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EXTENSION PRINCIPLE AND CONTROLLER DESIGN FOR SYSTEMS WITH DISTRIBUTED TIME-DELAY

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Extension principle is defined for systems with distributed time-delay and the necessary and sufficient conditions for one system being an extension of the other are presented. Preservation of stability properties between two systems, one of which is an extension of the other is also discussed and it is shown that when the expanded system is an extension of the original system, (i) the original system is bounded-input bounded-output stable if and only if the expanded system is bounded-input bounded-output stable and (ii) the original system is exponentially stable if the expanded system is exponentially stable. Controller design using the extension principle is then considered. It is shown that, if the expanded system is an extension of the original system, then any controller designed for the expanded system can be contracted for implementation on the original system. Furthermore, if the controller designed for the expanded system stabilizes the expanded system and satisfies certain performance requirements, then the contracted controller stabilizes the original system and satisfies corresponding performance requirements for the original system. Finally, overlapping decompositions and controller design using overlapping decompositions are demonstrated. A highway traffic congestion control problem is then considered to demonstrate a possible application of the presented controller design approach.

Keywords: large-scale systems, time-delay systems, distributed time-delay, overlapping decompositions, decentralized control, controller design

Classification: 93A14, 93A15, 93C23

1. INTRODUCTION

Many physical systems usually involve time-delays [31]. This is especially true for large-scale systems [4, 10, 28]. Systems which involve time-delay are usually termed as time-delay systems [42]. Such systems are infinite-dimensional, since their state can not be represented by a finite number of state variables [12]. Time-delays in a system can be discrete or distributed [9]. Systems with distributed time-delays can appear in many applications such as traffic flow [30, 44], combustion [47], logistics [8], neural networks [2, 40], and biology [32, 41]. Furthermore, many systems may involve both discrete and distributed time-delays together [29]. Both discrete and distributed time-delays, however, can be combined in the same model by using Dirac delta functions [41]. In this sense, distributed-time-delay systems are more general than discrete-time-delay systems.

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Decomposition techniques are usually necessary when designing controllers for large-scale systems [43]. Many systems, however, may involve subsystems which are strongly interconnected through certain dynamics but loosely interconnected otherwise. The approach of *overlapping decompositions* has first been introduced by Ikeda and Šiljak [24] in order to obtain useful decompositions for such systems. Since its introduction, this approach has been used successfully to design controllers for many large-scale systems (e. g., [1, 3, 15, 19, 21, 23, 26, 35, 45]). The theoretical framework of the overlapping decompositions is the *inclusion principle.* This principle was first introduced for finite-dimensional systems whose state-space is overlappingly decomposed by Ikeda et. al. [27]. This principle was then extended to finite-dimensional systems whose input and output spaces are also overlappingly decomposed by Ikeda and Šiljak [25]. A special case of inclusion, called *extension*, was then introduced by İftar and Özgüner [22]. The advantage of using extension, rather than the more general inclusion, is that any controller designed in the expanded spaces can be contracted for implemention on the original system if extension is used. When inclusion (or another special case, called unrestriction, which was introduced in [25]) is used, there is no guarantee of contractibility of controllers designed in the expanded spaces. Although extension was first introduced for systems whose state and input spaces are overlappingly decomposed, it was later extended to systems whose output space is also overlappingly decomposed by İftar [13]. Overlappingly decentralized controller and estimator design using extension was also discussed in [13].

Although overlapping decompositions and the inclusion and extension principles for finite-dimensional systems have been well discussed in the literature, as described in the previous paragraph, consideration of these concepts for time-delay systems have been very recent (e. g., [5, 6, 7, 16]). Furthermore, most of this literature has been restricted to discrete-time-delay systems. To the author's best knowledge, overlapping decompositions and the inclusion principle were first considered for distributed-time-delay systems in [17], where the inclusion principle was defined and overlapping decompositions of distributed-time-delay systems were presented. Robust controller design for large-scale distributed-time-delay systems using inclusion and overlapping decompositions was then presented in [18]. Both in [17] and [18], however, only the overlapping decompositions of the state space was considered. In the present work, we extend the results of [17] to the case where the input and the output spaces are also overlappingly decomposed besides the state space. We first define the principle of extension in Section 2 and present the necessary and sufficient conditions for one system being an extension of the other. We then show in Section 3 that, certain stability properties are preserved between two systems if one is an extension of the other. Controller design is then considered in Section 4, where we show that any controller designed in the expanded spaces is contractible to a controller for implementation on the original system if the expanded system is an extension of the original system. We further show in Section 4 that, if the controller designed for the expanded system stabilizes the expanded system and satisfies certain performance requirements, then the contracted controller stabilizes the original system and satisfies corresponding requirements for the original system. The overlapping decompositions and controller design using overlapping decompositions are then demonstrated in Section 5. A possible application of the presented results is demonstrated in Section 6, where a highway traffic congestion control problem is considered. Finally, some concluding remarks are included in Section 7.

Throughout the paper, $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of, respectively, real numbers and non-negative real numbers. For $a, b \in \mathbb{R}$ with $a \leq b$, $[a, b] := \{\theta \in \mathbb{R} \mid a \leq \theta \leq b\}$. For positive integers k and l, $\mathbb{R}^k$ and $\mathbb{R}^{k\times l}$ denote the spaces of, respectively, k-dimensional real vectors and $k \times l$-dimensional real matrices. For $x \in \mathbb{R}^k$, $\|x\|$ is the 2-norm of x, for $X \in \mathbb{R}^{k\times l}$, $\|X\|$ is the induced 2-norm (i. e., the maximum singular value) of X, and for $\xi : [a, b] \to \mathbb{R}^k$, $\|\xi\| := \int_a^b \|\xi(\theta)\| \,\mathrm{d}\theta$. I_k denotes the $k \times k$-dimensional identity matrix. 0 may denote either the scalar zero, a zero vector, a zero matrix, or a matrix function which is identically zero. $\cdot^T$ is the transpose of $\cdot$ and i denotes the imaginary unit. For a vector function $x(\cdot)$, $\dot{x}(t)$ is the derivative of $x(t)$ with respect to t.

2. EXTENSION

In this section, we introduce the principle of *extension* for systems with distributed time-delay. Consider the linear time-invariant (LTI) retarded distributed-time-delay systems, Σ:

$$\dot{x}(t) = \int_{-\tau}^{0} (A(\theta)x(t+\theta) + B(\theta)u(t+\theta)) \,\mathrm{d}\theta \tag{1}$$

$$y(t) = \int_{-\tau}^{0} C(\theta)x(t+\theta) \,\mathrm{d}\theta \tag{2}$$

and $\hat{\Sigma}$:

$$\dot{\hat{x}}(t) = \int_{-\tau}^{0} \left(\hat{A}(\theta)\hat{x}(t+\theta) + \hat{B}(\theta)\hat{u}(t+\theta)\right) \mathrm{d}\theta \tag{3}$$

$$\hat{y}(t) = \int_{-\tau}^{0} \hat{C}(\theta)\hat{x}(t+\theta) \,\mathrm{d}\theta \tag{4}$$

where $\tau > 0$ is the maximum time-delay in Σ and $\hat{\Sigma}$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, and $y(t) \in \mathbb{R}^q$ are, respectively, the state, the input, and the output vectors of Σ, and $\hat{x}(t) \in \mathbb{R}^{\hat{n}}$, $\hat{u}(t) \in \mathbb{R}^{\hat{p}}$, and $\hat{y}(t) \in \mathbb{R}^{\hat{q}}$ are, respectively, the state, the input, and the output vectors of $\hat{\Sigma}$ at time t. Here, it is assumed that the outputs, $y(t)$ and $\hat{y}(t)$ are measurable and the inputs $u(t)$ and $\hat{u}(t)$ are accessible. It is also assumed that the dimensions of the state, input, and output vectors of $\hat{\Sigma}$ are greater than or equal to the dimensions of the state, input, and output vectors of Σ, respectively; i. e., $\hat{n} \geq n$, $\hat{p} \geq p$, and $\hat{q} \geq q$. In the sequel, Σ and $\hat{\Sigma}$ are respectively referred to as the *original* and the *expanded* systems. Similarly, the state/input/output spaces of Σ are referred to as the *original spaces* and those of $\hat{\Sigma}$ are referred to as the *expanded spaces*. It is further assumed that $A(\cdot) : [-\tau, 0] \to \mathbb{R}^{n\times n}$, $B(\cdot) : [-\tau, 0] \to \mathbb{R}^{n\times p}$, $C(\cdot) : [-\tau, 0] \to \mathbb{R}^{q\times n}$, $\hat{A}(\cdot) : [-\tau, 0] \to \mathbb{R}^{\hat{n}\times\hat{n}}$, $\hat{B}(\cdot) : [-\tau, 0] \to \mathbb{R}^{\hat{n}\times\hat{p}}$, and $\hat{C}(\cdot) : [-\tau, 0] \to \mathbb{R}^{\hat{q}\times\hat{n}}$ are bounded matrix functions, except that they may contain Dirac delta terms, $\delta(\cdot)$. By this assumption, Σ and $\hat{\Sigma}$ are allowed to have discrete time-delays besides distributed time-delays, since a discrete time-delay of h, where $0 \leq h \leq \tau$, in, e. g., $x(\cdot)$, can be represented as

$$x(t-h) = \int_{-\tau}^{0} \delta(\theta + h) x(t+\theta)\, \mathrm{d}\theta \,. \tag{5}$$

Finally, the initial states for Σ and $\hat{\Sigma}$ are assumed to be given as:

$$x(\theta) = \phi(\theta) \quad \text{and} \quad \hat{x}(\theta) = \hat{\phi}(\theta) \,, \quad \theta \in [-\tau, 0] \,, \tag{6}$$

for some functions $\phi : [-\tau, 0] \to \mathbb{R}^n$ and $\hat{\phi} : [-\tau, 0] \to \mathbb{R}^{\hat{n}}$, respectively. We also note that the *state*s of Σ and $\hat{\Sigma}$ at time $t \geq 0$ are respectively given by the functions $x_t : [-\tau, 0] \to \mathbb{R}^n$ and $\hat{x}_t : [-\tau, 0] \to \mathbb{R}^{\hat{n}}$, which are respectively defined as $x_t(\theta) := x(t+\theta)$ and $\hat{x}_t(\theta) := \hat{x}(t+\theta)$, for $\theta \in [-\tau, 0]$, where $x(\cdot)$ and $\hat{x}(\cdot)$ are the unique solutions to (1) and (3), respectively, with the initial states as in (6) [12].

Now, we can define extension.

Definition 1. $\hat{\Sigma}$ is said to be an *extension* of Σ and Σ is said to be a *disextension* of $\hat{\Sigma}$ if there exist full-rank matrices

$$T \in \mathbb{R}^{\hat{n} \times n} \,, \quad R \in \mathbb{R}^{p \times \hat{p}} \,, \quad \text{and} \quad S \in \mathbb{R}^{\hat{q} \times q} \tag{7}$$

such that for all $\phi(\cdot)$ and for all $\hat{u}(\cdot)$, the choice

$$\hat{\phi}(\theta) = T\phi(\theta) \,, \quad \theta \in [-\tau, 0] \tag{8}$$

and

$$u(t) = R\hat{u}(t) \,, \quad t \geq -\tau \tag{9}$$

implies

$$\hat{x}(t) = Tx(t) \,, \quad t \geq -\tau \tag{10}$$

and

$$\hat{y}(t) = Sy(t) \,, \quad t \geq 0 \,. \tag{11}$$

Remark 1. The extension defined in Definition 1 is a generalization of *extension*, which was first defined for finite-dimensional systems in [13]. As indicated in [13], extension is a generalization of *unrestriction*, first defined for finite-dimensional systems in [27] and for time-delay systems in [17], to the case where input and output spaces are also expanded besides the state space. However, it is different than the unrestriction defined in [25], which was defined for an arbitrary input $u(t)$ in the original input space, and the input in the expanded space is obtained by a transformation of the form $\hat{u}(t) = \tilde{R}u(t)$, where $\tilde{R} \in \mathbb{R}^{\hat{p} \times p}$ is a full-rank matrix. Here, however, as in [13], the extension is defined for an arbitrary input $\hat{u}(t)$ in the expanded input space and the input in the original space is obtained by (9). This choice results in *contractibility* of any controller designed in the expanded spaces to a controller which can be implemented on the original system (see Corollary 2 below). As it was shown in [25], if unrestriction is used, a controller designed in the expanded spaces may not be contractible even in the case of finite-dimensional systems.

The following theorem gives the necessary and sufficient conditions for $\hat{\Sigma}$ to be an *extension* of Σ.

Theorem 1. $\hat{\Sigma}$ is an *extension* of Σ and Σ is a *disextension* of $\hat{\Sigma}$ if and only if there exist full-rank matrices as in (7) such that

$$\hat{A}(\theta)T = TA(\theta)\ , \qquad \forall\theta \in [-\tau, 0]\ , \tag{12}$$

$$\hat{B}(\theta) = TB(\theta)R\ , \qquad \forall\theta \in [-\tau, 0]\ , \tag{13}$$

and

$$\hat{C}(\theta)T = SC(\theta)\ , \qquad \forall\theta \in [-\tau, 0]\ . \tag{14}$$

Proof. To prove the if part, first note that (10) for $t \in [-\tau, 0]$ is implied by (8). To establish (10) for $t \geq 0$, premultiply both sides of (1) by T and use (9) to obtain

$$T\dot{x}(t) = \int_{-\tau}^{0} (TA(\theta)x(t+\theta) + TB(\theta)R\hat{u}(t+\theta))\,\mathrm{d}\theta\ . \tag{15}$$

Next, use (12) and (13) in the right-hand-side of (15) and compare with (3). Together with (8), by the uniqueness of solutions [12], this establishes (10) for $t \geq 0$. Next, to establish (11), premultiply both sides of (2) by S to obtain

$$Sy(t) = \int_{-\tau}^{0} SC(\theta)x(t+\theta)\,\mathrm{d}\theta \tag{16}$$

Next, use (14) and (10) in the right-hand-side of (16) and compare with (4). This establishes (11), which concludes the proof of the if part.

Next, to prove the only if part, consider (15) at $t = 0$, in which case, $x(t+\theta)$ in the right-hand-side is replaced by $\phi(\theta)$. Also, consider (3) at $t = 0$, in which case, $\hat{x}(t+\theta)$ in the right-hand-side is replaced by $\hat{\phi}(\theta)$, which, by (8), is $T\phi(\theta)$. Now, suppose that $\hat{u}(\theta) = 0$, for $\theta \in [-\tau, 0]$, and that (12) is not satisfied. Then this implies, for some $\phi(\cdot)$, $\dot{\hat{x}}(0) \neq T\dot{x}(0)$, which implies that (10) does not hold. Since $\phi(\cdot)$ is arbitrary, this establishes the necessity of (12). Next, to estbalish the necessity of (13), suppose that $\phi(\theta) = 0$, for $\theta \in [-\tau, 0]$, and that (13) is not satisfied. Then this implies, for some $\hat{u}(\cdot)$, $\dot{\hat{x}}(0) \neq T\dot{x}(0)$, which implies that (10) does not hold. Since $\hat{u}(\cdot)$ is arbitrary, this establishes the necessity of (13). Finally, to estbalish the necessity of (14), consider (16) at $t = 0$, in which case, $x(t+\theta)$ in the right-hand-side is replaced by $\phi(\theta)$. Also, consider (4) at $t = 0$, in which case, $\hat{x}(t+\theta)$ in the right-hand-side is replaced by $\hat{\phi}(\theta)$, which, by (8), is $T\phi(\theta)$. Now, suppose that (14) is not satisfied. Then this implies, for some $\phi(\cdot)$, $\hat{y}(0) \neq Sy(0)$, which implies that (11) does not hold. Since $\phi(\cdot)$ is arbitrary, this establishes the necessity of (14). This completes the proof. □

For finite-dimensional systems, it is common to relate the matrices of the two systems by using complementary matrices. Similarly, here we can relate the matrix functions of Σ and $\hat{\Sigma}$ by using *complementary matrix functions.* Using T, R, and S, which are introduced in (7), and $\tilde{T} \in \mathbb{R}^{n\times\hat{n}}$, which satisfies $\tilde{T}T = I_n$ (there exists such $\tilde{T}$, since T is of full-rank and $\hat{n} \geq n$), the matrix functions of Σ and $\hat{\Sigma}$ can be related as

$$\hat{A}(\theta) = TA(\theta)\tilde{T} + M(\theta) \tag{17}$$

$$\hat{B}(\theta) = TB(\theta)R + N(\theta) \tag{18}$$

$$\hat{C}(\theta) = SC(\theta)\tilde{T} + L(\theta) \tag{19}$$

for $\theta \in [-\tau, 0]$. Here, $M(\cdot)$, $N(\cdot)$, and $L(\cdot)$ are called as complementary matrix functions, which are bounded matrix functions, except that they may contain Dirac delta terms. As will be shown in Section 5, by defining these matrix functions appropriately, it will be possible to expand an overllappingly decomposed system into a system where the subsystems appear as disjoint. For the expanded system to be an extension of the original system, however, these matrix functions must satisfy certain relations, which are given by the following result.

Corollary 1. $\hat{\Sigma}$ is an *extension* of Σ and Σ is a *disextension* of $\hat{\Sigma}$ if and only if

$$M(\theta)T = 0\ , \quad N(\theta) = 0\ , \quad \text{and} \quad L(\theta)T = 0\ , \quad \forall \theta \in [-\tau, 0]\ . \tag{20}$$

Proof. Postmultiply both sides of (17) by T, use $\tilde{T}T = I_n$, and compare with (12). This establishes the first condition in (20). The second condition in (20) directly follows from (13). To establish the last condition in (20), postmultiply both sides of (19) by T, use $\tilde{T}T = I_n$, and compare with (14). This completes the proof. □

3. STABILITY

When $\hat{\Sigma}$ is an extension of Σ, some important properties, like stability, are preserved between the two systems. Here we consider both input-output stability and internal stability. For the case of input-output stability, we use the most common definition, *bounded-input bounded-output (BIBO) stability.* For internal stability, we use *exponential stability*, since it is the strongest internal stability definition commonly used for linear systems. For the sake of completeness, let us first present these definitions. In Definition 3, by ψ we indicate the initial state of the system (thus, for Σ, $\psi(\theta) = \phi(\theta)$, $\theta \in [-\tau, 0]$, and for $\hat{\Sigma}$, $\psi(\theta) = \hat{\phi}(\theta)$, $\theta \in [-\tau, 0]$) and by ξ_t we indicate the state of the system at time t (thus, for Σ, $\xi_t(\theta) = x_t(\theta)$, $\theta \in [-\tau, 0]$, and for $\hat{\Sigma}$, $\xi_t(\theta) = \hat{x}_t(\theta)$, $\theta \in [-\tau, 0]$).

Definition 2. A system, such as Σ or $\hat{\Sigma}$, is said to be *BIBO stable* if, in response to any bounded input, it produces a bounded output when its initial state is zero.

Definition 3. A system, such as Σ or $\hat{\Sigma}$, is said to be *exponentially stable* if there exist $\alpha, \beta > 0$, such that, when the system input is zero, for any bounded initial state, $\|\xi_t\| \le \alpha \|\psi\| e^{-\beta t}$, $\forall t \ge 0$.

Now we can establish the following.

Theorem 2. If $\hat{\Sigma}$ is an extension of Σ, then Σ is BIBO stable if and only if $\hat{\Sigma}$ is BIBO stable.

Proof. When Σ and $\hat{\Sigma}$ have both zero initial states, then (8) holds. Now, suppose that Σ is BIBO stable. For any bounded $\hat{u}(\cdot)$, any $u(\cdot)$, which satisfies (9), will be bounded. Then, since Σ is BIBO stable, $y(\cdot)$ will be bounded. Then, (11) implies that $\hat{y}(\cdot)$ will be bounded. Thus, $\hat{\Sigma}$ will be BIBO stable. Next, suppose that $\hat{\Sigma}$ is BIBO stable. Since S is of full-rank and $\hat{q} \geq q$, there exists $\tilde{S} \in \mathbb{R}^{q \times \hat{q}}$ such that $\tilde{S}S = I_q$. Then, (11) implies

$$y(t) = \tilde{S}\hat{y}(t) \; , \qquad t \geq 0 \; . \tag{21}$$

Furthermore, since R is of full-rank and $\hat{p} \geq p$, there exists $\tilde{R} \in \mathbb{R}^{\hat{p} \times p}$ such that $R\tilde{R} = I_p$. Then, for any bounded $u(\cdot)$, $\hat{u}(\cdot) := \tilde{R}u(\cdot)$ is bounded and satisfies (9). Then, since $\hat{\Sigma}$ is BIBO stable, corresponding $\hat{y}(\cdot)$ will be bounded. Then, (21) implies that $y(\cdot)$ will be bounded. Thus, Σ will be BIBO stable. □

Theorem 3. If $\hat{\Sigma}$ is an extension of Σ, then Σ is exponentially stable if $\hat{\Sigma}$ is exponentially stable.

Proof. When Σ and $\hat{\Sigma}$ have both zero inputs, then (9) holds. Now, suppose that $\hat{\Sigma}$ is exponentially stable. Then, there exist $\alpha, \beta > 0$, such that, for any bounded $\hat{\phi}(\cdot)$,

$$\|\hat{x}_t\| \leq \alpha \|\hat{\phi}\| e^{-\beta t} \; , \qquad \forall t \geq 0 \; . \tag{22}$$

Since T is of full-rank and $\hat{n} \geq n$, there exists $\tilde{T} \in \mathbb{R}^{n \times \hat{n}}$ such that $\tilde{T}T = I_n$. Then, (10) implies $x(t) = \tilde{T}\hat{x}(t)$, for $t \geq -\tau$, which implies

$$\|x_t\| \leq \|\tilde{T}\| \|\hat{x}_t\| \; , \qquad \forall t \geq 0 \; . \tag{23}$$

Furthermore, for any bounded $\phi(\cdot)$, by (8), $\hat{\phi}(\cdot)$ is bounded and $\|\hat{\phi}\| \leq \|T\| \|\phi\|$. Now, use (22) in (23) to obtain

$$\|x_t\| \leq \hat{\alpha} \|\phi\| e^{-\beta t} \; , \qquad \forall t \geq 0 \; , \tag{24}$$

where $\hat{\alpha} := \alpha \|\tilde{T}\| \|T\|$. This, however, implies that Σ is exponentially stable. □

Remark 2. Converse of Theorem 3 does not hold in general, since $\hat{x}_t$ may not be bounded for some $\hat{\phi}$ which is not in the range space of T. However, since the controllers are designed in the expanded space to satisfy stability of $\hat{\Sigma}$ and then contracted for implementation on Σ (see the next section), the direction in Theorem 3 is the important result.

4. CONTROLLER DESIGN

In this section, we consider output feedback controller design for a LTI distributed-time-delay system, like Σ or $\hat{\Sigma}$. For this purpose, we consider LTI controllers, which are dynamic, in general, and which may be subject to distributed time-delays. Thus, we consider a controller, to be denoted by Γ, of the form

$$\dot{z}(t) = \int_{-\sigma}^{0} \left(F(\theta) z(t+\theta) + G(\theta) w(t+\theta) \right) \mathrm{d}\theta \tag{25}$$

$$v(t) = \int_{-\sigma}^{0} \left(H(\theta) z(t+\theta) + K(\theta) w(t+\theta) \right) \mathrm{d}\theta \tag{26}$$

for Σ, and a controller, to be denoted by $\hat{\Gamma}$, of the form

$$\dot{\hat{z}}(t) = \int_{-\sigma}^{0} \Big(\hat{F}(\theta)\hat{z}(t+\theta) + \hat{G}(\theta)\hat{w}(t+\theta)\Big)\,\mathrm{d}\theta \tag{27}$$

$$\hat{v}(t) = \int_{-\sigma}^{0} \Big(\hat{H}(\theta)\hat{z}(t+\theta) + \hat{K}(\theta)\hat{w}(t+\theta)\Big)\,\mathrm{d}\theta \tag{28}$$

for $\hat{\Sigma}$. Here, $\sigma > 0$ is the maximum time-delay in Γ and $\hat{\Gamma}$, $z(t) \in \mathbb{R}^m$ and $\hat{z}(t) \in \mathbb{R}^{\hat{m}}$ are the state vectors, $w(t) \in \mathbb{R}^q$ and $\hat{w}(t) \in \mathbb{R}^{\hat{q}}$ are the input vectors, and $v(t) \in \mathbb{R}^p$ and $\hat{v}(t) \in \mathbb{R}^{\hat{p}}$ are the output vectors of, respectively, Γ and $\hat{\Gamma}$ at time t. Here, it is assumed that $\hat{m} \geq m$. This assumption is justified since Σ is, in general, forms a part of $\hat{\Sigma}$, and thus should not require a controller with larger dimensional state vector. Furthermore, it is assumed that $F(\cdot) : [-\sigma, 0] \to \mathbb{R}^{m\times m}$, $G(\cdot) : [-\sigma, 0] \to \mathbb{R}^{m\times q}$, $H(\cdot) : [-\sigma, 0] \to \mathbb{R}^{p\times m}$, $K(\cdot) : [-\sigma, 0] \to \mathbb{R}^{p\times q}$, $\hat{F}(\cdot) : [-\sigma, 0] \to \mathbb{R}^{\hat{m}\times\hat{m}}$, $\hat{G}(\cdot) : [-\sigma, 0] \to \mathbb{R}^{\hat{m}\times\hat{q}}$, $\hat{H}(\cdot) : [-\sigma, 0] \to \mathbb{R}^{\hat{p}\times\hat{m}}$, and $\hat{K}(\cdot) : [-\sigma, 0] \to \mathbb{R}^{\hat{p}\times\hat{q}}$ are bounded matrix functions, except that they may contain Dirac delta terms. Finally, the initial states for Γ and $\hat{\Gamma}$ are assumed to be given as:

$$z(\theta) = \zeta(\theta) \quad \text{and} \quad \hat{z}(\theta) = \hat{\zeta}(\theta)\,, \quad \theta \in [-\sigma, 0]\,, \tag{29}$$

for some functions $\zeta : [-\sigma, 0] \to \mathbb{R}^m$ and $\hat{\zeta} : [-\sigma, 0] \to \mathbb{R}^{\hat{m}}$, respectively.

The controllers Γ and $\hat{\Gamma}$ are to be applied to Σ and $\hat{\Sigma}$, respectively, by letting

$$w(t) = y(t) - r(t) \quad \text{and} \quad \hat{w}(t) = \hat{y}(t) - \hat{r}(t) \tag{30}$$

and

$$u(t) = v(t) + e(t) \quad \text{and} \quad \hat{u}(t) = \hat{v}(t) + \hat{e}(t) \tag{31}$$

for $t \geq 0$, where $r(t) \in \mathbb{R}^q$ and $\hat{r}(t) \in \mathbb{R}^{\hat{q}}$ are the *reference inputs* and $e(t) \in \mathbb{R}^p$ and $\hat{e}(t) \in \mathbb{R}^{\hat{p}}$ are the *external inputs* (which may be, e. g., actuator disturbance) for, respectively, Σ and $\hat{\Sigma}$ at time t.

Remark 3. The controllers of the form (25)–(26) and (27)–(28) are quite general. They include all LTI retarded output feedback controllers with distributed and/or discrete (by using Dirac delta terms) time-delays. They also include finite-dimensional dynamic and static LTI output feedback controllers. By letting $\sigma = 0^+$, $F(\theta) = F_o\delta(\theta)$, $G(\theta) = G_o\delta(\theta)$, $H(\theta) = H_o\delta(\theta)$, and $K(\theta) = K_o\delta(\theta)$, where F_o, G_o, H_o, and K_o are appropriately dimensioned constant matrices, (25)–(26) reduces to a finite-dimensional dynamic LTI controller. Furthermore, if $m = 0$, then (25)–(26) further reduces to a finite-dimensional static LTI controller of the form $v(t) = K_o w(t)$.

Since the outputs, v and $\hat{v}$, of Γ and $\hat{\Gamma}$ are respectively to be applied to the inputs of Σ and $\hat{\Sigma}$, to satisfy condition (9) following the application of the controllers, the following property must be satisfied.

Definition 4. Suppose that the connection in (30) is made but the connection in (31) is not made. The controller $\hat{\Gamma}$ for $\hat{\Sigma}$ is said to be *contractible* to the controller Γ for Σ if there exist full-rank matrices

$$T \in \mathbb{R}^{\hat{n}\times n}\,, \quad R \in \mathbb{R}^{p\times\hat{p}}\,, \quad S \in \mathbb{R}^{\hat{q}\times q}\,, \quad \text{and} \quad P \in \mathbb{R}^{m\times\hat{m}} \tag{32}$$

such that for all initial states $\phi(\cdot)$ of Σ, for all inputs $\hat{u}(\cdot)$ of $\hat{\Sigma}$, for all reference inputs $r(\cdot)$ of Σ, and for all initial states $\hat{\zeta}(\cdot)$ of $\hat{\Gamma}$, the choice (8), (9),

$$\zeta(\theta) = P\hat{\zeta}(\theta) , \qquad \theta \in [-\sigma, 0] \tag{33}$$

and

$$\hat{r}(t) = Sr(t) , \qquad t \geq -\sigma \tag{34}$$

implies

$$z(t) = P\hat{z}(t) , \qquad t \geq -\sigma \tag{35}$$

and

$$v(t) = R\hat{v}(t) , \qquad t \geq 0 . \tag{36}$$

Remark 4. Although we require $u(t) = R\hat{u}(t)$, for $t \geq -\tau$ in (9), we only require $v(t) = R\hat{v}(t)$, for $t \geq 0$ in (36). This is because the controllers are assumed to be applied at time $t = 0$.

The following theorem presents the conditions for $\hat{\Gamma}$ to be contractible to Γ when $\hat{\Sigma}$ is an extension of Σ.

Theorem 4. Suppose that $\hat{\Sigma}$ is an extension of Σ. Then, the controller $\hat{\Gamma}$ for $\hat{\Sigma}$ is contractible to the controller Γ for Σ if there exists a full-rank matrix $P \in \mathbb{R}^{m \times \hat{m}}$ such that

$$F(\theta)P = P\hat{F}(\theta) , \qquad G(\theta) = P\hat{G}(\theta)S , \qquad H(\theta)P = R\hat{H}(\theta) , \tag{37}$$

and

$$K(\theta) = R\hat{K}(\theta)S , \tag{38}$$

for $\theta \in [-\sigma, 0]$, where R and S are as in (7).

P r o o f. Since the connection (30) is made at time $t = 0$, we have

$$w(t) = \begin{cases} -r(t) , & t < 0 \\ y(t) - r(t) , & t \geq 0 \end{cases} \qquad \text{and} \qquad \hat{w}(t) = \begin{cases} -\hat{r}(t) , & t < 0 \\ \hat{y}(t) - \hat{r}(t) , & t \geq 0 \end{cases} .$$

Then, (11) and (34) implies

$$\hat{w}(t) = Sw(t) , \qquad t \geq -\sigma . \tag{39}$$

Now, note that (35) for $t \in [-\sigma, 0]$ is implied by (33). To establish (35) for $t \geq 0$, premultiply both sides of (27) by P and use (39) to obtain

$$P\dot{\hat{z}}(t) = \int_{-\sigma}^{0} \Big(P\hat{F}(\theta)\hat{z}(t+\theta) + P\hat{G}(\theta)Sw(t+\theta) \Big) \, \mathrm{d}\theta . \tag{40}$$

Next, use the first two equations in (37) in the right-hand-side of (40) and compare with (25). Together with (33), by the uniqueness of solutions, this establishes (35) for $t \geq 0$.

Next, to establish (36), premultiply both sides of (28) by R and use (39) and the last equation in (37) to obtain

$$R\hat{v}(t) = \int_{-\sigma}^{0} \Big(H(\theta)P\hat{z}(t+\theta) + R\hat{K}(\theta)Sw(t+\theta)\Big)\,\mathrm{d}\theta\ . \tag{41}$$

Next, use (35) and (38) in the right-hand-side of (41) and compare with (26). This establishes (36), which concludes the proof. □

As mentioned in Remark 1, since the controllers are to be designed in the expanded spaces and then to be contrated for implementation on the original system, it is important that any controller $\hat{\Gamma}$ for $\hat{\Sigma}$ to be contractible to a controller Γ for Σ. The following corollary proves that this is indeed the case if $\hat{\Sigma}$ is an extension of Σ.

Corollary 2. If $\hat{\Sigma}$ is an extension of Σ, then any controller $\hat{\Gamma}$ of the form (27)–(28) for $\hat{\Sigma}$ is contractible to a controller Γ of the form (25)–(26) for Σ with

$$F(\theta) = \hat{F}(\theta)\ , \qquad G(\theta) = \hat{G}(\theta)S\ , \qquad H(\theta) = R\hat{H}(\theta)\ , \tag{42}$$

and

$$K(\theta) = R\hat{K}(\theta)S\ , \tag{43}$$

for $\theta \in [-\sigma, 0]$, where R and S are as in (7).

Proof. With $m = \hat{m}$ and $P = I_m$, (42)–(43) are equivalent to (37)–(38). □

Now, suppose that Γ is applied to Σ and $\hat{\Gamma}$ is applied to $\hat{\Sigma}$ by making the connections in (30) and (31) at time $t = 0$. Let us denote the closed-loop system obtained by applying Γ to Σ by Σ_c and the closed-loop system obtained by applying $\hat{\Gamma}$ to $\hat{\Sigma}$ by $\hat{\Sigma}_c$. The input to Σ_c is $\begin{bmatrix} e(t) \\ r(t) \end{bmatrix}$ and the output from Σ_c is $\begin{bmatrix} y(t) \\ v(t) \end{bmatrix}$. Similarly, the input to $\hat{\Sigma}_c$ is $\begin{bmatrix} \hat{e}(t) \\ \hat{r}(t) \end{bmatrix}$ and the output from $\hat{\Sigma}_c$ is $\begin{bmatrix} \hat{y}(t) \\ \hat{v}(t) \end{bmatrix}$. The total time-delay in both Σ_c and $\hat{\Sigma}_c$ is $2\tau + \sigma$. However, since the loops are assumed to be closed at time $t = 0$, the effect of the initial state of the controller on the plant and vice-versa, for $t < 0$, is zero. Therefore, the initial states of Σ_c and $\hat{\Sigma}_c$ can be represented as

$$\psi(\theta) = \begin{bmatrix} \bar{\phi}(\theta) \\ \bar{\zeta}(\theta) \end{bmatrix} \quad \text{and} \quad \hat{\psi}(\theta) = \begin{bmatrix} \bar{\hat{\phi}}(\theta) \\ \bar{\hat{\zeta}}(\theta) \end{bmatrix}, \qquad \theta \in [-2\tau - \sigma, 0]\ , \tag{44}$$

respectively, where

$$\bar{\phi}(\theta) := \begin{cases} \phi(\theta)\ , & \theta \in [-\tau, 0] \\ 0\ , & \theta < -\tau \end{cases}\ , \qquad \bar{\zeta}(\theta) := \begin{cases} \zeta(\theta)\ , & \theta \in [-\sigma, 0] \\ 0\ , & \theta < -\sigma \end{cases} \tag{45}$$

and, similarly,

$$\bar{\hat{\phi}}(\theta) := \begin{cases} \hat{\phi}(\theta)\ , & \theta \in [-\tau, 0] \\ 0\ , & \theta < -\tau \end{cases}\ , \qquad \bar{\hat{\zeta}}(\theta) := \begin{cases} \hat{\zeta}(\theta)\ , & \theta \in [-\sigma, 0] \\ 0\ , & \theta < -\sigma \end{cases}\ . \tag{46}$$

The states of Σ_c and $\hat{\Sigma}_c$ at time $t \geq 0$, on the other hand, can be represented as

$$\xi_t(\theta) = \begin{bmatrix} x(t+\theta) \\ z(t+\theta) \end{bmatrix} \quad \text{and} \quad \hat{\xi}_t(\theta) = \begin{bmatrix} \hat{x}(t+\theta) \\ \hat{z}(t+\theta) \end{bmatrix} , \qquad \theta \in [-2\tau - \sigma, 0] , \tag{47}$$

respectively. The following two theorems show that, when $\hat{\Sigma}$ is an extension of Σ and $\hat{\Gamma}$ is contractible to Γ, stability of $\hat{\Sigma}_c$ implies stability of Σ_c.

Theorem 5. If $\hat{\Sigma}$ is an extension of Σ and $\hat{\Gamma}$ is contractible to Γ, then Σ_c is BIBO stable if $\hat{\Sigma}_c$ is BIBO stable.

Proof. Since the connection (31) is made at time $t = 0$, we have

$$u(t) = \begin{cases} e(t) , & t < 0 \\ v(t) + e(t) , & t \geq 0 \end{cases} \quad \text{and} \quad \hat{u}(t) = \begin{cases} \hat{e}(t) , & t < 0 \\ \hat{v}(t) + \hat{e}(t) , & t \geq 0 \end{cases} .$$

Then, (36) and

$$e(t) = R\hat{e}(t) , \qquad t \geq -\tau , \tag{48}$$

implies (9). Furthermore, when the initial states of Σ_c and $\hat{\Sigma}_c$ are both zero, (8) and (33) are both satisfied. Then, since $\hat{\Sigma}$ is an extension of Σ and $\hat{\Gamma}$ is contractible to Γ, as long as all the initial states are zero and $e(\cdot)$ and $\hat{r}(\cdot)$ respectively satisfy (48) and (34), (11) and (36) are satisfied. Furthermore, as in the proof of Theorem 2, (11) implies (21). Now, suppose that $\hat{\Sigma}_c$ is BIBO stable. Since R is of full-rank and $\hat{p} \geq p$, there exists $\tilde{R} \in \mathbb{R}^{\hat{p} \times p}$ such that $R\tilde{R} = I_p$. Then, for any bounded $e(\cdot)$, $\hat{e}(\cdot) := \tilde{R}e(\cdot)$ is bounded and satisfies (48). Furthermore, for any bounded $r(\cdot)$, any $\hat{r}(\cdot)$ which satisfies (34) is bounded. Then, since $\hat{\Sigma}_c$ is BIBO stable, corresponding $\hat{y}(\cdot)$ and $\hat{v}(\cdot)$ will be bounded. Then, (21) and (36) respectively implies that $y(\cdot)$ and $v(\cdot)$ will be bounded. Thus, Σ_c will be BIBO stable. □

Theorem 6. If $\hat{\Sigma}$ is an extension of Σ and $\hat{\Gamma}$ is contractible to Γ, then Σ_c is exponentially stable if $\hat{\Sigma}_c$ is exponentially stable.

Proof. First note that, when the inputs of both Σ_c and $\hat{\Sigma}_c$ are zero, i. e., when $e(t) = 0$, $r(t) = 0$, $\hat{e}(t) = 0$, and $\hat{r}(t) = 0$, $\forall t$, (48) and (34) are both satisfied. Then, by the arguments used in the proof of Theorem 5, as long as (8) and (33) are satisfied, (10) and (35) are satisfied. Then, the states of Σ_c and of $\hat{\Sigma}_c$ at time t will be related as

$$\xi_t(\theta) = Q\hat{\xi}_t(\theta) , \qquad \forall\theta \in [-2\tau - \sigma, 0] , \qquad \forall t \geq 0 , \tag{49}$$

where $Q := \begin{bmatrix} \tilde{T} & 0 \\ 0 & P \end{bmatrix}$, where $\tilde{T}$ satisfies $\tilde{T}T = I_n$ (there exists such $\tilde{T}$, since T is of full-rank and $\hat{n} \geq n$). Now, let $\tilde{Q} := \begin{bmatrix} T & 0 \\ 0 & \tilde{P} \end{bmatrix}$, where $\tilde{P}$ satisfies $P\tilde{P} = I_m$ (there exists such $\tilde{P}$, since P is of full-rank and $\hat{m} \geq m$). Then, for any initial state $\psi(\cdot)$ of Σ_c, the initial state

$$\hat{\psi}(\theta) = \tilde{Q}\psi(\theta) , \qquad \theta \in [-2\tau - \sigma, 0] , \tag{50}$$

of $\hat{\Sigma}_c$ is bounded and satisfies (44)–(46), where (8) and (33) are satisfied. Now, let $\hat{\Sigma}_c$ be exponentially stable. Then, there exist $\alpha, \beta > 0$, such that, for any bounded $\hat{\psi}(\cdot)$,

$$\|\hat{\xi}_t\| \leq \alpha \|\hat{\psi}\| e^{-\beta t}, \qquad \forall t \geq 0. \tag{51}$$

Now, for any bounded $\psi(\cdot)$, let $\hat{\psi}(\cdot)$ be given by (50). Then, $\hat{\psi}(\cdot)$ will be bounded and will satisfy $\|\hat{\psi}\| \leq \|\tilde{Q}\| \|\psi\|$. Then, using (51), (49) will give

$$\|\xi_t\| \leq \|Q\| \|\hat{\xi}_t\| \leq \hat{\alpha} \|\psi\| e^{-\beta t}, \qquad \forall t \geq 0, \tag{52}$$

where $\hat{\alpha} := \alpha \|\tilde{Q}\| \|Q\|$. This, however, implies that Σ_c is exponentially stable. □

In controller design, besides stability, some performance requirements are also sought. One common performance requirement is the tracking of certain reference inputs despite certain disturbances. We can define this requirement as follows.

Definition 5. A closed-loop system, such as Σ_c, is said to achieve *good tracking* for reference inputs $r(\cdot)$, with respect to a *tolerance function* $g : \mathbb{R}_+ \to \mathbb{R}_+$ and a *disturbance bound* $f : [-\tau, \infty) \to \mathbb{R}_+$, if, assuming that the initial state is zero, the output, $y(\cdot)$, satisfies $\|y(t) - r(t)\| \leq g(t)$, $\forall t \geq 0$, for all external inputs which satisfy $\|e(t)\| \leq f(t)$, $\forall t \geq -\tau$.

The following theorem shows that, when $\hat{\Sigma}$ is an extension of Σ, if $\hat{\Gamma}$ is designed to achieve good tracking for the expanded system, then the contracted controller Γ achieves good tracking for the original system.

Theorem 7. Let $\hat{\Sigma}$ be an extension of Σ and $\hat{\Gamma}$ be contractible to Γ. Suppose that $\hat{\Sigma}_c$ achieves good tracking for reference inputs $\hat{r}(t) = Sr(t)$, $t \geq -\sigma$, for some $r(\cdot)$, where S is as in (7), with respect to $\hat{g} : \mathbb{R}_+ \to \mathbb{R}_+$ and $\hat{f} : [-\tau, \infty) \to \mathbb{R}_+$. Then, Σ_c achieves good tracking for reference inputs $r(\cdot)$, with respect to $g(\cdot) := \|\tilde{S}\| \hat{g}(\cdot)$ and $f(\cdot) := \frac{1}{\|\tilde{R}\|} \hat{f}(\cdot)$, where, for R is as in (7), $\tilde{R}$ and $\tilde{S}$ respectively satisfy $R\tilde{R} = I_p$ and $\tilde{S}S = I_q$ (there exist such $\tilde{R}$ and $\tilde{S}$ since R and S are of full rank and $\hat{p} \geq p$ and $\hat{q} \geq q$ - note that it is advantageous to choose $\tilde{R}$ and $\tilde{S}$ as such matrices with minimum norm; i. e., it is advantageous to choose $\tilde{R}$ and $\tilde{S}$ as the Moore–Penrose inverse [39] of R and S respectively).

Proof. First, note that both (8) and (33) are satisfied when both Σ and $\hat{\Sigma}$ has zero initial states. Furthermore, (34) is satisfied by assumption. Then, since $\hat{\Gamma}$ is contractible to Γ, as discussed in the proof of Theorem 5, to satisfy (9), it is sufficient to have (48). Then, since $\hat{\Sigma}$ is an extension of Σ, when (48) is satisfied, (11) is satisfied. Now, for any $e(t)$, which satisfy $\|e(t)\| \leq f(t)$, $\forall t \geq -\tau$, let $\hat{e}(t) := \tilde{R}e(t)$, $t \geq -\tau$. Then, (48) is satisfied. Furthermore, $\|\hat{e}(t)\| \leq \|\tilde{R}\| \|e(t)\| \leq \|\tilde{R}\| f(t) = \hat{f}(t)$, $\forall t \geq -\tau$. Then, since $\hat{\Sigma}_c$ achieves good tracking, $\|\hat{y}(t) - \hat{r}(t)\| \leq \hat{g}(t)$, $\forall t \geq 0$. Then, by (11) and (34), $y(t) - r(t) = \tilde{S}(\hat{y}(t) - \hat{r}(t))$, $\forall t \geq 0$. Hence, $\|y(t) - r(t)\| \leq \|\tilde{S}\| \|\hat{y}(t) - \hat{r}(t)\| \leq \|\tilde{S}\| \hat{g}(t) = g(t)$, $\forall t \geq 0$, which proves the desired result. □

The above results show that, when $\hat{\Sigma}$ is an extension of Σ, a controller can first be designed for the expanded system to achieve stability and/or good tracking. Then, this controller can be contracted for implementation on the original system. The original closed-loop system will then be stable and/or achieve good tracking. This approach is particularly useful when the original system has an overlapping structure. In such a case overlapping decompositions can be used to obtain the expanded system for which a decentralized controller is first designed. This controller is then contracted for implementation on the original system. This approach is presented in the next section.

5. OVERLAPPING DECOMPOSITIONS

As pointed out in [14], large-scale systems may have subsystems which may overlap in many different ways. The simplest and the most investigated case in the literature is two overlapping subsystems. The state, the input, and the output vectors of the system in this case can be decomposed as follows:

$$x = \begin{bmatrix} x_1 \\ x_c \\ x_2 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_c \\ u_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_c \\ y_2 \end{bmatrix}, \tag{53}$$

where, for $i = 1, 2$, $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{p_i}$, and $y_i \in \mathbb{R}^{q_i}$ are, respectively, the state, the input, and the output vectors of the i^{th} subsystem only, and $x_c \in \mathbb{R}^{n_c}$, $u_c \in \mathbb{R}^{p_c}$, and $y_c \in \mathbb{R}^{q_c}$ are, respectively, the state, the input, and the output vectors of the overlapping part. Then, to obtain an *extension* of this system, the matrices in (7) can be chosen as

$$T = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_c} & 0 \\ 0 & I_{n_c} & 0 \\ 0 & 0 & I_{n_2} \end{bmatrix}, \quad R = \begin{bmatrix} I_{p_1} & 0 & 0 & 0 \\ 0 & I_{p_c} & I_{p_c} & 0 \\ 0 & 0 & 0 & I_{p_2} \end{bmatrix}, \quad S = \begin{bmatrix} I_{q_1} & 0 & 0 \\ 0 & I_{q_c} & 0 \\ 0 & I_{q_c} & 0 \\ 0 & 0 & I_{q_2} \end{bmatrix}. \tag{54}$$

Now, consider the system Σ, described by (1)–(2). Suppose that the state, the input, and the output vectors of Σ are decomposed as in (53). Partition the matrix functions in (1)–(2) compatible with (53):

$$A(\cdot) = \begin{bmatrix} A_{11}(\cdot) & A_{1c}(\cdot) & A_{12}(\cdot) \\ A_{c1}(\cdot) & A_{cc}(\cdot) & A_{c2}(\cdot) \\ A_{21}(\cdot) & A_{2c}(\cdot) & A_{22}(\cdot) \end{bmatrix}, \quad B(\cdot) = \begin{bmatrix} B_{11}(\cdot) & B_{1c}(\cdot) & B_{12}(\cdot) \\ B_{c1}(\cdot) & B_{cc}(\cdot) & B_{c2}(\cdot) \\ B_{21}(\cdot) & B_{2c}(\cdot) & B_{22}(\cdot) \end{bmatrix},$$

and

$$C(\cdot) = \begin{bmatrix} C_{11}(\cdot) & C_{1c}(\cdot) & C_{12}(\cdot) \\ C_{c1}(\cdot) & C_{cc}(\cdot) & C_{c2}(\cdot) \\ C_{21}(\cdot) & C_{2c}(\cdot) & C_{22}(\cdot) \end{bmatrix}.$$

Then, an extension $\hat{\Sigma}$ of Σ can be described by (3)–(4), where the matrix functions in (3)–(4) are chosen as in (17)–(19), where

$$\tilde{T} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & \frac{1}{2}I_{n_c} & \frac{1}{2}I_{n_c} & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix}, \quad M(\cdot) = \begin{bmatrix} 0 & \frac{1}{2}A_{1c}(\cdot) & -\frac{1}{2}A_{1c}(\cdot) & 0 \\ 0 & \frac{1}{2}A_{cc}(\cdot) & -\frac{1}{2}A_{cc}(\cdot) & 0 \\ 0 & -\frac{1}{2}A_{cc}(\cdot) & \frac{1}{2}A_{cc}(\cdot) & 0 \\ 0 & -\frac{1}{2}A_{2c}(\cdot) & \frac{1}{2}A_{2c}(\cdot) & 0 \end{bmatrix},$$

$$N(\cdot) = 0\,, \quad \text{and} \quad L(\cdot) = \begin{bmatrix} 0 & \frac{1}{2}C_{1c}(\cdot) & -\frac{1}{2}C_{1c}(\cdot) & 0 \\ 0 & \frac{1}{2}C_{cc}(\cdot) & -\frac{1}{2}C_{cc}(\cdot) & 0 \\ 0 & -\frac{1}{2}C_{cc}(\cdot) & \frac{1}{2}C_{cc}(\cdot) & 0 \\ 0 & -\frac{1}{2}C_{2c}(\cdot) & \frac{1}{2}C_{2c}(\cdot) & 0 \end{bmatrix}.$$

Here, $\tilde{T}$ is chosen as the Moore–Penrose inverse of T and the matrix functions $M(\cdot)$ and $L(\cdot)$ are chosen to minimize the interactions between the subsystems among all complementary matrix functions which satisfy (20). $N(\cdot)$, however, is chosen as zero, to satisfy (20). By Corollary 1, then, $\hat{\Sigma}$ is an extension of Σ. The state, the input, and the output vectors of $\hat{\Sigma}$ can now be decomposed as:

$$\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}, \quad \hat{u} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix}, \tag{55}$$

where, for $i = 1, 2$, $\hat{x}_i \in \mathbb{R}^{\hat{n}_i}$, $\hat{n}_i := n_i + n_c$, $\hat{u}_i \in \mathbb{R}^{\hat{p}_i}$, $\hat{p}_i := p_i + p_c$, and $\hat{y}_i \in \mathbb{R}^{\hat{q}_i}$, $\hat{q}_i := q_i + q_c$. Then, $\hat{\Sigma}$ is composed of two subsystems: $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$, where $\hat{x}_i$, $\hat{u}_i$, and $\hat{y}_i$ respectively form the state, the input, and the output vectors of $\hat{\Sigma}_i$, for $i = 1, 2$.

Now, suppose that a local controller $\hat{\Gamma}_i$ of the form

$$\dot{\hat{z}}_i(t) = \int_{-\sigma}^{0} \Big(\hat{F}_i(\theta)\hat{z}_i(t+\theta) + \hat{G}_i(\theta)\hat{w}_i(t+\theta)\Big)\,\mathrm{d}\theta \tag{56}$$

$$\hat{v}_i(t) = \int_{-\sigma}^{0} \Big(\hat{H}_i(\theta)\hat{z}_i(t+\theta) + \hat{K}_i(\theta)\hat{w}_i(t+\theta)\Big)\,\mathrm{d}\theta \tag{57}$$

is designed for $\hat{\Sigma}_i$, for $i = 1, 2$, which is meant to be applied to $\hat{\Sigma}_i$ by letting

$$\hat{w}_i(t) = \hat{y}_i(t) - \hat{r}_i(t) \quad \text{and} \quad \hat{u}_i(t) = \hat{v}_i(t) + \hat{e}_i(t) \tag{58}$$

for $t \geq 0$, where $\hat{r}_i(t) \in \mathbb{R}^{\hat{q}_i}$ is the *reference input* and $\hat{e}_i(t) \in \mathbb{R}^{\hat{p}_i}$ is the *external input* for $\hat{\Sigma}_i$ at time t. A decentralized controller $\hat{\Gamma}$ for $\hat{\Sigma}$ can then be obtained as in (27)–(28), where, for $\theta \in [-\sigma, 0]$,

$$\hat{F}(\theta) = \begin{bmatrix} \hat{F}_1(\theta) & 0 \\ 0 & \hat{F}_2(\theta) \end{bmatrix}, \quad \hat{G}(\theta) = \begin{bmatrix} \hat{G}_1(\theta) & 0 \\ 0 & \hat{G}_2(\theta) \end{bmatrix},$$

$$\hat{H}(\theta) = \begin{bmatrix} \hat{H}_1(\theta) & 0 \\ 0 & \hat{H}_2(\theta) \end{bmatrix}, \quad \text{and} \quad \hat{K}(\theta) = \begin{bmatrix} \hat{K}_1(\theta) & 0 \\ 0 & \hat{K}_2(\theta) \end{bmatrix}.$$

This controller can then be contracted to a controller Γ, which is described by (25)–(26), where the matrix functions are given by (42)–(43). This controller can then be implemented on the original system as shown in Figure 1 (where e and r are decomposed compatible with, respectively, u and y in (53); v_{1_1} (w_{1_1}) indicates the first p_1 (q_1) elements of $\hat{v}_1$ ($\hat{w}_1$) and v_{1_2} (w_{1_2}) indicates the last p_c (q_c) elements of $\hat{v}_1$ ($\hat{w}_1$); similarly, v_{2_1} (w_{2_1}) indicates the first p_c (q_c) elements of $\hat{v}_2$ ($\hat{w}_2$) and v_{2_2} (w_{2_2}) indicates the last p_2 (q_2) elements of $\hat{v}_2$ ($\hat{w}_2$)). The *overlapping decentralized structure* of the controller is evident from the figure. Assuming that the controller $\hat{\Gamma}$ stabilizes the system $\hat{\Sigma}$ (respectively in BIBO and exponential sense), by Theorems 5 and 6, the contracted

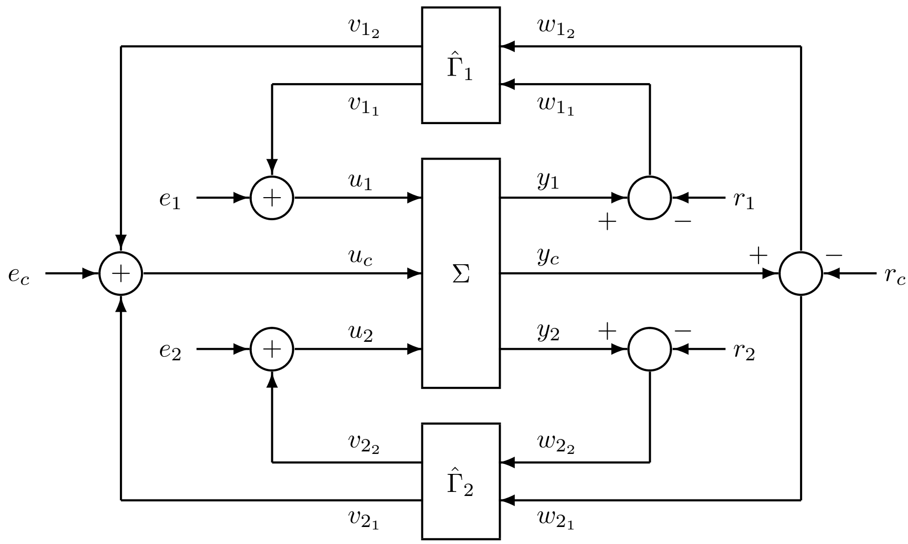

Fig. 1. Implementation of the contracted controller on the original system.

controller Γ stabilizes the original system Σ. Furthermore, if $\hat{\Sigma}_c$ (obtained by applying $\hat{\Gamma}$ to $\hat{\Sigma}$) achieves good tracking for reference inputs $\hat{r}(t) = Sr(t)$, $t \geq -\sigma$, for some $r(\cdot)$, with respect to some $\hat{g} : \mathbb{R}_+ \to \mathbb{R}_+$ and $\hat{f} : [-\tau, \infty) \to \mathbb{R}_+$, then, by Theorem 7, Σ_c (obtained by applying Γ to Σ, as shown in Figure 1) achieves good tracking for reference inputs $r(\cdot)$, with respect to $g(\cdot) := \|\tilde{S}\|\hat{g}(\cdot)$ and $f(\cdot) := \frac{1}{\|\tilde{R}\|}\hat{f}(\cdot)$.

Remark 5. If each of the local controllers $\hat{\Gamma}_i$ $(i = 1, 2)$ is designed by considering only the subsystem $\hat{\Sigma}_i$ (i. e., by ignoring the interactions between the subsystems), then, naturally, there is no guarantee that the combined controller $\hat{\Gamma}$ will stabilize and/or achieve good performance for $\hat{\Sigma}$. However, if the overlapping decomposition and the expansion are done properly such that the interactions between the subsystems are weak (which is the whole idea of overlapping decompositions and expansions), there is a good chance that the overall controller $\hat{\Gamma}$ will stabilize and achieve good performance for $\hat{\Sigma}$ (e. g., see [4]–[7], [19]–[27], [43], [45]). Furthermore, while designing $\hat{\Gamma}_i$ $(i = 1, 2)$, the interactions between the subsystems may also be taken into account. Such an approach was suggested in [18], where a *robustness bound*, which accounts for the neglected interactions between the subsystems and any uncertainties in the system model is first obtained. Then, the stability of the overall closed-loop system $\hat{\Sigma}_c$ is guaranteed, if each of the local controllers $\hat{\Gamma}_i$ is designed to stabilize the subsystem $\hat{\Sigma}_i$ and to satisfy a certain constraint involving the robustness bound (only the decomposition of the state-space was considered in [18]; however, similar results can also be obtained for the present case). We should perhaps note that, as it was shown in [11], there exist local controllers $\hat{\Gamma}_i$ such that the overall closed-loop system $\hat{\Sigma}_c$ is stable, if and only if the open-loop system $\hat{\Sigma}$ does not have any *unstable decentralized fixed modes* (although only systems with discrete time-delays were considered in [11], its results can be extended to the present case). We should also indicate that, even though the expanded closed-loop system $\hat{\Sigma}_c$ is not stable, the original closed-loop system Σ_c, under the contracted controllers may be stable (e. g., see the example in [20]). Therefore, rather than ensuring the stability of $\hat{\Sigma}_c$, one may also design local controllers $\hat{\Gamma}_i$ independently (counting on weak interactions between the subsystems) and check whether the resulting actual closed-loop system Σ_c is stable and achieves desired preformance.

6. EXAMPLE

In this section, we consider a simplified model of a highway traffic congestion control system to demonstrate a possible application of our results. The rate of change of traffic density at a point i on the highway is given by [37]:

$$\dot{\rho}_i(t) = f_i^{\text{in}}(t) - f_i^{\text{out}}(t) \tag{59}$$

where $\rho_i(t)$ (measured in v/ul, where v and ul respectively stand for "vehicles" and "unit length") is the traffic density at point i at time t and $f_i^{\text{in}}(t)$ and $f_i^{\text{out}}(t)$ (both measured in v/tu, where tu stands for "time unit") are respectively the in-flow and the out-flow traffic rates at time t at a section of infinitesimally small unit length centered at point i. The in-flow traffic rate at time t at point i is simply a delayed version of the out-flow traffic rate at point $i-1$, which precedes point i, plus the sum of the delayed versions of the traffic in-flow rates from the on-ramps between point $i-1$ and i, minus the sum of the delayed versions of the traffic out-flow rates from the off-ramps between point $i-1$ and i. Although, many times in the literature, these delays are assumed to be discrete time-delays, it is more realistic to use distributed time-delays, since different vehicles typically travel at different speeds. In the present example, we will assume that there is only one on-ramp and one off-ramp between any two points, which are at the same position on the highway. Thus, we let

$$f_i^{\text{in}}(t) = \int_{-\tau/2}^{0} \big(\alpha_i(\theta) f_{i-1}^{\text{out}}(t+\theta) + b_i(\theta) u_i(t+\theta)\big)\,\mathrm{d}\theta \tag{60}$$

where $\tau/2$ is the maximum time of travel between any two points (due to a convolution, to be given below, the maximum delay in the system becomes twice that value, this is why we use $\tau/2$, rather than τ), $u_i(t)$ is the net in-flow rate at the ramps between points $i-1$ and i (i. e., the in-flow rate from the on-ramp minus the out-flow rate from the off-ramp) at time t, and $\alpha_i(\cdot)$ and $b_i(\cdot)$ are functions which depend on the traffic characteristics between the two points and satisfy $\int_{-\tau/2}^{0} \alpha_i(\theta)\,\mathrm{d}\theta = \int_{-\tau/2}^{0} b_i(\theta)\,\mathrm{d}\theta = 1$.

The out-flow traffic rate at time t at point i, on the other hand, is given as $f_i^{\text{out}}(t) = v_i(t)\rho_i(t)$, where $v_i(t)$ is the mean traffic speed at point i at time t. When there is no downstream congestion, this speed is simply equal to the *free speed*, v_f, on the highway.

When the downstream traffic densities are higher than a critical value, however, this speed is significantly reduced [36]. Although there is a nonlinear relation between the downstream traffic density and mean speed, thus between the downstream traffic density and the out-flow traffic rate, we take a simplified, linearized model as:

$$f_i^{\text{out}}(t) = v_f \rho_i(t) - \int_{-\tau/2}^{0} \beta_i(\theta)\rho_{i+1}(t+\theta)\,\mathrm{d}\theta \tag{61}$$

where we take $v_f = 1\ ul/tu$ and $\beta_i(\cdot)$ is a function which depends on the traffic characteristics and the current operating conditions. Using (60) and (61) in (59), we obtain

$$\begin{aligned}\dot{\rho}_i(t) = -\rho_i(t) - \int_{-\tau}^{0} \gamma_i(\theta)\rho_i(t+\theta)\,\mathrm{d}\theta \\ + \int_{-\tau/2}^{0} \left(\alpha_i(\theta)\rho_{i-1}(t+\theta) + \beta_i(\theta)\rho_{i+1}(t+\theta) + b_i(\theta)u_i(t+\theta)\right)\mathrm{d}\theta\end{aligned} \tag{62}$$

where

$$\gamma_i(\theta) := \int_{-\tau/2}^{0} \alpha_i(\nu)\beta_{i-1}(\theta-\nu)\,\mathrm{d}\nu. \tag{63}$$

Now, let us consider a highway where congestion is present on a stretch which includes three consecutive points, labelled as "1", "c" and "2". Let us assume that there is a constant traffic density, which equals to ρ_0 at point "0" which immediately precedes point "1". Let us also assume that the maximum time of travel between any two points is 10 tu (thus, $\tau = 20\ tu$) and

$$\alpha_1(\theta) = \alpha_c(\theta) = \alpha_2(\theta) := \alpha(\theta) = \begin{cases} 10+\theta\,, & \text{for } -10 \le \theta \le -9 \\ -8-\theta\,, & \text{for } -9 \le \theta \le -8 \\ 0\,, & \text{otherwise} \end{cases}$$

$$b_1(\theta) = b_c(\theta) = b_2(\theta) := b(\theta) = \begin{cases} 3+\theta\,, & \text{for } -3 \le \theta \le -2 \\ -1-\theta\,, & \text{for } -2 \le \theta \le -1 \\ 0\,, & \text{otherwise} \end{cases}$$

$$\beta_0(\theta) = \beta_1(\theta) = \beta_c(\theta) := \beta(\theta) = \begin{cases} (10+\theta)/50\,, & \text{for } -10 \le \theta \le -5 \\ -\theta/50\,, & \text{for } -5 \le \theta \le 0 \\ 0\,, & \text{otherwise.} \end{cases}$$

Furthermore, we let $\beta_2(\theta) = 0$, since there is no congestion downstream of point "2". We then have $\gamma_1(\theta) = \gamma_c(\theta) = \gamma_2(\theta) := \gamma(\theta) = \int_{-10}^{0} \alpha(\nu)\beta(\theta-\nu)\,\mathrm{d}\nu$.

We assume that the traffic densities at each point can be measured directly (which is possible, e. g., by using inductive loops [36]) and, for $i = 1, c, 2$, let

$$\dot{\xi}_i(t) = \rho_i(t) - r_i \tag{64}$$

where r_i is the desired traffic density at section i and $\xi_i(t)$ is the state of an integrator added for the purpose of achieving tracking of the desired traffic density. Then, by defining $x_i(t) := \begin{bmatrix} \rho_i(t) & \xi_i(t) \end{bmatrix}^T$, the overall system dynamics can then be represented as in (1) with an additional constant vector $\begin{bmatrix} \rho_0 & -r_1 & 0 & -r_c & 0 & -r_2 \end{bmatrix}^T$ on the right-hand side (the term ρ_0 represents the effect of the upstream traffic density on the part of the system under consideration and, for control purposes, can be treated as a constant disturbance), where

$$A(\theta) := \begin{bmatrix} -\delta(\theta)-\gamma(\theta) & 0 & \beta(\theta) & 0 & 0 & 0 \\ \delta(\theta) & 0 & 0 & 0 & 0 & 0 \\ \alpha(\theta) & 0 & -\delta(\theta)-\gamma(\theta) & 0 & \beta(\theta) & 0 \\ 0 & 0 & \delta(\theta) & 0 & 0 & 0 \\ 0 & 0 & \alpha(\theta) & 0 & -\delta(\theta)-\gamma(\theta) & 0 \\ 0 & 0 & 0 & 0 & \delta(\theta) & 0 \end{bmatrix},$$

$$B(\theta) := \begin{bmatrix} b(\theta) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b(\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b(\theta) & 0 \end{bmatrix}^T,$$

where $\delta(\cdot)$ is the Dirac delta function, and x and u are as defined in (53). Furthermore, since the traffic densities can be measured and the integrator outputs are available, the output of the system is given as $y(t) = x(t)$, which can be represented as in (2), where $C(\theta) := \delta(\theta)I_6$. Moreover, we assume that the traffic out-flow rates from each off-ramp can also be measured and the traffic in-flow rates from each on-ramp can be adjusted using *on-ramp metering* [38]. Thus, assuming that there is sufficient out-flow (to allow negative net in-flow) and sufficient demand for in-flow, the net in-flow rates $u_1(t)$, $u_c(t)$, and $u_2(t)$ can be set as desired. The problem, then, is to choose $u(t) := \begin{bmatrix} u_1(t) & u_c(t) & u_2(t) \end{bmatrix}^T$ so that the traffic density vector $\rho(t) := \begin{bmatrix} \rho_1(t) & \rho_c(t) & \rho_2(t) \end{bmatrix}^T$ converges to a desired constant vector $r(t) := \begin{bmatrix} r_1 & r_c & r_2 \end{bmatrix}^T$.

To design controllers, we first obtain an extension of this system as explained in Section 5. We then obtain two identical local design models as:

$$\dot{\hat{x}}_i(t) = \int_{-\tau}^{0} \left(\hat{A}_i(\theta)\hat{x}_i(t+\theta) + \hat{B}_i(\theta)\hat{u}_i(t+\theta)\right) \mathrm{d}\theta \tag{65}$$

$$\hat{y}_i(t) = \hat{x}_i(t) = \int_{-\tau}^{0} \hat{C}_i(\theta)\hat{x}_i(t+\theta)\,\mathrm{d}\theta \tag{66}$$

for $i = 1, 2$, where

$$\hat{A}_1(\theta) = \hat{A}_2(\theta) = \begin{bmatrix} -\delta(\theta)-\gamma(\theta) & 0 & \beta(\theta) & 0 \\ \delta(\theta) & 0 & 0 & 0 \\ \alpha(\theta) & 0 & -\delta(\theta)-\gamma(\theta) & 0 \\ 0 & 0 & \delta(\theta) & 0 \end{bmatrix},$$

$$\hat{B}_1(\theta) = \hat{B}_2(\theta) = \begin{bmatrix} b(\theta) & 0 & 0 & 0 \\ 0 & 0 & b(\theta) & 0 \end{bmatrix}^T,$$

and $\hat{C}_1(\theta) = \hat{C}_2(\theta) = \delta(\theta)I_4$.

To stabilize the overall system and achieve desired tracking, then, for each $i = 1, 2$, we design a local static controller $\hat{\Gamma}_i$ of the form

$$\hat{u}_i(t) = \bar{K}_i \left(\hat{y}_i(t) - \hat{r}_i(t)\right) \tag{67}$$

where $\hat{r}_1(t) = \left[\begin{array}{cccc} r_1 & 0 & r_c & 0 \end{array}\right]^T$ and $\hat{r}_2(t) = \left[\begin{array}{cccc} r_c & 0 & r_2 & 0 \end{array}\right]^T$. Note that, this controller can be represented in the form (56)–(58) with $\dim(\hat{z}_i) = 0$, $\hat{K}_i(\theta) = \delta(\theta)\bar{K}_i$, and $\hat{e}_i(t) = 0$. We note that the open-loop systems are unstable, due to the integral action used in (64), which is required to achieve desired tracking. We then choose the gains $\bar{K}_i$ to move the right-most mode(s) of the closed-loop system as far left as possible by using the non-smooth optimization approach of [46] (see [34, 33] for the decentralized implementation of this approach). We determine

$$\bar{K}_1 = \bar{K}_2 = \left[\begin{array}{cccc} -0.7212 & -0.2434 & 0.0570 & 0.0463 \\ -0.3095 & -0.3680 & -0.4026 & -0.1042 \end{array}\right]$$

which move the right-most modes of each local closed-loop system to -0.0586 ± 0.3104i. Although it is possible to move these modes further left by using dynamic (whether finite-dimensional or time-delay) controllers, we find these locations sufficient. We then apply these controllers to the original system as described in Section 5 and as illustrated in Figure 1. The right-most modes of the original closed-loop system turn out to be -0.0480 ± 0.5463i. Thus, the designed controllers stabilize the original system and achieve tracking of the desired traffic density.

Although, in this example, we considered a highway with only three points, it is possible to extend this methodolgy when there are, say, N consecutive points on the highway whose traffic densities are to be controlled. In such a case this system can be decomposed into $N-1$ ovelapping subsystems, where subsystem i contains points i and $i+1$ (now, we assume that the points are labelled as "1", "2", ..., "N"). Furthermore, when the congestion proceeds upstream or downstream, new subsystems, thus new controllers, can be added to the system without changing the already designed controllers. Therefore, besides offering an easier design approach, the proposed methodology also offers a modularity in design.

7. CONCLUSION

Extension principle has been defined for LTI retarded distributed-time-delay systems. Preservation of stability properties between two systems, one of which is an extension of the other has also been discussed. Controller design using the extension principle has then been considered. It has been shown that, if the expanded system is an extension of the original system, then any controller designed for the expanded system can be contracted for implementation on the original system. Furthermore, if the controller designed for the expanded system stabilizes the expanded system and satisfies certain performance requirements, such as *good tracking*, then the contracted controller stabilizes the original system and satisfies corresponding performance requirements for the original system. Finally, overlapping decompositions of LTI retarded distributed-time-delay systems, whose input, output, and state spaces are overlappingly decomposed, has been discussed. Controller design using overlapping decompositions has also been demonstrated. We in particular note that, when such decompositions are used, the resulting controllers will also have an overlappingly decentralized structure as shown in Figure 1. Although, for the sake of brevity, we only considered the case of two overlapping subsystems, different overlapping structures can also be considered (e. g., see [14] for the finite-dimensional case). Extensions of the present results to non-linear, and/or time-varying, and/or neutral systems can be undertaken as further research topics.

The main contribution of the present work is that, using the proposed approach, it is possible to decompose a large-scale system, which may be subject to distributed time-delays, into a number of overlapping subsystems, then design a controller for each subsystem, and finally apply the designed controllers to the actual system. As demonstrated in the previous section, such an approach provides both ease and modularity in design. This is important in many large-scale systems, such as traffic control, polution control, energy distribution, among others, where the system under consideration may involve overlapping subsystems some of which may require control at certain times but may be passive at certain other times.

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