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CONSENSUS-BASED STATE ESTIMATION FOR MULTI-AGENT SYSTEMS WITH CONSTRAINT INFORMATION

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This paper considers a distributed state estimation problem for multi-agent systems under state inequality constraints. We first give a distributed estimation algorithm by projecting the consensus estimate with help of the consensus-based Kalman filter (CKF) and projection on the surface of constraints. The consensus step performs not only on the state estimation but also on the error covariance obtained by each agent. Under collective observability and connective assumptions, we show that consensus of error covariance is bounded. Based on the Lyapunov method and projection, we provide and prove convergence conditions of the proposed algorithm and demonstrate its effectiveness via numerical simulations.

Keywords: multi-agent systems, distributed Kalman filter, state constraints, stability

Classification: 90B10

1. INTRODUCTION

Multi-agent systems have attracted much interest recently due to its broad range of applications in engineering systems such as aerospace, sensor networks and public transportation. As one of the important problem, distributed estimation cares about designing distributed algorithm to estimate the state of a process in multi-agent system. Among the distributed estimation methods, Kalman-filter-based estimation has received great attention due to its ability of tracking a dynamic process.

Most of existing methods in distributed Kalman-filter-based estimation are based on consensus strategy [5, 12, 13, 19, 20, 26, 28–30], which combine the consensus idea with state update based on the Kalman filter, called Kalman-consensus filter (KCF). For example, a distributed Kalman filter was developed in [19] using two identical consensus filters. Since the optimal KCF is not scalable and needs all-to-all communication, a scalable suboptimal KCF was reported in [20]. Moreover, an algorithm for overlapping decentralized estimator was developed based on the consensus strategy in [26] assuming intermittent observations and communication faults. Additionally, some results were obtained for switching communication topologies [12, 30] and nonlinear distributed filtering [10, 11]. However, most of the results [5, 12, 13, 20, 26, 28–30] required the local observ-

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able condition (i. e., each agent should be observable), which is not suitable in practical application. Different from the consensus approach to find the optimal Kalman filter gain and consensus gain in the distributed filtering, another kind of approach is to fuse the local estimate obtained by the Kalman filter [2, 5, 13], which can be treated as fusion-based distributed estimation. In [5], a distributed diffusion-based Kalman filter was realised by fusing estimate from neighbors', and the authors discussed convergence property under local observable condition. In [13], a diffusion Kalman filter based on a covariance intersection scheme was proposed by incorporating the covariance information. Note that how to find the optimal weights for fusion is the main challenge in the design of such distributed Kalman filters.

In many practical applications, we may obtain some priori knowledge about a system that Kalman filter dose not incorporate directly. The priori knowledge often formulate as equlity or inequality constraints about state variables, and we may use constraints information to improve estimate performance. There are various schemes incorporating state constraints information in the Kalman filter structure [3, 8, 9, 16, 24, 25]. [16] studied constrained filter by system projection. It use the fact that the process noise is also constrained, and constructed the optimal estimator which also satisfy the constraints. In [24], the constrained estimation obtained by projecting the unconstrained estimation onto the constrained surface, and it has proved that the projection approach with linear equality constraints performs better than unconstrained estimation by choosing suitable weights. In [9], the author studied the Kalman filter with inequality constraints. Among those approaches, the projection one has simple and clear interpretations by projecting the unconstrained estimation onto the constrained surface. In distributed scenario, we also wish to obtain state estimates that take advantage of constraint information to get better estimate than those in the absence of information.

The objective of this paper is to study distributed Kalman filter design with state inequality constraints, which is an important problem. To solve the problem, we propose algorithm combining the ideas of the consensus-based Kalman filter with the projection method. Each agent communicates with its neighbors to obtain an estimate based on consensus strategy, and then projects the unconstrained consensus-based estimate onto the surface defined by the inequality constraints only when the estimate does not satisfy the constraints. The contributions of this paper are summarized as follows. We study the distributed consensus-based Kalman filter with state constraints over a sensor network and propose distributed estimation algorithm. The work can be viewed as a distributed extension of the conventional ones, and an extension of some conventional results by studying inequalities rather than equalities. Under collective observability and connection assumption, the proposed distributed Kalman filter algorithm with inequality constraints design by projecting the unconstrained estimation onto the surface of constraints. It is worth noting that our algorithm is fully distributed and under collective observable condition, i. e., each agent does not need to be observable, which is much more suitable in application. Also, we provide and prove the stability conditions of the distributed projection-based algorithms using the Lyapunov method.

The remainder of the paper is organized as follows. Necessary preliminaries and the problem formulation are given in section 2. Consensus-basd distributed estimation algorithm with projection method is given in section 3. Then stability analysis of the proposed algorithm are provided and convergence results are obtained in section 4. Numerical simulation is shown in section 5, which shows that the algorithm proposed with constraints performs better than the existing unconstrained distributed estimation algorithm. Finally, some concluding remarks are provided in section 6.

Notations: The set of real number is denoted by $\mathbb{R}$. For a symmetric matrix M, $M \geq 0$ ($M > 0$) means that the matrix is positive semi-definite (definite), and $M_1 \geq M_2$ means that $M_1 - M_2 \geq 0$. $\lambda_{\max}(\cdot)$ and $\lambda_{min}(\cdot)$ denote maximum and minimum eigenvalue, respectively. $\| M \|$ represents the spectral norm of M and $\det(M)$ denotes its determinant. Furthermore, $diag\{M_1, M_2, \dots, M_n\}$ represents the block-diagonal matrix. $\mathbb{E}\{\cdot\}$ represents mathematical expectation. $Tr(M)$ represents the trace of matrix M, where $M \in \mathbb{R}^{m\times m}$.

2. PROBLEM FORMULATION

2.1. Preliminaries

We introduce some concepts of graph theory [7]. An undirected graph is denoted as $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, where $\mathscr{V} = \{1, 2, \dots, N\}$ is the node set, and $\mathscr{E} = \{(i, j) : i, j \in \mathscr{V}\}$ is the edge set. If node i and node j are connected by an edge, then these two vertices are called adjacent. The neighbor set of node i is defined by $\mathscr{N}_i = \{j : (i, j) \in \mathscr{E}\}$, which includes node i itself. The size of $\mathscr{N}_i$ is denoted as $|\mathscr{N}_i|$. In this paper, node i can be regarded as agent i, and the communication link can be treated as edge. A path is a sequence of edges of the form $(i_1, i_2), (i_2, i_3), \dots,$ where $i_j \in \mathscr{V}$. The graph $\mathscr{G}$ is connected if there exists a path between any two vertices of graph $\mathscr{G}$. The weighted adjacency matrix of graph $\mathscr{G}$ is defined as $\mathscr{A} = (a_{ij})_{NN} \in R^{N\times N}$, where $\sum_{j=1}^{N} a_{ij} = 1$ and $a_{ij} > 0$.

Next, we present some preliminaries about convex analysis [4]. A function $f(\cdot) : \mathbb{R}^m \to \mathbb{R}^n$ is said to be convex if $f(ax+(1-a)y) \leq af(x)+(1-a)f(y)$ for any $x, y \in \mathbb{R}^m$ and $0 < a < 1$. A set $X \in \mathbb{R}^n$ is convex if $ax+(1-a)y \in X$ for any $x, y \in X$ and $0 \leq a \leq 1$. For any closed and convex subset X of $\mathbb{R}^n$, define the projection onto X, denoted by $\mathbf{P}_X : \mathbb{R}^n \to X$, as follows:

$$\mathbf{P}_X(z) = arg \min_{x\in X} \| x - z \| .$$

Projection is widely used to deal with constraints (referring to [18] and [22])

The following lemma shows some projection properties for the following analysis (which can be found in [18]).

Lemma 2.1. Let X be a closed convex set in $\mathbb{R}^m$. Then

(i) $(\mathbf{P}_X(x) - x)^\top (z - \mathbf{P}_X(x)) \geq 0$, for all $z \in X$;

(ii) $\| \mathbf{P}_X(x) - \mathbf{P}_X(z) \| \leq \| x - z \|$, for all x and z;

(iii) $\| \mathbf{P}_X(x) - z \|^2 \leq \| x - z \|^2 - \| \mathbf{P}_X(x) - x \|^2$, for any $z \in X$.

Then we introduce some concepts related to stochastic processes, which are useful in the following convergence analysis ([1, 27]).

Definition 2.2. The stochastic process ζ_k is said to be exponential bounded in mean square, if there are real numbers, $\eta > 0$, $\vartheta > 0$ and $0 < \nu < 1$ such that

$$\mathbb{E}\{\|\zeta_k\|^2\} \leq \eta\|\zeta_0\|^2\vartheta^k + \nu \tag{1}$$

holds for every $k \geq 0$.

Definition 2.3. The stochastic process ζ_k is said to be bounded with probability one, if

$$\sup_{k\geq 0} \|\zeta_k\| < \infty \tag{2}$$

holds with probability one.

2.2. Formulation

Consider the dynamics of a target described by

$$x_{k+1} = Ax_k + w_k, \tag{3}$$

where $x_k \in \mathbb{R}^m$ is the states, w_k is additive process noises and modeled as m-dimensional Gaussian white noise with zero-mean and covariance $Q > 0$. We assume that A is nonsingular.

Sometimes, additional information such as state constraints may be known as priori knowledge in practice. In fact, some limitations/constraints for the states may be known according to the design specification or physical laws, for example, a tank with a limited liquid level and a moving car along a given road (and more engineering applications can be found in [23]). Here, for the dynamics of the target (3), the state constraints are given as the following inequalities

$$q_t(x) \leq 0, \quad t = 1, \dots, s, \tag{4}$$

where $q_t(x) : \mathbb{R}^m \to \mathbb{R}$ is a convex function and s is the number of the constraints. Many practical examples can be formulated with state inequality constraints [3, 8, 25]. For instance, in target tracking problem, the norm of acceleration should be below a bound, and in process control key variables are known to lie in certain regions.

A network consisting of N agents is used to estimate x_k. The measurement equation of the ith agent is given by:

$$y_{i,k} = C_i x_k + v_{i,k}, \tag{5}$$

where $C_i \in \mathbb{R}^{q_i \times m}$, $v_{i,k}$ is the measurement noises by sensor i, which is assumed to be zero-mean white Gaussian with covariance $R_i > 0$. $v_{i,k}$ is independent of w_k $\forall k, i$ and is independent of $v_{j,s}$ when $i \neq j$ or $k \neq s$.

The communication between agents is described by an undirected graph $\mathcal{G}$ with $\mathscr{V} = 1, 2, \dots, N$ and Laplacian L. The edge (i, j) means that the ith agent can receive information from the jth agent. The number of neighboring agents of the ith agent is denoted by d_i.

Denote $C = [C_1^\top, \dots, C_N^\top]^\top$, the following two standard assumptions are adopted, which have been widely used.

Assumption 2.4. The undirected graph $\mathcal{G}$ is connected.

Assumption 2.4 is a basic assumption in most of works [6, 29, 30]. To ensure the double stochastic of weight matrix $\mathscr{A}$, a possible choise of the weights [28] is

$$\begin{aligned} a_{ij} &= \frac{1}{\max\{|\mathscr{N}_i|, |\mathscr{N}_j|\}}, \ j \in \mathscr{N}_i, i{\neq}j, \\ a_{ii} &= 1 - \sum_{j \in \mathscr{N}_i, j \neq i} a_{ij}. \end{aligned}$$

Moreover, when we consider a directed graph, a necessary condition for the weight matrix $\mathscr{A}$ to be primitive is that the associated graph $\mathcal{G}$ is strong connected[1].

Assumption 2.5. (A, C) is observable.

Remark 2.6. Assumption 2.5 is different from that given in many existing works [5, 20, 26, 30], which rely on local observability, i. e., each agent is observable. Our work only need collective observable, which is much more suitable in applications. Works in [14, 19] developed distributed estimation algorithms under collective observability assumption. However, they need to communicate infinity time between any two sampling instance. In [6, 15], authors dealt with the problem under collective observable condition. While, some global information is needed in order to guarantee the stability of the algorithms. In section 3, we will propose a fully distributed algorithm, which does not rely on any global information and only need to communicate once between any two sampling instance.

Using the measurement and neighbors' information, the following consensus-based Kalman estimator can be constructed for agent i,

$$\bar{x}_{i,k} = A\tilde{x}_{i,k-1}, \tag{6}$$

$$\epsilon_{i,k} = A\bar{x}_{i,k-1} + K_{i,k}(y_{i,k} - C_i\bar{x}_{i,k-1}), \tag{7}$$

$$\tilde{x}_{i,k} = \sum_{j=1}^{N} a_{ij}\epsilon_{j,k}, \tag{8}$$

where $K_{i,k}$ is the estimator gain, and $\tilde{x}_{i,k}$ is the consensus estimate by agent i. The aim of estimator (6), (7) and (8) is to find the estimator gain $K_{i,k}$ to minimize the mean-squared estimation error $\sum_{i=1}^{N} \| \tilde{x}_{i,k} - x_k \|^2$.

This paper deals with the problem of distributed estimation with inequality constraints (4). In other words, the constrained estimation problem can be written as follows:

$$\begin{aligned} \min_{\hat{x}_{i,k}} \quad & (\hat{x}_{i,k} - \tilde{x}_{i,k})^\top (\hat{x}_{i,k} - \tilde{x}_{i,k}), \\ s.t. \quad & q_t(\hat{x}_{i,k}) \leq 0, \quad t = 1, \ldots, s. \end{aligned} \tag{9}$$

[1]A graph is called strong connected if for any pair of node (i, j) there exists a path from i to j and vice versa.

Here, the information about constraint (4) is shared by all agents, and therefore, for each agent, the constrained estimation can be obtained by projection,

$$\hat{x}_{i,k} = \mathbf{P}_X(\tilde{x}_{i,k}), \tag{10}$$

with $X = \{x \mid q_t(x) \leq 0, t = 1, \ldots, s\}$.

Most of the distributed Kalman filter designs did not consider any state constraints. Here we consider a constrained distributed estimation problem, which is much more difficult than those without constraints. In [24], the authors proposed an estimation method by projecting the unconstraint estimation onto the constrained surface. When it comes to distributed system, we need to guarantee that the estimate from all the agents, obtained by exchanging information with neighbors, should satisfy the given constraints. Hence, to solve our problem, we have to show how to design the Kalman filter gain $K_{i,k}$ such that the estimation error of (10) is stable.

In the following section, we present a distributed estimation algorithm, and analyze the stochastic stability of the proposed algorithm.

3. DISTRIBUTED ALGORITHM

The proposed consensus-based Kalman filter with constraints is described in Algorithm 1.

Algorithm 1 CKF with constraints

Initialization
$\hat{x}_{i,0}$, $P_{i,0}$;
Local Estimation
$\bar{x}_{i,k} = A\hat{x}_{i,k-1}$,
$\bar{P}_{i,k} = AP_{i,k-1}A^\top + Q$,
$\epsilon_{i,k} = \bar{x}_{i,k-1} + K_{i,k}(y_{i,k} - C_i\bar{x}_{i,k-1})$,
$K_{i,k} = \bar{P}_{i,k}C_i^\top(C_i\bar{P}_{i,k}C_i^\top + R_i)^{-1}$,
$\tilde{P}_{i,k} = (I - K_{i,k}C_i)\bar{P}_{i,k}$;
Consensus
$\tilde{x}_{i,k} = \sum_{j=1}^N a_{ij}\epsilon_{j,k}$,
$P_{i,k} = \sum_{j=1}^N a_{ij}\tilde{P}_{j,k}$;
Projection
$\hat{x}_{i,k} = \mathbf{P}_X(\tilde{x}_{i,k})$.

Note that local estimation step is performed according to the classical Kalman filter. The unconstrained estimation is then obtained by the consensus operation with its neighbors. Finally, the constrained estimation of each agent is achieved by projection. If we ignore the consensus on error covariance and projection step in the case without constraints, our algorithm can be reduced to the DKF described in [17], still with the advantages of low computational complexity and easy implementation.

According to Lemma 2.1,

$$Tr(\mathbb{E}\{(x_k - \hat{x}_{i,k})(x_k - \hat{x}_{i,k})^\top\}) \leq Tr(\mathbb{E}\{(x_k - \tilde{x}_{i,k})(x_k - \tilde{x}_{i,k})^\top\}),$$

which indicates that the estimator can achieve better performance using the constraint information in the design. In [24], the authors proved that the projection approach with linear equality constraints performs better than unconstrained estimation by choosing suitable weights.

Define error terms $e_{i,k} = x_k - \hat{x}_{i,k}$, $\tilde{e}_{i,k} = x_k - \tilde{x}_{i,k}$, $\tilde{\epsilon}_{i,k} = x_k - \epsilon_{i,k}$ and $\bar{e}_{i,k} = x_k - \bar{x}_{i,k}$. The error dynamics $\tilde{e}_{i,k}$ can, therefore, be written as

$$\tilde{e}_{i,k} = \sum_{j=1}^{N} a_{ij}\tilde{\epsilon}_{j,k}. \tag{11}$$

Notice that $P_{i,k}$ in the algorithm does not represent the estimation error covariance with respect to $\hat{x}_{i,k}$ any longer. In section IV, we will show that $P_{i,k}$ is bounded and error term $e_{i,k}$ is stable.

Remark 3.1. In this paper, the constraints are formulated as inequalities $q_t(x) \leq 0$, $t = 1, \ldots, s$, where $q_t(x), t = 1, \ldots, s$ is convex, including the case of linear equality and inequality constraints studied in [24] and [9], respectively. In section 4, the closed-form solution will be given under linear equality constraints $D_k x_k = d_k$ as a special case, consistent with the case in [24]. Notice that the projection method may not be suitable for the nonlinear equality constraints $q_t(x) = 0,\ t = 1, \ldots, s$, because $\{x|\, q_t(x) = 0, t = 1, \ldots, s\}$ may not be a convex set.

Remark 3.2. To facilitate the convergence analysis of the algorithms, we provide some notations here. In fact, without loss of generality, there are positive scalars $\bar{b}$, $\bar{c}$, $\underline{q}$ and $\underline{r}_i$ such that

$$\|A\| \leq \bar{a}, \quad \|C_i\| \leq \bar{c}_i, \quad Q \geq \underline{q}I, \quad R_i \geq \underline{r}_i I. \tag{12}$$

Since A, C_i, Q, R_i are given, the inequalities are obvious.

4. MAIN RESULTS

In this section, we analyze the convergence of the proposed algorithm. We first give two lemmas, which are useful in convergence analysis.

Lemma 4.1. (Lemma 2.1, Reif et al. [21]) Suppose that there is a stochastic process $V_k(\xi_k)$ as well as positive numbers $\bar{\theta}$, $\underline{\theta}$, μ, and $0 < \alpha \leq 1$ such that

$$\underline{\theta} \parallel \xi_k \parallel^2 \leq V_k(\xi_k) \leq \bar{\theta} \parallel \xi_k \parallel^2 \tag{13}$$

and

$$\mathbb{E}\{V_{k+1}(\xi_{k+1})|\xi_k\} - V_k(\xi_k) \leq \mu - \alpha V_k(\xi_k) \tag{14}$$

are fulfilled. Then the stochastic process is exponentially bounded in mean square, i. e.,

$$\mathbb{E}\{\parallel \xi_k \parallel^2\} \leq \frac{\bar{\theta}}{\underline{\theta}}\mathbb{E}\{\parallel \xi_0 \parallel^2\}(1-\alpha)^n + \frac{\mu}{u}\sum_{i=1}^{n-1}(1-\alpha)^i \tag{15}$$

for every $n \geq 0$. Moreover, the stochastic process is bounded with probability one.

Lemma 4.2. (Lemma 1, Battistelli and Chisci [2]) For any positive semidefinite matrix $\check{P}$, there exists a strictly positive real $\beta \leq 1$ such that $(APA^\top + Q)^{-1} \geq \beta A^{-\top} P^{-1} A^{-1}$ for any $P \leq \check{P}$.

We now present our first main result.

Theorem 4.3. (Boundedness) Consider algorithm 1, under Assumption 2.4 and 2.5, there exist a time instant $\bar{k} > 0$ and a positive definite matrix Π_i, such that $0 < P_{i,k} < \Pi_i$, $\forall i \in \mathscr{V}$ and $k \geq \bar{k}$.

Proof. By information form of Kalman filter, we can see that $\tilde{P}_{j,k}^{-1} = \bar{P}_{j,k}^{-1} + C_j^\top R_j^{-1} C_j$, thus $P_{i,k}$ can be written as

$$P_{i,k+1} = \sum_{j=1}^{N} a_{ij} (\bar{P}_{j,k+1}^{-1} + C_j^\top R_j^{-1} C_j)^{-1}. \tag{16}$$

Notice that $\bar{P}_{j,k+1} = AP_{i,k}A^\top + Q$, then by Lemma 4.2 one can obtain

$$P_{i,k+1} \leq \sum_{j=1}^{N} a_{ij} \beta^{-1} A P_{j,k} A^\top + \sum_{j=1}^{N} a_{ij} C_j^\top R_j^{-1} C_j, \tag{17}$$

where $0 < \beta < 1$.Taking inverse to both side of (17), and recursively applying (16) and (17) $\bar{k}$ times, one can obtain

$$P_{i,k+1}^{-1} \geq \left(\beta^{-\bar{k}} A^{\bar{k}} (\sum_{j=1}^{N} a_{ij,\bar{k}} P_{j,k-\bar{k}}) (A^{\bar{k}})^\top + \sum_{\tau=1}^{\bar{k}} \beta^{1-\tau} A^{\tau-1} (\sum_{j=1}^{N} a_{ij,\bar{k}} C_j^\top R_j^{-1} C_j) (A^{\tau-1})^\top \right)^{-1} \tag{18}$$

where $a_{ij,k}$ denote the (i,j)th element of $\mathcal{A}^k$. We need to show $P_{i,k+1}^{-1}$ is lower bounded by a positive definite matrix.

Define

$$\Delta_1 = \beta^{-\bar{k}} A^{\bar{k}} (\sum_{j=1}^{N} a_{ij,\bar{k}} P_{j,k-\bar{k}}) (A^{\bar{k}})^\top, \tag{19}$$

$$\Delta_2 = \sum_{\tau=1}^{\bar{k}} \beta^{1-\tau} A^{\tau-1} (\sum_{j=1}^{N} a_{ij,\tau} C_j^\top R_j^{-1} C_j) (A^{\tau-1})^\top. \tag{20}$$

Notice that $\Delta_1 \geq 0$, we need to show $\Delta_2 > 0$. Let $\varrho_{\min} = arg \min_\varrho a_{ij,\varrho} > 0$. Actually, by global observability assumption and positive definitness of $R_i, \forall i \in \mathscr{V}$, it not hard to verify that $\sum_{\tau=1}^{\bar{k}} \beta^{1-\tau} A^{\tau-1} (\sum_{j=1}^{N} a_{ij,\tau} C_j^\top R_j^{-1} C_j)(A^{\tau-1})^\top$ is positive definite if $\bar{k} > \varrho_{\min} + m$. The result is concluded. □

Remark 4.4. It should be pointed out that Theorem 4.3 rely on collective observable condition, which has broader application compare with local observable condition. It also should noting that algorithm proposed in this paper without projection step is consistent with the results [17], where it is shown that collective observability is indeed to design distributed estimation algorithm. In [2], a distributed estimation algorithm obtained by covariance intersection scheme, in which covariance matrix inversion is needed to obtain state estimate. Our proposed algorithm does not need inverse operation, which has the benefit of low computational complexity.

Another important feature of proposed algorithm is the convergence results. We give the convergence result in the following theorem.

Theorem 4.5. (Convergence) Under Assumption 2.4 and 2.5, the error dynamics for Algorithm 1 is exponentially bounded in mean square and bounded with probability one for some $\bar{k} > 0$.

Proof. Based on Lemma 2.1, the error term satisfies

$$\|e_{i,k+1}\|^2 \leq \|\tilde{e}_{i,k+1}\|^2 = \| \sum_{j=1}^{N} a_{ij}\tilde{\epsilon}_{j,k+1}\|^2 \leq \sum_{j=1}^{N} a_{ij}\|\tilde{\epsilon}_{j,k+1}\|^2. \tag{21}$$

The last inequality caused by Jenson's inequality.

We define the following Lyapunov function

$$V_k(e_k) = \sum_{i=1}^{N} e_{i,k}^{\top} P_{i,k}^{-1} e_{i,k}. \tag{22}$$

In what follows, we show that $V_k(e_k)$ satisfies Lemma 4.1, which leads to the conclusion.

By Theorem 4.3, there exists a time instance $\bar{k} > 0$, such that $\underline{\Pi} < P_{i,k} < \Pi_i$, $\forall i \in \mathscr{V}$ and $k > \bar{k}$, where $\underline{\Pi}$ is the solution of the centralized filter. Assume that $\underline{p}I_m \leq P_{i,k} \leq \bar{p}_i I_m$, we have

$$\sum_{i=1}^{N} \frac{1}{\bar{p}_i}\|e_{i,k}\|^2 \leq V_k(e_k) \leq \sum_{i=1}^{N} \frac{1}{\underline{p}}\|e_{i,k}\|^2, \tag{23}$$

which satisfies (13). for $k > \bar{k}$ one has

$$V_{k+1}(e_{k+1}) = \sum_{i=1}^{N} e_{i,k+1}^{\top} P_{i,k+1}^{-1} e_{i,k+1} \tag{24}$$

$$\leq \sum_{i=1}^{N} \tilde{e}_{i,k+1}^{\top} P_{i,k+1}^{-1} \tilde{e}_{i,k+1} \tag{25}$$

$$\leq \sum_{i=1}^{N} \left(\sum_{j=1}^{N} a_{ij}\tilde{\epsilon}_{j,k+1} \right)^{\top} P_{i,k+1}^{-1} \left(\sum_{j=1}^{N} a_{ij}\tilde{\epsilon}_{j,k+1} \right) \tag{26}$$

$$\leq \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\tilde{\epsilon}_{j,k+1}^{\top} P_{i,k+1}^{-1}\tilde{\epsilon}_{j,k+1}. \tag{27}$$

The last inequality caused by the fact that

$$\left(\sum_{i=1}^{N} n_i m_i\right)^{\top} M \left(\sum_{i=1}^{N} n_i m_i\right) \leq \sum_{i=1}^{N} n_i m_i^{\top} M m_i,$$

for a positive scale N, a set of non-negative weight $\{n_i\}_{i=1}^{N}$, $\sum_{i=1}^{N} n_i = 1$, a set of vectors m_i and a positive definite matrix M. By changing the summation order, one has

$$V_{k+1}(e_{k+1}) \leq \sum_{i=1}^{N} \tilde{\epsilon}_{i,k+1}^{\top}\left(\sum_{j=1}^{N} a_{ij} P_{i,k+1}^{-1}\right)\tilde{\epsilon}_{i,k+1} \tag{28}$$

$$\leq \sum_{i=1}^{N} \tilde{\epsilon}_{i,k+1}^{\top}\left(a_{ii} P_{i,k+1}^{-1}\right)\tilde{\epsilon}_{i,k+1} \tag{29}$$

$$\leq \sum_{i=1}^{N} \tilde{\epsilon}_{i,k+1}^{\top}\tilde{P}_{i,k+1}^{-1}\tilde{\epsilon}_{i,k+1}. \tag{30}$$

Notice that $\tilde{\epsilon}_{i,k+1} = (I_m - K_{i,k+1}C_i)Ae_{i,k} - K_{i,k+1}v_{i,k+1} + (I_m - K_{i,k+1}C_i)w_k$, and recalling the white noise property and taking expectation on $V_{k+1}(e_{k+1})$, one has

$$\begin{aligned}
&\mathbb{E}\{V_{k+1}(e_{k+1})\} \\
&\leq \mathbb{E}\{\sum_{i=1}^{N} e_{i,k}^{\top}F_{i,k+1}^{\top}\tilde{P}_{i,k+1}^{-1}F_{i,k+1}e_{i,k} + v_{i,k+1}^{\top}K_{i,k+1}^{\top}\tilde{P}_{i,k+1}^{-1}K_{i,k+1}v_{i,k+1} \\
&\qquad + w_k^{\top}F_{i,k+1}^{\top}\tilde{P}_{i,k+1}^{-1}F_{i,k+1}w_k\},
\end{aligned} \tag{31}$$

where $F_{i,k+1} = (I_m - K_{i,k+1}C_i)A$. Similar with Lemma 3.3 in [21], there exists a positive scale δ_i, such that

$$\mathbb{E}\{v_{i,k+1}^{\top}K_{i,k+1}^{\top}\tilde{P}_{i,k+1}^{-1}K_{i,k+1}v_{i,k+1} + w_k^{\top}F_{i,k+1}^{\top}\tilde{P}_{i,k+1}^{-1}F_{i,k+1}w_k\} < \delta_i. \tag{32}$$

One can express $\tilde{P}_{i,k+1}$ as follows,

$$\tilde{P}_{i,k+1} = F_{i,k+1}AP_{i,k}A^{\top}F_{i,k+1}^{\top} + K_{i,k+1}R_iK_{i,k+1}^{\top} + F_{i,k+1}QF_{i,k+1}^{\top}. \tag{33}$$

Denote $\mathcal{Q}_{i,k+1} = F_{i,k+1}QF_{i,k+1}^{\top}$, and notice $Q > 0$, we have $\mathcal{Q}_{i,k+1} \geq 0$. By remark 3.2, one has

$$\| K_{i,k} \| \leq \frac{\bar{\rho}_i c_i}{r_i}. \tag{34}$$

Since $R_i > 0$, combining (33) and (34), one can obtain

$$\tilde{P}_{i,k+1} \;\geq\; F_{i,k+1}(P_{i,k} + F_{i,k+1}^{-1}\mathcal{Q}_{i,k+1}F_{i,k+1}^{-\top})F_{i,k+1}^{\top} \;\geq\; F_{i,k+1}(P_{i,k} + \frac{q}{\bar{a}^2}I_m)F_{i,k+1}^{\top}. \tag{35}$$

Notice that $P_{i,k}$ is bounded for $k > \bar{k}$, thus $\bar{P}_{i,k}$ is also bounded for $k > \bar{k}$. Denote $\underline{\rho}_i I_m < \bar{P}_{i,k} < \bar{\rho}_i I_m$. Taking inverse of both side of (35), multiplying from left and right with $F_{i,k+1}^\top$ and $F_{i,k+1}$, and then taking expectation operator, we obtain

$$\mathbb{E}\{F_{i,k+1}^\top \tilde{P}_{i,k+1}^{-1} F_{i,k+1}\} \leq (1 + \frac{q}{\bar{\rho}_i \bar{a}^2})^{-1} P_{i,k}^{-1}. \tag{36}$$

Substituting (32) and (36) into (31) yield

$$\mathbb{E}\{V_{k+1}(e_{k+1})\} \leq \mathbb{E}\{\sum_{i=1}^{N}(1 + \frac{q}{\bar{\rho}_i \bar{a}^2})^{-1} e_{i,k}^\top P_{i,k}^{-1} e_{i,k} + \sum_{i=1}^{N} \delta_i \tag{37}$$

$$= \sum_{i=1}^{N}(1 - \alpha_i) e_{i,k}^\top P_{i,k}^{-1} e_{i,k} + \sum_{i=1}^{N} \delta_i, \tag{38}$$

where $0 < \alpha_i < 1$, and $(1 - \alpha_i) = (1 + \frac{q}{\bar{\rho}_i \bar{a}^2})^{-1}$. Equation (37) can be written as follows,

$$\mathbb{E}\{V_{k+1}(e_{k+1})\} \leq (1 - \alpha) V_k(e_k) + \delta, \tag{39}$$

where $\bar{\alpha} = min\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$, and $\bar{\delta} = \sum_{i=1}^{N} \delta_i$. Thus, the conclusion follows. □

Remark 4.6. From the proof of Theorem 4.5, the stochastic stability of Algorithm 1 can be guaranteed for any $k > \bar{k}$. Different with [2, 20], which showed the asymptotic stability of the error dynamics (without noise terms), we show the stochastic stability of the error dynamics. Note that the stochastic stability studied in [21] was not for constraint and distributed case. It also should be noticed that the information requires to exchange is the local estimation pair $(\epsilon_{i,k}, \tilde{P}_{i,k})$, and it can reduce the use of network bandwidth.

In what follows, we study the distributed filter with linear equality constraints as a special case, which can be written in the following form:

$$D_k x_k = d_k, \tag{40}$$

where $D_k \in \mathbb{R}^{s \times m}$ is constraint matrix, $d_k \in \mathbb{R}^s$, and s is the number of constraints. Suppose that D_k is of full rank. If D_k is not of full rank, then we have redundant constraints. In this case, we can remove the linearly dependent rows from D_k until D_k is of full rank.

Denote $X = \{x | D_k x = d_k\}$, which is clearly a convex set. Thus, we can project the local unconstrained estimate onto X. For each individual agent, we can obtain the constrained estimate by projection as follows:

$$\hat{x}_{i,k+1} = \tilde{x}_{i,k+1} - D_{k+1}^\top (D_{k+1} D_{k+1}^\top)^{-1} (D_{k+1} \tilde{x}_{i,k+1} - d_{k+1}), \tag{41}$$

where $\tilde{x}_{i,k+1}$ is the estimate obtained by the consensus step. Therefore, we have the following result.

Corollary 4.7. Under Assumptions 1 and 2, the error dynamics of the Algorithm 1 with constraints (40) is exponentially bounded in mean square and bounded with probability one for some $k > \bar{k}$. Moreover, the constrained estimate is

$$\hat{x}_{i,k+1} = \tilde{x}_{i,k+1} - D_{k+1}^{\top}(D_{k+1}D_{k+1}^{\top})^{-1}(D_{k+1}\tilde{x}_{i,k+1} - d_{k+1}).$$

Clearly, Corollaries 4.7 can easily be applied to the linear state inequality constraints $D_k x_k \leq d_k$. If the estimation by the consensus step satisfies the inequality constraints (i. e., $D_k\tilde{x}_{i,k} \leq d_k$), then the projected estimation $\hat{x}_{i,k}$ and consensus estimation $\tilde{x}_{i,k}$ will be the same. Otherwise, the solutions of linear inequality constraints can be obtained by projection the consensus estimation onto $D_k x_k = d_k$.

Above discussions are restrict to linear time-invariant dynamics (3) and observation model (5). Considering the following nonlinear time-invariant dynamics and observation model,

$$x_{k+1} = f(x_k) + w_k, \tag{42}$$
$$y_{i,k} = h_i(x_k) + v_{i,k}. \tag{43}$$

The algorithm 1 can be applied to nonlinear case by linearizing about current state estimate. The detail of distributed nonlinear filtering design can be found in [11]. The linearizion lead to the case with A_k and $C_{i,k}$ being the Jacobian of the partial derivatives of $f(x_k)$ and $h_i(x_k)$ w.r.t x. Thus, we can obtain the following linear time-varying model,

$$x_{k+1} = A_k x_k + w_k, \tag{44}$$
$$y_{i,k} = C_{i,k} x_k + v_{i,k}. \tag{45}$$

From the proof of Theorem 4.3, it is not hard to check that $P_{i,k}$ is bounded, if A_k is nonsingular. However, under the same assumptions, it is not easy to obtain the convergence property due to the time-varing A_k and $C_{i,k}$.

5. SIMULATIONS

In this section, we illustrate the proposed algorithms by some numerical simulations. We compare the estimation performances of the proposed estimator with the CIDKF in [2] and the CBDKF in [17] without constraints. CIDKF is a fusion-based distributed estimation algorithm, in which the consensus estimates obtained by covariance intersection sheme. CBDKF is achieved by consensus on eatimates and covariance matrix.

Consider a network with $n = 6$ agents to track a target that moves along a line with a constant velocity. The topology of the network is shown in Figure 1. The target dynamics is given by the following equation

$$x_{k+1} = \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x_k + w_k, \tag{46}$$

where T is the sampling period, which is chosen as $T = 1s$. Assume that the position of the target can be measured by each agent, i. e., the first two components of x_k. The last two components of x_k can be treated as the velocity along different directions.

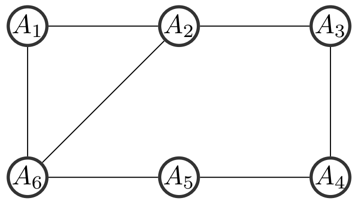

Fig. 1. Topology of the multi-agent network.

Denote $x_k = [x_k(1), x_k(2), x_k(3), x_k(4)]^\top$. Then the measurement matrix of different agent are

$$C_1 = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}, \ C_2 = \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}, \ C_3 = \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}, \tag{47}$$

$$C_4 = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}, \ C_5 = \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}, \ C_6 = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}. \tag{48}$$

Notice that each agent is not observable, but the collective (A, C) is observable, where $C = [C_1^\top, \dots, C_6^\top]^\top$.

As stated in [24], the vehicle may be travelling off-road, or on an unknown road, where the problem is unconstrained. Most of the time it may be traveling along a given road, where the estimation problem is constrained. Here we consider that the target is travelling on a road with a heading of η, which means $\tan\eta = \frac{x_k(2)}{x_k(1)} = \frac{x_k(4)}{x_k(3)}$. Then matrix D and vector d can be written as:

$$\begin{aligned} D &= \begin{bmatrix}1 & -\tan\eta & 0 & 0\\ 0 & 0 & 1 & -\tan\eta\end{bmatrix}\\ d &= \begin{bmatrix}0 & 0\end{bmatrix}^\top \end{aligned} \tag{49}$$

Take $\eta = 60\ deg$ and $Q = diag(0.1,\ 0.1,\ 0.1,\ 0.1)$. Denote $e_0 = [5, 5, 0.3, 0.3]^\top$, $x_0 = [0, 0, \tan\eta, 1]^\top$, and initial conditions by agent i is set to $x_{i,k} = x_0 + (-1)^i i e_0$, $R_1 = diag(90,\ 90)$,$R_2 = diag(80,\ 80)$, $R_3 = diag(70,\ 70)$, $R_4 = diag(75,\ 75)$, $R_5 = diag(85,\ 85)$, $R_6 = diag(95,\ 95)$.

We consider the total mean square estimation errors (TMSEE), which is widely used to indicate the performance of the estimator, which is defined as

$$TMSEE_k = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(\hat{x}_{i,k} - x_k)^\top(\hat{x}_{i,k} - x_k).$$

In both cases, 200 times independent Monte Carlo simulations are carried out to show the estimation performance. Figure 2 shows the TMSEE of Algorithm 1 and unconstrained CIDKF in [2] and compare to the CBDKF in [17], and $Tr(P) = \frac{1}{N}\sum_i^N Tr(P_i)$. It can be seen that the constrained estimation by Algorithm 1 is more accurate than the unconstrained CIDKF and CBDKF. Therefore, for this example based on the constraint information, the proposed CKF with the constraints outperforms unconstrained CIDKF and CBDKF.

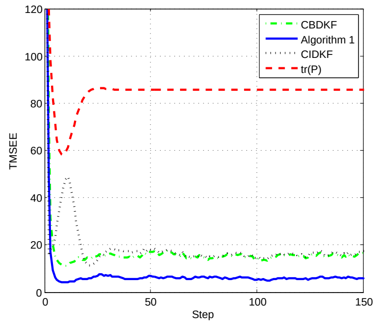

Fig. 2. Estimation performance.

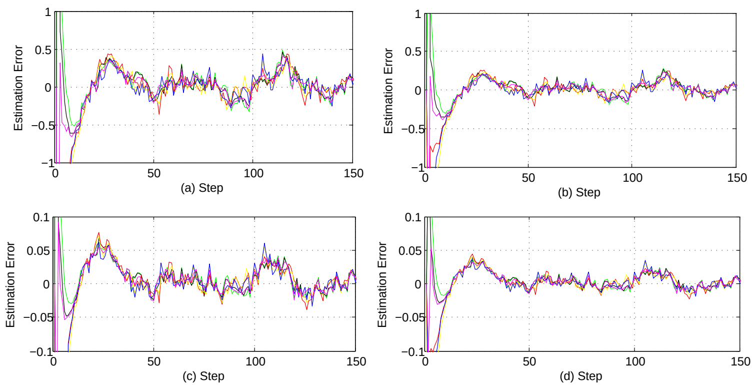

Fig. 3. Performance of agents in Algorithm 1. (a), (b), (c) and (d) represent the estimation error of $x_k(1)$, $x_k(2)$, $x_k(3)$ and $x_k(4)$, respectively.

6. CONCLUSIONS

Distributed estimation based on the consensus strategy with state constraints was proposed in this paper. A fully distributed algorithm was proposed. Under collective observability and connectivity assumptions, the convergence of the proposed algorithms was verified, and the conditions for the corresponding stability were given. Moreover, it was shown that the information of additional state constraints is useful to improve the performance of distributed estimators.

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