, M_\alpha) \cong \operatorname{Ext}_R^n(C, \varinjlim M_\alpha)$ *for any n-presented left R-module C and direct system $(M_\alpha)_{\alpha\in A}$ of left R-modules.*

(3) $\operatorname{Tor}_n^R(\prod N_\alpha, C) \cong \prod \operatorname{Tor}_n^R(N_\alpha, C)$ *for any family $\{N_\alpha\}$ of right R-modules and any n-presented left R-module C.*

(4) *Any direct product of copies of R_R is n-flat.*

(5) *Any direct product of n-flat right R-modules is n-flat.*

(6) *Any direct limit of n-FP-injective left R-modules is n-FP-injective.*

(7) *Any direct limit of injective left R-modules is n-FP-injective.*

(8) *A left R-module M is n-FP-injective if and only if M^+ is n-flat.*

(9) *A left R-module M is n-FP-injective if and only if M^{++} is n-FP-injective.*

(10) *A right R-module M is n-flat if and only if M^{++} is n-flat.*

(11) *For any ring S,* $\operatorname{Tor}_n^R(\operatorname{Hom}_S(B, E), C) \cong \operatorname{Hom}_S(\operatorname{Ext}_R^n(C, B), E)$ *for the situation $({}_RC, {}_RB_S, E_S)$ with C n-presented and E_S injective.*

(12) *Every right R-module has an n-flat preenvelope.*

We note that the equivalences of (1)–(6), (8)–(11) in Corollary 5.4 appeared in [3], Theorem 3.1.

Lemma 5.5. *Let A be an $(n-1)$-presented left R-module. Then A is n-presented if and only if $\operatorname{Ext}_R^n(A, M) = 0$ for any FP-injective module M.*

P r o o f. Let A have a finite $(n-1)$-presentation $F_{n-1} \xrightarrow{d_{n-1}} \ldots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} A \longrightarrow 0$. Write $K_{n-2} = \operatorname{Ker}(d_{n-2})$. Then K_{n-2} is finitely generated. By the isomorphism $\operatorname{Ext}_R^n(A, M) \cong \operatorname{Ext}_R^1(K_{n-2}, M)$, we have that $\operatorname{Ext}_R^n(A, M) = 0$ for any FP-injective module M if and only if $\operatorname{Ext}_R^1(K_{n-2}, M) = 0$ for any FP-injective module M. So, by [5], we have that $\operatorname{Ext}_R^n(A, M) = 0$ for any FP-injective module M if and only if K_{n-2} is finitely presented, that is, A is n-presented. $\square$

Theorem 5.6. *The following statements are equivalent for a ring R.*

(1) *R is $(\mathcal{T}, n)$-coherent.*

(2) *$\mathrm{Ext}_R^{n+1}(A, N) = 0$ for any $(\mathcal{T}, n+1)$-presented left R-module A and any FP-injective left R-module N.*

(3) *If N is a $(\mathcal{T}, n)$-injective left R-module, N_1 is an FP-injective submodule of N, then N/N_1 is $(\mathcal{T}, n)$-injective.*

(4) *For any FP-injective left R-module N, $E(N)/N$ is $(\mathcal{T}, n)$-injective, where $E(N)$ is the injective hull of N.*

Proof. (1) $\Rightarrow$ (2). For any $(\mathcal{T}, n+1)$-presented left R-module A, there exists an exact sequence of left R-modules $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$, where F is finitely generated free and K is $(\mathcal{T}, n)$-presented. Since R is $(\mathcal{T}, n)$-coherent, K is n-presented, and so from the exact sequence

$$0 = \mathrm{Ext}_R^n(F, N) \longrightarrow \mathrm{Ext}_R^n(K, N) \longrightarrow \mathrm{Ext}_R^{n+1}(A, N) \longrightarrow \mathrm{Ext}_R^{n+1}(F, N) = 0$$

we have $\mathrm{Ext}_R^{n+1}(A, N) \cong \mathrm{Ext}_R^n(K, N) = 0$ by Lemma 5.5 since N is FP-injective.

(2) $\Rightarrow$ (3). For any $(\mathcal{T}, n+1)$-presented left R-module A, the exact sequence $0 \longrightarrow N_1 \longrightarrow N \longrightarrow N/N_1 \longrightarrow 0$ induces the exactness of the sequence

$$0 = \mathrm{Ext}_R^n(A, N) \longrightarrow \mathrm{Ext}_R^n(A, N/N_1) \longrightarrow \mathrm{Ext}_R^{n+1}(A, N_1) = 0.$$

Therefore $\mathrm{Ext}_R^n(A, N/N_1) = 0$, as required.

(3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (1). Let A be a $(\mathcal{T}, n+1)$-presented left R-module. Then there exists an exact sequence of left R-modules $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$, where F is finitely generated free and K is $(n-1)$-presented. For any FP-injective module N, $E(N)/N$ is $(\mathcal{T}, n)$-injective by (4). From the exactness of the two sequences

$$0 = \mathrm{Ext}_R^n(F, N) \longrightarrow \mathrm{Ext}_R^n(K, N) \longrightarrow \mathrm{Ext}_R^{n+1}(A, N) \longrightarrow \mathrm{Ext}_R^{n+1}(F, N) = 0$$

and

$$0 = \mathrm{Ext}_R^n(A, E(N)) \to \mathrm{Ext}_R^n(A, E(N)/N) \to \mathrm{Ext}_R^{n+1}(A, N) \to \mathrm{Ext}_R^{n+1}(A, E(N)) = 0$$

we have $\mathrm{Ext}_R^n(K, N) \cong \mathrm{Ext}_R^{n+1}(A, N) \cong \mathrm{Ext}_R^n(A, E(N)/N) = 0$. Thus, K is n-presented by Lemma 5.5, and so A is $(n+1)$-presented. Therefore, R is $(\mathcal{T}, n)$-coherent. $\square$

Corollary 5.7. *The following statements are equivalent for a ring R:*

(1) *R is left n-coherent.*

(2) $\mathrm{Ext}_R^{n+1}(A, N) = 0$ *for any n-presented left R-module A and any FP-injective left R-module N.*

(3) *If N is an n-FP-injective left R-module, N_1 is an FP-injective submodule of N, then N/N_1 is n-FP-injective.*

(4) *For any FP-injective left R-module N, $E(N)/N$ is n-FP-injective.*

Corollary 5.8. *Let R be a $(\mathcal{T}, n)$-coherent ring. Then every left R-module has a $(\mathcal{T}, n)$-injective cover.*

Proof. Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a pure exact sequence of left R-modules with B $(\mathcal{T}, n)$-injective. Then $0 \longrightarrow C^+ \longrightarrow B^+ \longrightarrow A^+ \longrightarrow 0$ is split exact. Since R is $(\mathcal{T}, n)$-coherent, B^+ is $(\mathcal{T}, n)$-flat by Theorem 5.3 (8), so C^+ is $(\mathcal{T}, n)$-flat, and hence C is $(\mathcal{T}, n)$-injective by Remark 4.10. Thus, the class of $(\mathcal{T}, n)$-injective modules is closed under pure quotients. By [9], Theorem 2.5, and Proposition 4.6, every left R-module has a $(\mathcal{T}, n)$-injective cover. □

Corollary 5.9. *Let R be a left n-coherent ring. Then every left R-module has an n-FP-injective cover.*

Corollary 5.10. *The following statements are equivalent for a $(\mathcal{T}, n)$-coherent ring R:*

(1) *Every $(\mathcal{T}, n)$-flat right R-module is n-flat.*

(2) *Every $(\mathcal{T}, n)$-injective left R-module is n-FP-injective.*

In this case, R is left n-coherent.

Proof. (1) $\Rightarrow$ (2). Let M be any $(\mathcal{T}, n)$-injective left R-module. Then M^+ is a $(\mathcal{T}, n)$-flat right R-module by Theorem 5.3 (8) since R is $(\mathcal{T}, n)$-coherent, and so M^+ is n-flat by (1). Thus M^{++} is n-FP-injective. Since M is a pure submodule of M^{++} , and a pure submodule of an n-FP-injective module is n-FP-injective, so M is n-FP-injective.

(2) $\Rightarrow$ (1). Let M be any $(\mathcal{T}, n)$-flat right R-module. Then M^+ is a $(\mathcal{T}, n)$-injective left R-module by Theorem 4.8, and so M^+ is n-FP-injective by (2). Thus M is n-flat.

In this case, any direct product of n-flat right R-modules is n-flat by Theorem 5.3 (5), and so R is left n-coherent by Corollary 5.4 (5). □

Proposition 5.11. *The following statements are equivalent for a ring R:*

(1) *Every right R-module has a monic $(\mathcal{T}, n)$-flat preenvelope.*

(2) *R is $(\mathcal{T}, n)$-coherent and $_RR$ is $(\mathcal{T}, n)$-injective.*

(3) *R is $(\mathcal{T}, n)$-coherent and every left R-module has an epic $(\mathcal{T}, n)$-injective cover.*

(4) *R is $(\mathcal{T}, n)$-coherent and every injective right R-module is $(\mathcal{T}, n)$-flat.*

(5) *R is $(\mathcal{T}, n)$-coherent and every flat left R-module is $(\mathcal{T}, n)$-injective.*

Proof. (1) $\Rightarrow$ (4). Assume (1). Then it is clear that R is a $(\mathcal{T}, n)$-coherent ring by Theorem 5.3 (12). Let E be any injective right R-module. E has a monic $(\mathcal{T}, n)$-flat preenvelope F, so E is isomorphic to a direct summand of F, and thus E is $(\mathcal{T}, n)$-flat.

(4) $\Rightarrow$ (5). Let M be a flat left R-module. Then M^+ is injective, and so M^+ is $(\mathcal{T}, n)$-flat by (4). Hence M is $(\mathcal{T}, n)$-injective by Theorem 5.3 (8).

(5) $\Rightarrow$ (2). It is obvious.

(2) $\Rightarrow$ (1). Let M be any right R-module. Then M has a $(\mathcal{T}, n)$-flat preenvelope $f\colon M \to F$ by Theorem 5.3 (12). Since $(_RR)^+$ is a cogenerator, there exists an exact sequence $0 \longrightarrow M \xrightarrow{g} \prod(_RR)^+$. Since $_RR$ is $(\mathcal{T}, n)$-injective, by Theorem 5.3, $\prod(_RR)^+$ is $(\mathcal{T}, n)$-flat, and so there exists a right R-homomorphism $h\colon F \to \prod(_RR)^+$ such that $g = hf$, which shows that f is monic.

(2) $\Rightarrow$ (3). Let M be a left R-module. Then M has a $(\mathcal{T}, n)$-injective cover $\varphi\colon C \to M$ by Corollary 5.8. On the other hand, there is an exact sequence $F \xrightarrow{\alpha} M \longrightarrow 0$ with F free. Since F is $(\mathcal{T}, n)$-injective by (2) and Proposition 4.6, there exists a homomorphism $\beta\colon F \to C$ such that $\alpha = \varphi\beta$. It follows that φ is epic.

(3) $\Rightarrow$ (2). Let $f\colon N \longrightarrow {_RR}$ be an epic $(\mathcal{T}, n)$-injective cover. Then the projectivity of $_RR$ implies that $_RR$ is isomorphic to a direct summand of N, and so $_RR$ is $(\mathcal{T}, n)$-injective. □

Corollary 5.12. *The following statements are equivalent for a ring R:*

(1) *Every right R-module has a monic n-flat preenvelope.*

(2) *R is left n-coherent and $_RR$ is n-FP-injective.*

(3) *R is left n-coherent and every left R-module has an epic n-FP-injective cover.*

(4) *R is left n-coherent and every injective right R-module is n-flat.*

(5) *R is left n-coherent and every flat left R-module is n-FP-injective.*

Acknowledgment. The author wishes to thank the referee for careful reading of the paper and giving a detailed and helpful report.

References

[1] *S. U. Chase*: Direct products of modules. Trans. Am. Math. Soc. *97* (1960), 457–473. zbl MR doi
[2] *T. J. Cheatham, D. R. Stone*: Flat and projective character modules. Proc. Am. Math. Soc. *81* (1981), 175–177. zbl MR doi
[3] *J. Chen, N. Ding*: On n-coherent rings. Commun. Algebra *24* (1996), 3211–3216. zbl MR doi
[4] *D. L. Costa*: Parameterizing families of non-Noetherian rings. Commun. Algebra *22* (1994), 3997–4011. zbl MR doi
[5] *E. Enochs*: A note on absolutely pure modules. Canad. Math. Bull. *19* (1976), 361–362. zbl MR doi
[6] *E. E. Enochs, O. M. G. Jenda*: Relative Homological Algebra. De Gruyter Expositions in Mathematics 30, Walter de Gruyter, Berlin, 2000. zbl MR doi
[7] *E. E. Enochs, O. M. G. Jenda, J. A. Lopez-Ramos*: The existence of Gorenstein flat covers. Math. Scand. *94* (2004), 46–62. zbl MR doi
[8] *M. Finkel Jones*: Coherence relative to an hereditary torsion theory. Commun. Algebra *10* (1982), 719–739. zbl MR doi
[9] *H. Holm, P. Jørgensen*: Covers, precovers, and purity. Illinois J. Math. *52* (2008), 691–703. zbl MR
[10] *L. Mao, N. Ding*: Relative coherence of rings. J. Algebra Appl. *11* (2012), 1250047, 16 pages. zbl MR doi
[11] *C. Megibben*: Absolutely pure modules. Proc. Am. Math. Soc. *26* (1970), 561–566. zbl MR doi
[12] *J. J. Rotman*: An Introduction to Homological Algebra. Pure and Applied Mathematics 85, Academic Press, Harcourt Brace Jovanovich Publishers, New York-London, 1979. zbl MR
[13] *B. Stenström*: Coherent rings and FP-injective modules. J. Lond. Math. Soc., II. Ser. *2* (1970), 323–329. zbl MR doi
[14] *B. Stenström*: Rings of Quotients. An Introduction to Methods of Ring Theory. Die Grundlehren der mathematischen Wissenschaften, Band 217, Springer, New York, 1975. zbl MR doi
[15] *J. Trlifaj*: Cover, Envelopes, and Cotorsion Theories. Lecture notes for the workshop. Homological Methods in Module Theory, Cortona, 2000.
[16] *R. Wisbauer*: Foundations of Module and Ring Theory. A Handbook for Study and Research. Algebra, Logic and Applications 3, Gordon and Breach Science Publishers, Philadelphia, 1991. zbl MR
[17] *X. Yang, Z. Liu*: n-flat and n-FP-injective modules. Czech. Math. J. *61* (2011), 359–369. zbl MR doi
[18] *D. Zhou*: On n-coherent rings and (n, d)-rings. Commun. Algebra *32* (2004), 2425–2441. zbl MR doi

Author's address: Zhanmin Zhu, Department of Mathematics, Jiaxing University, 118 Jiahang Rd, Nanhu, 314001 Jiaxing, Zhejiang, P. R. China e-mail: `zhuzhanminzjxu@hotmail.com`.