Mathematica Bohemica

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Mathematica Bohemica, Vol. 142 (2017), No. 3, 299–307

Persistent URL: http://dml.cz/dmlcz/146827

COEFFICIENT MULTIPLIERS ON SPACES OF VECTOR-VALUED ENTIRE DIRICHLET SERIES

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Received February 17, 2016. First published January 3, 2017.
Communicated by Grigore Sălăgean

Abstract. The spaces of entire functions represented by Dirichlet series have been studied by Hussein and Kamthan and others. In the present paper we consider the space X of all entire functions defined by vector-valued Dirichlet series and study the properties of a sequence space which is defined using the type of an entire function represented by vector-valued Dirichlet series. The main result concerns with obtaining the nature of the dual space of this sequence space and coefficient multipliers for some classes of vector-valued Dirichlet series.

Keywords: vector-valued Dirichlet series; analytic function; entire function; dual space; norm

MSC 2010: 30B50, 30D15, 46E40

1. Introduction

Let

$$f(s)=\sum_{n=1}^{\infty}a_n\mathrm{e}^{s\lambda_n}, \tag{1.1}$$

where $s=\sigma+\mathrm{i}t$, σ and t are real variables, a_n's belong to a complex Banach algebra E with the unit element ω and $\{\lambda_n\}$ is an increasing sequence such that $0<\lambda_1<\lambda_2<\lambda_3<\ldots<\lambda_n\ldots$; $\lim\limits_{n\to\infty}\lambda_n=\infty$ and

$$\limsup_{n\to\infty}\frac{\log n}{\lambda_n}=D<\infty. \tag{1.2}$$

Let $\sigma_c(f)$ and $\sigma_a(f)$ be the abscissa of convergence and abscissa of absolute convergence, respectively, of the series in (1.1). Then under the condition (1.2), we have

DOI: 10.21136/MB.2017.0026-16

(see [3], page 59)

$$0 \leqslant \sigma_c(f) - \sigma_a(f) \leqslant D.$$

Further, if $D = 0$, then (see [3], page 62),

$$\sigma_c = \sigma_a = -\limsup_{n\to\infty} \frac{\ln \|a_n\|}{\lambda_n}. \tag{1.3}$$

Much earlier, Mandelbrojt (see [2], page 166) had obtained a result similar to (1.3) for the classical Dirichlet series $\sum_{n=1}^{\infty} a_n \mathrm{e}^{-s\lambda_n}$ under the condition

$$\limsup_{n\to\infty} \frac{n}{\lambda_n} < \infty. \tag{1.4}$$

It is evident that if (1.4) holds, then $D = 0$.

Suppose that the sequence $\{\lambda_n\}$ in the vector-valued Dirichlet series (1.1) given above satisfies the condition (1.4) so that (1.3) holds. If $\sigma_c(f) = \sigma_a(f) = \infty$, then $f(s)$ is a vector-valued entire function represented by the Dirichlet series in (1.1). We define its maximum modulus as

$$M(\sigma) = \sup_{-\infty<t<\infty} \|f(\sigma + \mathrm{i}t)\|.$$

The concepts of order and type of an entire function represented by vector-valued Dirichlet series of one complex variable were first introduced in [3] by Srivastava. Thus the order ϱ of the entire function $f(s)$ is defined as

$$\varrho = \limsup_{\sigma\to\infty} \frac{\log\log M(\sigma)}{\sigma}, \quad 0 \leqslant \varrho \leqslant \infty. \tag{1.5}$$

When $0 < \varrho < \infty$, the type T of $f(s)$ is defined as

$$T = \limsup_{\sigma\to\infty} \frac{\log M(\sigma)}{\exp(\varrho\sigma)}, \quad 0 \leqslant T \leqslant \infty. \tag{1.6}$$

Srivastava in [3] also obtained the coefficient characterizations of order and type. Thus $f(s)$ is an entire function of order ϱ if and only if

$$\varrho = \limsup_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\log \|a_n\|^{-1}}. \tag{1.7}$$

Further, if $f(s)$ is an entire of order ϱ, then it is of type T if and only if

$$T = \limsup_{n\to\infty} \frac{\lambda_n \|a_n\|^{\varrho/\lambda_n}}{\varrho \mathrm{e}}. \tag{1.8}$$

Let X denote the linear space of all entire functions $f(s)$ defined by vector-valued Dirichlet series (1.1) over the complex field and satisfying

$$\limsup_{n\to\infty} \frac{\lambda_n \|a_n\|^{\varrho/\lambda_n}}{\varrho \mathrm{e}} \leqslant T. \tag{1.9}$$

Lê Hai Khôi in [1] introduced various concepts of duality for sequence spaces which we state below.

Let A and B be two sequence spaces. We denote the sequence space of "multipliers" from A to B by (A, B) such that

$$(A, B) = \{u = (u_n)\colon\ (u_n a_n) \in B,\ \forall\,(a_n) \in A\}.$$

A sequence space A is said to be normal, if whenever A contains (a_n) it also contains the sequence (b_n) satisfying $\|b_n\| \leqslant \|a_n\|$ for $n = 1, 2, \dots$ Equivalently, A is normal if $l^\infty \subset (A, A)$. If D is a fixed sequence space, then the D-dual of a sequence space A is defined to be (A, D), the space of multipliers from A to D, and denoted by A^D. Some duals are defined with some conditions such as Köthe dual or Abel dual. The Köthe dual is obtained when $D = l^1$, and will be denoted by A^α (it is also denoted by A^K). The Abel dual is obtained when D is the space of Abel-summable sequences, that is, the space of sequences (d_n) for which $\lim\limits_{r\to 1} \sum\limits_{n=1}^{\infty} d_n r^n$ exists. Note that when $d_n \geqslant 0$, the existence of this limit is equivalent to the condition $\sum d_n < \infty$. We denote the Abel dual of A by A^a. It is clear that $A^\alpha \subseteq A^a$. The reverse inclusion is true if space A is normal.

The aim of this paper is to introduce a new sequence space using the type of entire functions represented by vector-valued Dirichlet series (VVDS) and obtain some auxiliary conditions of convergence of VVDS given in (1.1). In what follows we shall always consider E to be a complex Banach algebra and assume that the sequence $\{\lambda_n\}$ satisfies the condition (1.4). Consequently, (1.2) also holds and $D = 0$.

2. Main results

We denote by E_T the sequence space

$$E_T = \{(a_n)\colon\ a_n \in E \text{ and } (a_n) \text{ satisfies (1.9)}\}.$$

In this section, we study some dual spaces of the space E_T. We note that if the sequence $\{\lambda_n\}$ satisfies condition (1.2), then

$$\sum_{n=1}^{\infty} r_n^{\lambda} < \infty \quad \forall\, r \in (0, 1). \tag{2.1}$$

The Köthe dual of the space E_T is defined as

$$E_T^\alpha = \Big\{(u_n)\colon \sum_{n=1}^{\infty} \|u_n a_n\| \text{ converges } \forall\,(a_n) \in E_T\Big\}.$$

Now we introduce another sequence space E_T^β defined as

$$E_T^\beta = \Big\{(u_n)\colon \sum_{n=1}^{\infty} u_n a_n \text{ converges } \forall\,(a_n) \in E_T\Big\}.$$

It can be easily verified that $E_T^\alpha \subseteq E_T^\beta$. We now find the criteria for the reverse inclusion relation to be true

We now prove the following statement.

Theorem 1. *If $(u_n) \in E_T^\beta$, then we have*

$$\liminf_{n\to\infty} \frac{\lambda_n \|u_n\|^{-\varrho/\lambda_n}}{\varrho \mathrm{e}} \geqslant T. \tag{2.2}$$

Conversely, if the sequence (u_n) satisfies (2.2), then $(u_n) \in E_T^\alpha$.

P r o o f. Let us assume that (2.2) does not hold, i.e.,

$$\liminf_{n\to\infty} \frac{\lambda_n \|u_n\|^{-\varrho/\lambda_n}}{\varrho \mathrm{e}} < T.$$

Then for a given $\varepsilon > 0$ there exists a sequence (n_k) of positive integers such that

$$\frac{\lambda_{n_k} \|u_{n_k}\|^{-\varrho/\lambda_{n_k}}}{\varrho \mathrm{e}} < T + \varepsilon \quad \forall\, k \geqslant 1.$$

Let (a_n) be a sequence defined as

$$a_n = \begin{cases} \dfrac{\omega}{\|u_n\|}, & \text{if } n = n_k,\ k = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\limsup_{n\to\infty} \frac{\lambda_n \|a_n\|^{\varrho/\lambda_n}}{\varrho \mathrm{e}} = \lim_{k\to\infty} \frac{\lambda_{n_k} \|a_{n_k}\|^{\varrho/\lambda_{n_k}}}{\varrho \mathrm{e}} = \lim_{k\to\infty} \frac{\lambda_{n_k} \|u_{n_k}\|^{-\varrho/\lambda_{n_k}}}{\varrho \mathrm{e}} \leqslant T.$$

It follows that $(a_n) \in E_T$. But $\|a_n u_n\| = 1$ for $n = n_k$, $k = 1, 2, \ldots$, that is, $\lim\limits_{n\to\infty} \|a_n u_n\| \neq 0$. So the series $\sum\limits_{n=1}^{\infty} \|u_n a_n\|$ does not converge. Hence if $(u_n) \in E_T^\beta$, then (2.2) will always be satisfied.

Conversely, suppose that (2.2) holds, that is,

$$\liminf_{n\to\infty} \frac{\lambda_n \|u_n\|^{-\varrho/\lambda_{n_k}}}{\varrho \mathrm{e}} = M \geqslant T.$$

Then for a given $\varepsilon > 0$, there exists N_1 such that for all $n \geqslant N_1$ we have

$$\frac{\lambda_n \|u_n\|^{-\varrho/\lambda_{n_k}}}{\varrho \mathrm{e}} \geqslant M - \varepsilon.$$

Also, for every sequence $(a_n) \in E_T$, using (1.9), we can find a positive integer N_2 such that for all $n \geqslant N_2$

$$\|a_n\|^{\varrho/\lambda_n} < \frac{(T+\varepsilon)\varrho \mathrm{e}}{\lambda_n} \quad \forall\, n \geqslant N_2.$$

Therefore for all $n \geqslant \max\{N_1, N_2\}$,

$$\|a_n u_n\|^{\varrho/\lambda_n} \leqslant \frac{T+\varepsilon}{M-\varepsilon}, \quad \text{i.e.,} \ \|a_n u_n\| \leqslant \Bigl(\frac{T+\varepsilon}{M-\varepsilon}\Bigr)^{\lambda_n/\varrho}.$$

For $M > T$, we choose any $\varepsilon > 0$ such that $M-\varepsilon > T+\varepsilon$. Then from (2.1) we can see that the series $\sum\limits_{n=1}^{\infty} \|a_n u_n\|$ converges. Hence $(u_n) \in E_T^{\beta}$. This proves Theorem 1. □

Theorem 2. *The space E_T is perfect, i.e., $E_T^{\alpha\alpha} = E_T$.*

P r o o f. Let the sequence $(a_n) \notin E_T$. Then we have

$$\limsup_{n\to\infty} \frac{\lambda_n \|a_n\|^{\varrho/\lambda_n}}{\varrho \mathrm{e}} \geqslant T.$$

We denote by T' the left-hand side limit if it is finite and a number $> T$ if the limit is infinite. Then for a given arbitrarily small $\varepsilon > 0$, there exists a sequence (n_k) of positive integers such that

$$\|a_{n_k}\|^{\varrho/\lambda_{n_k}} \geqslant \frac{(T'-\varepsilon)\varrho \mathrm{e}}{\lambda_{n_k}}, \quad k = 1, 2, \ldots$$

Let us define a sequence

$$u_n = \begin{cases} \dfrac{\omega}{\|a_n\|} & \text{if } n = n_k,\ k = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\liminf_{n\to\infty} \frac{\lambda_n \|u_n\|^{-\varrho/\lambda_n}}{\varrho \mathrm{e}} = \lim_{k\to\infty} \frac{\lambda_{n_k} \|u_{n_k}\|^{-\varrho/\lambda_{n_k}}}{\varrho \mathrm{e}} = \lim_{k\to\infty} \frac{\lambda_{n_k} \|a_{n_k}\|^{\varrho/\lambda_{n_k}}}{\varrho \mathrm{e}} \geqslant T.$$

Hence from Theorem 1, $(u_n) \in E_T^{\alpha}$. But $\|a_n u_n\| = 1$ for $n = n_k$, i.e., $\sum a_n u_n$ does not converge. Therefore $(a_n) \notin E_T^{\alpha\alpha}$. Hence $E_T^{\alpha\alpha} \subseteq E_T$. The reverse inclusion always holds. Hence the space E_T is perfect. □

Theorem 3. *For the sequence space E_T defined as above, we have*

$$(E_T, l^p) = E_T^\alpha \quad \forall\, 0 < p \leqslant \infty.$$

P r o o f. Suppose that a sequence $(u_n) \notin E_T^\alpha$. Then from Theorem 1, we have

$$\liminf_{n\to\infty} \frac{\lambda_n \|u_n\|^{-\varrho/\lambda_n}}{\varrho \mathrm{e}} \leqslant T.$$

Then for an arbitrarily small $\varepsilon > 0$, there exists a sequence (n_k) of positive integers such that

$$\frac{\lambda_n \|u_n\|^{-\varrho/\lambda_n}}{\varrho \mathrm{e}} \leqslant T + \varepsilon, \quad n = n_k \ \forall\, k \geqslant 1.$$

Let $0 < p < \infty$. We consider the sequence

$$a_n = \begin{cases} \dfrac{\omega}{\|u_{n_k}\|} & \text{if } n = n_k,\ k = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\limsup_{n\to\infty} \frac{\lambda_n \|a_n\|^{\varrho/\lambda_n}}{\varrho \mathrm{e}} = \limsup_{k\to\infty} \frac{\lambda_{n_k} \|a_{n_k}\|^{\varrho/\lambda_n}}{\varrho \mathrm{e}} = \limsup_{k\to\infty} \frac{\lambda_{n_k} \|u_{n_k}\|^{-\varrho/\lambda_{n_k}}}{\varrho \mathrm{e}} \leqslant T.$$

Hence we get $(a_n) \in E_T$. By the definition of (E_T, l^p), $\sum\limits_{n=1}^{\infty} \|a_n u_n\|^p$ should be convergent. But $\|a_n u_n\| = 1$, $n = 1, 2, \ldots$ This implies $(a_n u_n) \notin l^p$.

For the case when $p = \infty$, consider a sequence

$$a_n = \begin{cases} \dfrac{\omega n_k}{\|u_{n_k}\|} & \text{if } n = n_k,\ k = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} \limsup_{n\to\infty} \frac{\lambda_n \|a_n\|^{\varrho/\lambda_n}}{\varrho \mathrm{e}} &= \limsup_{k\to\infty} \frac{\lambda_{n_k} \|a_{n_k}\|^{\varrho/\lambda_{n_k}}}{\varrho \mathrm{e}} \\ &= \limsup_{k\to\infty} \frac{n_k^{\varrho/\lambda_{n_k}} \lambda_{n_k} \|u_{n_k}\|^{-\varrho/\lambda_{n_k}}}{\varrho \mathrm{e}} \leqslant T, \end{aligned}$$

since $\lim\limits_{k\to\infty} n_k^{1/n_k} = 1$. Hence $(a_n) \in E_T$. Since $\lim\limits_{k\leftrightarrow\infty} \|a_{n_k} u_{n_k}\| = \infty$, this implies that $(a_n u_n) \notin l^\infty$. Hence we conclude that for $0 < p \leqslant \infty$, $(u_n) \notin E_T^\alpha \Rightarrow (u_n) \notin (E_T, l^p)$ or equivalently, $(E_T, l^p) \subseteq E_T^\alpha$, $0 < p \leqslant \infty$.

Conversely, assume that $(u_n) \in E_T^\alpha$. Then for a given $M > T$, there exists N_1 such that
$$\|u_n\| \leqslant \Bigl(\frac{\varrho \mathrm{e} M}{\lambda_n}\Bigr)^{-\lambda_n/\varrho} \qquad \forall\, n \geqslant N_1.$$
Suppose that $(a_n) \in E_T$, then for $\delta \in (0, M-T)$ there exists N_2 such that for all $n \geqslant N_2$
$$\|a_n\| \leqslant \Bigl(\frac{\varrho \mathrm{e}(T+\delta)}{\lambda_n}\Bigr)^{\lambda_n/\varrho} \qquad \forall\, n \geqslant N_2.$$
Consequently, for all $n \geqslant N = \max\{N_1, N_2\}$, we have
$$\|a_n u_n\| \leqslant \|a_n\| \|u_n\| < \Bigl(\frac{T+\delta}{M}\Bigr)^{\lambda_n}.$$
If $0 < p < \infty$, then since $(T+\delta)/M < 1$, we have by condition (2.1),
$$\sum_{n=N}^{\infty} \|a_n u_n\|^p \leqslant \sum_{n=N}^{\infty} \Bigl(\frac{T+\delta}{M}\Bigr)^{p\lambda_n/\varrho} < \infty,$$
which implies that $(a_n u_n) \in l^p$.

Now let us take $p = \infty$, then we have $\|a_n u_n\| \leqslant ((T+\delta)/M)^{\lambda_n/\varrho} < 1$ for all $n \geqslant N$, which shows that $(a_n u_n) \in l^\infty$. Thus in both cases, $(u_n) \in (E_T, l^p)$ and consequently, $E_T^\alpha \subset (E_T, l^p)$, $0 < p \leqslant \infty$. This completes the proof of Theorem 3. □

In the next theorem we obtain the sequence space of multipliers from l^p to E_T.

Theorem 4. *A sequence (u_n) is a multiplier from l^p to E_T if*
$$(l^p, E_T) = E_T, \quad 0 < p \leqslant \infty.$$

Proof. Let $(u_n) \in (l^p, E_T)$, $0 < p \leqslant \infty$ and suppose that $(u_n) \notin E_T$. Then we have
$$\liminf_{n\to\infty} \frac{\lambda_n \|u_n\|^{-\varrho/\lambda_n}}{\varrho \mathrm{e}} = M < T.$$
Then for a given number δ, $0 < 2\delta < T - M$, there exists a sequence (n_k) of positive integers such that $\lambda_{n_k} \|u_{n_k}\|^{-\varrho/\lambda_{n_k}} \varrho^{-1} \mathrm{e}^{-1} \leqslant M + \delta$ for all $k \geqslant 1$. This implies $\|u_{n_k}\|^{-1} \leqslant ((M+\delta)\varrho \mathrm{e} \lambda_{n_k}^{-1})^{\lambda_{n_k}/\varrho}$ for all $k \geqslant 1$.

Define a new sequence (b_n) such that
$$b_n = \begin{cases} \dfrac{\omega((M+2\delta)\varrho \mathrm{e} \lambda_{n_k}^{-1})^{-\lambda_{n_k}/\varrho}}{\|u_{n_k}\|} & \text{if } n = n_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have by (2.1),

$$\sum_{n=1}^{\infty}\|b_n\|^p=\sum_{k=1}^{\infty}\|b_{n_k}\|^p=\sum_{k=1}^{\infty}\|u_{n_k}\|^{-p}\Big\|\omega\Big(\frac{(M+2\delta)\varrho\mathrm{e}}{\lambda_{n_k}}\Big)\Big\|^{-\lambda_{n_k}p/\varrho}$$
$$\leqslant\sum_{k=1}^{\infty}\Big(\frac{M+\delta}{M+2\delta}\Big)^{\lambda_{n_k}p/\varrho}<\infty,$$

which shows that $(b_n)\in l^p$. Now consider

$$\liminf_{n\to\infty}\frac{\lambda_n\|b_nu_n\|^{-\varrho/\lambda_n}}{\varrho\mathrm{e}}=\liminf_{k\to\infty}\frac{\lambda_{n_k}\|b_{n_k}u_{n_k}\|^{-\varrho/\lambda_{n_k}}}{\varrho\mathrm{e}}=(M+2\delta)<T.$$

In the second case, i.e., for $p=\infty$, we define a sequence (c_n) such that

$$c_n=\begin{cases}\dfrac{\omega((M+\delta)\varrho\mathrm{e}\lambda_{n_k}^{-1})^{-\lambda_{n_k}/\varrho}}{\|u_{n_k}\|} & \text{if } \ n=n_k,\\ 0 & \text{otherwise.}\end{cases}$$

We can see that $\|c_n\|\leqslant 1$ for all $n\geqslant 1$, which shows that $(c_n)\in l^{\infty}$. Then we have

$$\liminf_{n\to\infty}\frac{\lambda_n\|c_nu_n\|^{-\varrho\lambda_n}}{\varrho\mathrm{e}}=\liminf_{k\to\infty}\frac{\lambda_{n_k}\|c_{n_k}u_{n_k}\|^{-\varrho\lambda_{n_k}}}{\varrho\mathrm{e}}=M+\delta<T.$$

Hence we see that in both cases, the sequences (b_nu_n) and (c_nu_n) do not belong to E_T even though $(b_n)\in l^p$ and $(c_n)\in l^{\infty}$. This is a contradiction. Thus $(l^p,E_T)\subset E_T$, $0<p\leqslant\infty$.

To prove the converse, assume that $(u_n)\in E_T$. Then we have

$$\liminf_{n\to\infty}\frac{\lambda_n\|u_n\|^{-\varrho/\lambda_n}}{\varrho\mathrm{e}}\geqslant T.$$

Let (d_n) be an arbitrary sequence such that $(d_n)\in l^p$, $0<p\leqslant\infty$. In both cases, there exists a constant P such that $\|d_n\|\leqslant P$ for all $n\geqslant 1$. Hence we have

$$\liminf_{n\to\infty}\frac{\lambda_n\|d_nu_n\|^{-\varrho/\lambda_n}}{\varrho\mathrm{e}}=\liminf_{k\to\infty}\frac{\lambda_{n_k}\|d_{n_k}u_{n_k}\|^{-\varrho/\lambda_{n_k}}}{\varrho\mathrm{e}}$$
$$=\liminf_{k\to\infty}\frac{\lambda_{n_k}P^{-\varrho/\lambda_{n_k}}\|u_{n_k}\|^{-\varrho/\lambda_{n_k}}}{\varrho\mathrm{e}}\leqslant T.$$

which shows that $(d_nu_n)\in E_T$. Thus $E_T\subset(l^p,E_T)$ for all $0<p\leqslant\infty$. Hence the result follows. □

A c k n o w l e d g e m e n t. The authors are very much thankful to the referee for his valuable comments which helped in improving the paper.

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