Jae-Hyouk Lee; Mang Xu; Jiajin Zhang

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POLYTOPES, QUASI-MINUSCULE REPRESENTATIONS AND RATIONAL SURFACES

Jae-Hyouk Lee, Seoul, Mang Xu, Jiajin Zhang, Chengdu

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Abstract. We describe the relation between quasi-minuscule representations, polytopes and Weyl group orbits in Picard lattices of rational surfaces. As an application, to each quasi-minuscule representation we attach a class of rational surfaces, and realize such a representation as an associated vector bundle of a principal bundle over these surfaces. Moreover, any quasi-minuscule representation can be defined by rational curves, or their disjoint unions in a rational surface, satisfying certain natural numerical conditions.

Keywords: rational surface; minuscule representation; polytope

MSC 2010: 14J26, 14N20

1. Introduction

It is well-known that Del Pezzo surfaces S are closely linked to the exceptional Lie groups E_n. For example, if S is of degree $9-n$, the orthogonal complement of the canonical class K in $H^2(S,\mathbb{Z})$, equipped with the natural intersection product, is the root lattice of E_n ([5], [17]), where we extend the exceptional E_n-series to $0 \leqslant n \leqslant 8$ by setting $E_0 = 0$, $E_1 = \mathbb{C}$, $E_2 = A_1 \times \mathbb{C}$, $E_3 = A_2 \times A_1$, $E_4 = A_4$, and $E_5 = D_5$. A curve l in X_n is a *line* under the anti-canonical embedding if and only if $l^2 = l \cdot K = -1$. The lines in S are related to a fundamental representation of E_n.

These structures are used in [6], [7], [8], [16], and so on, to study moduli spaces of rational surfaces with an anti-canonical cycle and moduli spaces of flat principal

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bundles over an elliptic curve. These moduli spaces are related to the duality between the string theory and the F-theory, see [9], [10]. Moreover, these studies are partially extended to rational surfaces S of type G, where G is a simple Lie group of any type or affine E_n-type (see [15], [14]).

For instance, an E_n-surface X_n is simply a *Del Pezzo surface* of degree $9-n$ and it is always a blowup of $\mathbb{P}^2$ at n points which are in general position.

Notice that in the Dynkin diagram of E_n presented as Figure 3, (i) the left-end node corresponds to the set of *lines* in S, (ii) the right-end node corresponds to the set of *rulings*, and (iii) the top node corresponds to the set of *twisted cubics*, that is, the pull-backs of the lines in $\mathbb{P}^2$ via the blowdown map $X_n \to \mathbb{P}^2$ (see Section 3, or [13]). For example, in the most famous case E_6, the set of 27 lines in X_6 corresponds to the left-end node $\alpha_L = \alpha_6$, the set of 27 rulings corresponds to the right-end node $\alpha_R = \alpha_2$, and the set of 72 twisted cubics corresponds to the top node $\alpha_T = \alpha_1$ (which also corresponds to the set of 72 (unordered) collections $l_1, \ldots, l_6$ determining the blowdown map $X_6 \to \mathbb{P}^2$), see Figure 3.

In these examples of curves, we have a transitive action of the Weyl group $W(E_n)$. On the other hand, the combinatorics of these curves is closely related to polytopes in Picard lattices, see [11], [18]. Motivated from these, for all simply laced Lie groups G, we consider all such curves in G-surfaces S (see Section 3) and all quasi-minuscule weights of G. We find that all these curves correspond precisely to quasi-minuscule weights of G, and certain polytopes in Picard lattices $\mathrm{Pic}(S)$. Moreover, we construct a principal G-bundle over S, such that these quasi-minuscule weights define associated vector bundles. Thus, many interesting curves (for instance, 27 lines, 27 rulings and 72 twisted cubics over a smooth cubic surface) can be explained as associated vector bundles of a principal G-bundle $\mathcal{G}$.

For this, given a simply laced simple Lie group G, we define a G-surface as a pair (S, C), where, roughly speaking, S is a rational surface and C is a smooth rational curve sitting in the negative part of the Mori cone of S, such that $\langle K_S, C\rangle^{\perp} \subseteq \mathrm{Pic}(S)$ is a root lattice of G (for details, see Section 3).

Given a G-surface (S, C), a curve D (reduced but possibly reducible) in S is called an (S, C)-curve if $DC = 0$. Recall that the (arithmetic) genus of D is defined by the formula $g(D) = 1 + (D^2 + DK_S)/2$. D is said to be *rational* if $g(D) = 0$. The integer $D(-K_S)$ is called the *degree* of D. D is called a *line* (a *ruling*, a *twisted cubic*, or a *twisted quartic*) if D is rational of degree 1 (2, 3, or 4, respectively) and $DC = 0$. D is called an (S, C)-*section* if $DC = 1$. For details, see Section 3.

For the Lie group G of ADE-type, we draw the Dynkin diagram of G, and label the nodes as in Figure 1, 2 or 3, respectively. Let λ_i be the fundamental weight corresponding to the node α_i.

Let $\mathfrak{a}_n$ be an n-dimensional simplex with equilateral edges, $\mathfrak{b}_n$ an n-dimensional crosspolytope, and k_{21}, $-1 \leqslant k \leqslant 4$, the $(k+4)$-dimensional Gosset polytope (see Section 4 or [11]).

We obtain the relations between (quasi-)minuscule fundamental weights, special curves, Weyl group orbits and special polytopes, listed in the following table (see Theorem 9, Theorem 11 and Proposition 14). For example, in (b1) of Table 1, we consider a fundamental representation V_{λ_n} with a highest weight λ_n of D_n which is minuscule, and relate the representation with the Picard group $\mathrm{Pic}(S)$ of a ruled surface S which has a D_n-type lattice structure. We obtain that its orbit $W(D_n)\lambda_n$ in $\mathrm{Pic}(S)$ is the set of lines in S and also the set of vertices of an n-dimensional crosspolytope $\mathfrak{b}_n$.

	G	λ	$W(G)\lambda$	Polytopes
(a)	A_n	λ_i $1 \leqslant i \leqslant n$	(S,C)-curves of genus $i-n$ $DK_S = i-n-1$	$\mathfrak{a}_{n-i}$ in $\mathfrak{a}_n$
(b1)	D_n	λ_n	lines in S	vertices of $\mathfrak{b}_n$
(b2)	D_n	λ_{n-1}	(S,C)-curves of genus -1 $DK_S = -2$	edges of $\mathfrak{b}_n$
(b3)	D_n	λ_2	(S,C)-sections of degree 1	$\mathfrak{a}_{n-1}$ in $\mathfrak{b}_n$
(b4)	D_n	λ_1	(S,C)-sections of degree 2	$\mathfrak{a}_{n-1}$ in $\mathfrak{b}_n$
(c1)	E_n $4 \leqslant n \leqslant 8$	λ_n	lines in S	vertices in $(n-4)_{21}$
(c2)	E_n $4 \leqslant n \leqslant 6$	λ_1	twisted cubics in S	$\mathfrak{a}_{n-1}$ in $(n-4)_{21}$ $n \geqslant 4$
(c3)	E_n $4 \leqslant n \leqslant 7$	λ_2	rulings in S	$\mathfrak{b}_n$ in $(n-4)_{21}$ $n \geqslant 4$
(c4)	E_n $4 \leqslant n \leqslant 5$	λ_3	twisted quartics in S	$\mathfrak{a}_{n-2}$ in $(n-4)_{21}$ $n \geqslant 5$

Table 1. Curves and polytopes associated with quasi-minuscule representations.

For further study, we see that for every nontrivial quasi-minuscule weight λ of any (simply laced) Lie group G, there is a G-surface (S,C) and a principal G-bundle over S, such that the associated bundle $\mathcal{V}_\lambda$ with the highest weight λ is defined by a set of certain special curves in S. Each of these curves is always an irreducible rational curve, or a disjoint union of such curves. These bundles give those special curves on rational surfaces a meaningful explanation, and can be applied to the study of homogeneous spaces associated with rational surfaces. Furthermore, they are corresponded to subpolytopes of certain polytopes with G-symmetry.

2. Notation, quasi-minuscule weights and quasi-minuscule representations

2.1. Notation. We first list some notation from the Lie theory.

- ▷ G: a complex Lie group, simply laced in this paper
- ▷ $W(G)$: the Weyl group of G
- ▷ $\mathrm{Lie}(G)$: the Lie algebra of G
- ▷ H: a Cartan subgroup of G, which is a torus
- ▷ $\mathfrak{X}(H)$: the characteristic group of H
- ▷ $\Lambda_r(G)$: the root lattice of G
- ▷ $\Lambda_w(G)$: the weight lattice of G
- ▷ $R(G)$: the root system of G
- ▷ $\Delta_r(G)$: the set of all simple roots in $R(G)$
- ▷ $\Delta_w(G)$: the set of all fundamental dominant weights in $\Lambda_w(G)$

2.2. Quasi-minuscule weights. A *quasi-minuscule representation* is an irreducible representation in which all the nonzero weights lie in a single Weyl group orbit. The highest weight of such a representation is said to be a *quasi-minuscule weight.* For example, the adjoint representations of type ADE are quasi-minuscule. If there is no zero weight in a quasi-minuscule representation, then such a quasi-minuscule representation is called *minuscule*, and the highest weight is called a *minuscule weight.* The highest weight of the adjoint representation of G is called the *adjoint weight of* G.

We know that the highest weight space is always one-dimensional. Also, if two weights are in the same Weyl group orbit, then their weight spaces must have the same dimension. Thus, all of the nonzero weight spaces of a quasi-minuscule representation are one-dimensional.

The Dynkin diagrams of simply laced Lie groups are drawn as in Figures 1, 2 and 3, where the simple roots are labeled as α_i.

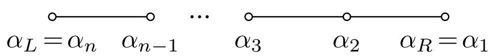

Figure 1. The root system A_n.

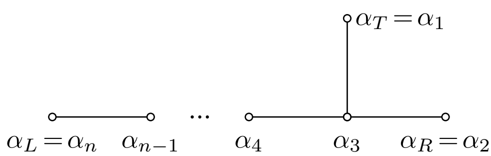

Figure 2. The root system D_n.

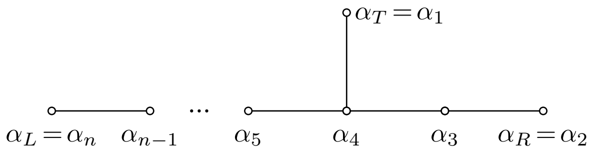

Figure 3. The root system E_n.

Let λ_i be the fundamental weight corresponding to α_i, and let V_{λ_i} be the fundamental representation with the highest weight λ_i. All nontrivial quasi-minuscule fundamental weights and their corresponding fundamental representations of simply laced Lie groups are listed as follows (see [1], [19]):

(1) $G = A_n$; $\lambda_1, \ldots, \lambda_n$; all are minuscule; V_{λ_n} is the standard representation of dimension $n+1$, and $V_{\lambda_i} = \bigwedge^{n+1-i} V_{\lambda_n}$, $i = 1, \ldots, n$.

(2) $G = D_n$; $\lambda_1, \lambda_2, \lambda_{n-1}, \lambda_n$; $\lambda_1, \lambda_2, \lambda_n$ are minuscule; V_{λ_1} and V_{λ_2} are the two spinor representations S^+ and S^- of dimension 2^{n-1}, $V_{\lambda_{n-1}}$ is the adjoint representation, and V_{λ_n} is the standard representation of dimension $2n$.

(3) $G = E_6$; $\lambda_1, \lambda_2, \lambda_6$; λ_2, λ_6 are minuscule; V_{λ_1} is the adjoint representation; V_{λ_2} and V_{λ_6} are of dimension 27 and dual to each other.

(4) $G = E_7$; λ_2, λ_7; λ_7 is minuscule; V_{λ_2} is the adjoint representation; V_{λ_7} is of dimension 56.

(5) $G = E_8$; λ_8; V_{λ_8} is the adjoint representation.

For compatibility with rational surfaces, we extend the exceptional E_n-series to $0 \leqslant n \leqslant 8$, by setting $E_0 = 0$, $E_1 = \mathbb{C}$, $E_2 = A_1 \times \mathbb{C}$, $E_3 = A_2 \times A_1$, $E_4 = A_4$ and $E_5 = D_5$. Since among these, $E_0, \ldots, E_3$ are not simple, we only consider the Lie groups $E_4, \ldots, E_8$.

3. Rational ADE-surfaces

3.1. Definitions. The definition of ADE-surfaces is motivated by the classical Del Pezzo surfaces, which we call *E_n-surfaces* ([15]). It is well-known that over a Del Pezzo surface X_n, $0 \leqslant n \leqslant 8$, of degree $9-n$, there is a root lattice structure of E_n-type on $K_{X_n}^{\perp}$ in $\mathrm{Pic}(X_n)$, and the lines and the rulings in X_n can be bijectively related to the fundamental representations associated with the right endpoint and the left endpoint of the Dynkin diagram presented as Figure 3, in a natural way. Inspired by this, we consider the so called G-surfaces with root lattice structures of G-type where G is a semisimple Lie group of ADE-type with the simply laced Dynkin diagram.

When the ADE-type Lie group G is simple, that is, G is of type E_n for $4 \leqslant n \leqslant 8$, A_n for $n \geqslant 1$, or D_n for $n \geqslant 3$, we gave a uniform definition of ADE-surfaces in [15], using the pair (S, C). It turns out that when $G = E_n$, we obtain the classical Del Pezzo surfaces X_n; when $G = D_n$ and $G = A_n$, we obtain rational surfaces with new structures.

Definition 1. Let (S, C) be a pair consisting of a smooth rational surface S and a smooth rational curve $C \subset S$ with $C^2 \neq 4$. The pair (S, C) is said to be of ADE-type (or an ADE-surface) if it satisfies the following two conditions:

(i) Any (smooth) rational curve on S has a self-intersection number at least -1.

(ii) The sub-lattice $\langle K_S, C\rangle^{\perp}$ of $\mathrm{Pic}(S)$ is an irreducible root lattice of rank equal to $r-2$, where r is the rank of $\mathrm{Pic}(S)$.

The following proposition shows that such surfaces can be classified into three types, and the curve C in fact sits in the negative part of the Mori cone.

Proposition 2 ([15])**.** *Let (S, C) be a rational surface of* ADE*-type. Let $n = \mathrm{rank}(\mathrm{Pic}(S)) - 2$. Then $C^2 \in \{-1, 0, 1\}$ and*

(i) *when $C^2 = -1$, $\langle K_S, C\rangle^{\perp}$ is of E_n-type, where $4 \leqslant n \leqslant 8$;*

(ii) *when $C^2 = 0$, $\langle K_S, C\rangle^{\perp}$ is of D_n-type, where $n \geqslant 3$;*

(iii) *when $C^2 = 1$, $\langle K_S, C\rangle^{\perp}$ is of A_n-type.*

Corollary 3. *Let (S, C) be an* ADE*-surface.*

(i) *In the E_n case, blowing down the (-1) curve C of S, we obtain a Del Pezzo surface X_n of degree $9 - n$.*

(ii) *In the D_n case, S is just a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ at n points in general position with C as the natural ruling.*

(iii) *In the A_n case, the linear system $|C|$ defines a birational map $\varphi_{|C|}\colon S \to \mathbb{P}^2$. Therefore S is just the blowup of $\mathbb{P}^2$ at $n+1$ points in general position, and C is a smooth curve which represents the class of the strict transformation of lines in $\mathbb{P}^2$.*

Let D be a curve over a surface S. Recall the (arithmetic) genus $g(D)$ is defined as

$$g(D) := 1 + \frac{1}{2}(D^2 + DK_S),$$

where K_S is the canonical divisor of S.

Definition 4. Given a G-surface (S, C), let D be a curve (reduced but possibly reducible) in S.

(1) D is called an (S, C)-curve (an (S, C)-section) if $DC = 0$ (respectively, $DC = 1$).

(2) D is said to be of degree d if $D(-K_S) = d$.

(3) D is called *rational* if $g(D) = 0$.

(4) If D is a rational (S, C)-curve of degree $1, 2, 3$, or 4, then D is respectively called a line, a ruling, a twisted cubic, or a twisted quartic.

Here a rational curve D of degree 2 is a ruling because $D^2 = 0$.

In the following, we recall the construction in [15] of ADE-surfaces as well as their root lattice structure.

3.2. Constructions. By Corollary 3, ADE-surfaces are constructed as follows. Most of the contents in this subsection are taken from [15].

3.2.1. A_n-surfaces. We take a slightly different but equivalent description from [15], according to Corollary 3. Let $S = Z_n$ be the blowup of $\mathbb{P}^2$ at $n+1$ points (say, $x_1, \ldots, x_{n+1}$) which are in general position. Let h be the class coming from the lines in $\mathbb{P}^2$, and l_i the exceptional class corresponding to the blowup at x_i, $i = 1, \ldots, n+1$. The Picard group of S is a lattice of rank $n+2$, generated by h, $l_1, \ldots, l_{n+1}$. The intersection form is:

$$h^2 = 1, \quad l_i^2 = -1, \quad hl_i = l_i l_j = 0, \quad 1 \leqslant i \neq j \leqslant n+1.$$

Let K_S be the canonical class of $S = Z_n$. Then $K_{Z_n} = -3h + l_1 + \ldots + l_{n+1}$.

The sub-lattice $\Lambda_r(A_n) := \langle K_{Z_n}, h\rangle^{\perp} \subset \mathrm{Pic}(Z_n)$ is a root lattice of A_n-type, with the root system $R(A_n) := \{\alpha \in \langle K_{Z_n}, h\rangle^{\perp} \colon \alpha^2 = -2\}$ and the set of simple roots $\Delta_r(A_n) := \{\alpha_1 = l_1 - l_2, \ldots, \alpha_n = l_n - l_{n+1}\}$. These simple roots are labeled in the Dynkin diagram as in Figure 1.

3.2.2. D_n-surfaces. It suffices to consider the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$, since for the Hirzebruch surface $\mathbb{F}_1$, the whole story is completely the same.

Let $Y_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Fix the ruling f (note that f defines a fibration: $Y_0 \to \mathbb{P}^1$) and the section s, where $f, s \in \mathrm{Pic}(Y_0) = H^2(Y_0, \mathbb{Z})$ such that $f^2 = s^2 = 0$, and $sf = 1$.

Let $S = Y_n$ be the blowup of Y_0 at n points $x_1, \ldots, x_n$ which are in general position. We also denote the classes of the fibers and the sections by f, s, respectively. Thus the Picard group $\mathrm{Pic}(Y_n)$ is a lattice of rank $n+2$ with the generators $f, s, l_1, \ldots, l_n$, where l_i is the exceptional class corresponding to the blowup at x_i. The intersection form is

$$f^2 = 0 = s^2, \quad sf = 1, \quad l_i^2 = -1, \quad fl_i = sl_i = l_i l_j = 0, \quad 1 \leqslant i \neq j \leqslant n.$$

Note that f also defines a fibration $Y_n \to \mathbb{P}^1$ which is factored as $Y_n \to Y_0 \to \mathbb{P}^1$. The canonical class of Y_n is $K_{Y_n} = -2f - 2s + l_1 + \ldots + l_n$.

The sub-lattice $\Lambda_r(D_n) := \langle K_{Y_n}, f\rangle^{\perp} \subset \mathrm{Pic}(Y_n)$ is a root lattice of D_n-type, with $R(D_n) := \{\alpha \in \mathrm{Pic}(Y_n)\colon\ \alpha^2 = -2,\ \alpha \in \langle K_{Y_n}, f\rangle^{\perp}\}$ ($\Delta_r(D_n) := \{\alpha_1 = f - l_1 - l_2,\ \alpha_2 = l_1 - l_2,\ \ldots,\ \alpha_n = l_{n-1} - l_n\}$) as the root system (respectively, the set of simple roots). These simple roots are labeled in the Dynkin diagram as in Figure 2.

3.2.3. E_n-surfaces. According to Proposition 2, it suffices to consider the surfaces X_n, where X_n is a blowup of $\mathbb{P}^2$ at n, $n \leqslant 8$, points $x_1, \ldots, x_n$ which are in general position. Thus, X_n is a Del Pezzo surface of degree $9 - n$. Recall that Del Pezzo surfaces are classified into ten types: X_n, $0 \leqslant n \leqslant 8$, and $\mathbb{P}^1 \times \mathbb{P}^1$. The Picard group $\mathrm{Pic}(X_n) \cong H^2(X_n, \mathbb{Z})$ is a lattice of rank $n+1$ generated by $h, l_1, \ldots, l_n$, where h is the class of the pull-back of lines in $\mathbb{P}^2$, and l_i is the exceptional class corresponding to the blowup at x_i, $i = 1, \ldots, n$. The intersection form is

$$h^2 = 1, \quad l_i^2 = -1, \quad hl_i = l_i l_j = 0, \quad 1 \leqslant i \neq j \leqslant n.$$

The canonical class is $K_{X_n} = -3h + l_1 + \ldots + l_n$.

For $n \leqslant 8$, the set $\Lambda_r(E_n) := \{\alpha \in \mathrm{Pic}(X_n)\colon\ \alpha K_{X_n} = 0\}$ (the set $R(E_n) := \{\alpha \in \mathrm{Pic}(X_n)\colon\ \alpha K_{X_n} = 0,\ \alpha^2 = -2\}$, the set $\Delta_r(E_n) := \{\alpha_1 = h - l_1 - l_2 - l_3,\ \alpha_i = l_{i-1} - l_i,\ i = 1, \ldots, n\}$) is a root lattice (respectively, root system, root basis) of E_n-type, where we extend the definition of exceptional Lie groups E_6, E_7, E_8 to all n, $0 \leqslant n \leqslant 8$, as in Section 2. See Figure 3 for the Dynkin diagram and the labeling of the simple roots.

When $4 \leqslant n \leqslant 8$, then X_n is an *E_n-surface*, if we blow down C, according to Definition 1. In this case, we will obtain a simple Lie group E_n.

3.3. Fundamental weights in terms of curves in S.

Proposition 5. *Let (S, C) be a rational surface of* ADE*-type, and G the corresponding (simply connected) semisimple Lie group. The lattice* $\mathrm{Pic}(S)/(\mathbb{Z}C + \mathbb{Z}K_S)$ *is the corresponding weight lattice $\Lambda_w(G)$. Hence the group* $\mathrm{Pic}(S)/(\mathbb{Z}C + \mathbb{Z}K_S) \otimes_{\mathbb{Z}} \mathbb{C}^*$ *is a maximal torus of G.*

P r o o f. According to Corollary 3, one can check that the radical of the intersection pairing

$$\langle C, K_S\rangle^{\perp} \times \mathrm{Pic}(S) \to \mathbb{Z}$$

is exactly $\mathbb{Z}C + \mathbb{Z}K_S$.

We take a D_n-surface $S = Y_n$ for example. Recall $\mathrm{Pic}(Y_n) = \mathbb{Z}\langle f = C, s, l_1, \ldots, l_n\rangle$, and $K_{Y_n} = -2f - 2s + \sum l_i$. Let $af + bs - \sum c_i l_i \in \mathrm{Pic}(Y_n)$ be an element in the radical of the intersection pairing. Then we have $(af + bs - \sum c_i l_i)\alpha_i = 0$ for all i. Hence $b = 2c$ and $c_1 = c_2 = \ldots = c_n = c$. Thus $af + bs - \sum c_i l_i = af + c(2s - \sum l_i) =$

$af - c(K_{Y_n} + 2f) \in \mathbb{Z}\langle f, K_{Y_n}\rangle$. Therefore the radical of the pairing is contained in $\mathbb{Z}\langle f, K_{Y_n}\rangle$. Then they are equal, since the other inclusion is obvious.

Therefore, the intersection pairing induces a perfect non-degenerate pairing

$$\langle C, K_{Y_n}\rangle^{\perp} \times \mathrm{Pic}(S)/(\mathbb{Z}C + \mathbb{Z}K_{Y_n}) \to \mathbb{Z}.$$

Since $\langle C, K_{Y_n}\rangle^{\perp}$ is the (simply laced) root lattice of G, $\mathrm{Pic}(S)/(\mathbb{Z}C + \mathbb{Z}K_{Y_n})$ is the weight lattice of G. And the last statement follows since G is simply connected. □

Proposition 6. *For the set $\Delta_w(G) := \{\bar{\lambda}_1, \ldots, \bar{\lambda}_n\}$ of the fundamental dominant weights, $\lambda_i \in \mathrm{Pic}(S)$ representing $\bar{\lambda}_i$ can be taken as follows:*

(i) *for $G = A_n$, $\lambda_i = l_{i+1} + \ldots + l_{n+1}$, $i = 1, \ldots, n$;*
(ii) *for $G = D_n$, $\lambda_1 = s$, $\lambda_2 = s - l_1$, $\lambda_i = l_i + \ldots + l_n$, $i = 3, \ldots, n$;*
(iii) *for $G = E_n$, $\lambda_1 = h$, $\lambda_2 = h - l_1, \lambda_3 = 2h - l_1 - l_2$, $\lambda_i = l_i + \ldots + l_n$, $i = 4, \ldots, n$.*

Moreover, each of these weights is effective and represented by an irreducible rational curve, or a disjoint union of such curves.

Proof. It follows from a direct computation that these elements map to a dual basis of α_i, $i = 1, \ldots, n$, since $\lambda_i \cdot \alpha_j = \delta_{ij}$ for all $i, j = 1, \ldots, n$. □

Corollary 7. *With $\lambda_i \in \mathrm{Pic}(S)$ taken as in Proposition 6, the quasi-minuscule fundamental weights are the following:*

(i) *for $G = A_n$, $\bar{\lambda}_i$, $i = 1, \ldots, n$;*
(ii) *for $G = D_n$, $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_{n-1}, \bar{\lambda}_n$;*
(iii) *for $G = E_4$, $\bar{\lambda}_1, \ldots, \bar{\lambda}_4$; for $G = E_5$, $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_5$; for $G = E_6$, $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_6$; for $G = E_7$, $\bar{\lambda}_2, \bar{\lambda}_7$; for $G = E_8$, $\bar{\lambda}_8$.*

Given a quasi-minuscule fundamental weight $\bar{\lambda}$ with $\lambda \in \mathrm{Pic}(S)$, we have the following three natural numerical invariants (under the action of the Weyl group $W(G)$): $\lambda \cdot C$, $\lambda \cdot K_S$ and λ^2. In the following, we show that the orbit $W(G) \cdot \lambda$ is uniquely determined by these three numerical invariants. We give a different proof in Section 4 via polytopes.

Before doing this, we first prove a simple lemma.

Lemma 8. *Let a, b be two integers. The system of Diophantine equations*

$$(1) \qquad \begin{cases} Y^2 = u + a^2 K_S^2 + b^2 C^2 + 2av + 2ab(CK_S) + 2bw, \\ YK = v + aK_S^2 + b(CK_S), \\ YC = w + a(CK_S) + bC^2, \end{cases}$$

is solvable if and only if the system

$$X^2 = u, \quad XK_S = v, \quad XC = w, \tag{2}$$

is solvable. And the sets of respective solutions of (1) *and* (2) *are in one-to-one correspondence via the relation* $Y_0 = X_0 + aK_S + bC$.

Moreover, the Weyl group $W(G)$ *acts on the set of the solutions of* (1) *transitively if and only if* $W(G)$ *acts on the set of the solutions of* (2) *transitively.*

Proof. For the first statement, if X_0 is a solution of (2), then $Y_0 = X_0 + aK_S + bC$ is obviously a solution of (1). On the other hand, if Y_0 is a solution of (1), then one can check that $X_0 = Y_0 - aK_S - bC$ is a solution of (2). For the second statement, one just notices that $W(G)$ preserves the bilinear form on $\mathrm{Pic}(S)$ and acts on $\mathbb{Z}\langle K_S, C\rangle$ trivially. □

Theorem 9. *Let* (S, C) *be a rational surface of* ADE-*type, let* G *be the corresponding (simply connected) semisimple Lie group and let* $\lambda \in \mathrm{Pic}(S)$ *be taken as in Proposition* 6. *For a subset* $I_{G,\lambda}$ *defined as*

$$I_{G,\lambda} := \{D \in \mathrm{Pic}(S)\colon\ DC = \lambda \cdot C,\ D^2 = \lambda^2,\ DK_S = \lambda \cdot K_S\},$$

we have

$$W(G) \cdot \lambda = I_{G,\lambda}.$$

Proof. We shall use the following elementary lemma several times, and leave its proof as an easy exercise.

Key lemma: Let $a_1, \ldots, a_n$ be n integers such that $\sum a_i^2 = \sum a_i$. Then for all i, $a_i = 0$ or 1.

According to Proposition 6 and Lemma 8, it suffices to consider the quasi-minuscule fundamental weights λ_i's as in Proposition 6.

(1) The A_n case. It is easy to see that

$$W(A_n)(l_{i+1} + \ldots + l_{n+1}) = \{l_{k_1} + \ldots + l_{k_{n-i+1}}\colon\ 1 \leqslant k_j < k_{j+1} \leqslant n+1\},$$

$$I_{A_n,\lambda_i} = \{X \in \mathrm{Pic}(S)\colon\ Xh = 0,\ X^2 = -(n-i+1) = XK_S\}$$

$Xh = 0$ implies that $X = \sum a_i l_i$. Further, $X^2 = -(n-i+1) = XK_S$ implies that

$$\begin{cases} \sum a_j^2 = n - i + 1, \\ \sum a_j = n - i + 1. \end{cases}$$

By Key lemma, we have $a_{k_1} = \ldots = a_{k_{n-i+1}} = 1$, and $a_j = 0$ for $j \notin \{k_1, \ldots, k_{n-i+1}\}$. Therefore we see that

$$W(A_n)(l_{i+1} + \ldots + l_{n+1}) = I_{A_n,\lambda_i}.$$

(2) The $G = D_n$ case.

(2.1) The adjoint representation. The corresponding quasi-minuscule weight is $\lambda_{n-1} = l_{n-1} + l_n$. It is easy to see that

$$W(D_n)(l_{n-1} + l_n) = \{l_i + l_j,\ 2f - l_i - l_j,\ f - l_i + l_j \colon\ i \neq j,\ i,j = 1, \ldots, n\}.$$

Further,

$$I_{D_n,\lambda_{n-1}} = \{X \in \mathrm{Pic}(S) \colon\ Xf = 0,\ X^2 = -2 = XK_S\}.$$

Let $X \in I_{D_n,\lambda_{n-1}}$. Then $Xf = 0$ implies that we can suppose $X = af + \sum a_i l_i$. $X^2 = -2 = XK_S$ implies that

$$\begin{cases} \sum a_i^2 = 2, \\ 2a + \sum a_i = 2. \end{cases}$$

Solving this system of equations, we obtain that there exist $i \neq j$ such that for all $k \neq i, j$, $(a, a_i, a_j, a_k) = (0, 1, 1, 0), (1, \pm 1, \mp 1, 0)$ or $(2, -1, -1, 0)$.

Thus we have

$$I_{D_n,\lambda_{n-1}} = \{l_i + l_j,\ 2f - l_i - l_j,\ f - l_i + l_j \colon\ i \neq j,\ i,j = 1, \ldots, n\}.$$

(2.2) The standard representation and the spinors. In these cases, we consider $\lambda_n = l_n$, $\lambda_1 = s$ and $\lambda_2 = s - l_1$. The proofs are similar, and the last two cases are reduced to solving equations of the type in Key lemma.

(3) The $G = E_n$ case. These results follow from direct computations. For $\lambda_n = l_n$, see [5] and Chapter IV of [17]. For $\lambda_2 = h - l_1$, see [15]. For $\lambda_1 = h$, we only need to consider the case $n = 4, 5$ or 6. We take $n = 6$ for example. The Diophantine equations for $\bar{\lambda} = \bar{h}$ are

$$X^2 = 1, \quad XK_S = -3.$$

They can be solved directly by a somehow cumbersome computation. Here is a simpler proof. Observe that we have a bijection between the following two sets:

$$\begin{gathered} \{X \colon\ X^2 = 1,\ XK_S = -3\} \to \{Y \colon\ Y^2 = -2,\ YK_S = 0\} \\ X \mapsto K_S + X. \end{gathered}$$

Since the set $\{Y \in \mathrm{Pic}(S)\colon Y^2 = -2,\ YK_S = 0\}$ is the root system of E_6 by Proposition 2, from the fact that there are 72 roots in the root system of E_6 with the transitive action of the Weyl group $W(E_6)$, it follows that the system of equations

$$X^2 = 1, \quad XK_S = -3$$

has exactly 72 solutions and $W(E_6)$ acts on the set of solutions transitively. For $\lambda_3 = 2h - l_1 - l_2$, we only need to consider the case $n = 4$. And the proof is an easy computation. □

Moreover, the second equality of Theorem 9 is independent of the choices of representatives of the coset $\bar{\lambda}$. Therefore the cardinality of the set $I_{G,\lambda}$ is exactly the dimension of the irreducible representation V_λ when λ is minuscule.

Thus, we obtain the relation between the Weyl groups orbits of a quasi-minuscule fundamental weights and special curves on S, listed in Table 1.

Lemma 10. *There are the following natural bijective maps of sets:*

(a) *For $G = A_n$.*

$$\begin{aligned} \phi^i_{A_n}\colon I_{A_n,\lambda_i} &\to I_{A_n,\lambda_{n+1-i}} \\ x &\mapsto 3h + K_S - x. \end{aligned}$$

(b1) *For $G = D_n$.*

$$\begin{aligned} \phi^{1,2}_{D_n}\colon I_{D_n,\lambda_1} &\to I_{D_n,\lambda_2} \\ x &\mapsto -x + \tfrac{1}{2}(n-5)f - K_S. \end{aligned}$$

(b2) *For $G = D_n$.*

$$\begin{aligned} \phi^{n-1}_{D_n}\colon I_{D_n,\lambda_{n-1}} &\to R(D_n) \\ x &\mapsto x - f. \end{aligned}$$

(c1) *For $G = E_8$.*

$$\begin{aligned} \phi^8_{E_8}\colon I_{E_8,\lambda_8} &\to R(E_8) \\ x &\mapsto K_S + x. \end{aligned}$$

(c2) *For $G = E_7$.*

$$\begin{aligned} \phi^2_{E_7}\colon I_{E_7,\lambda_2} &\to R(E_7) \\ x &\mapsto K_S + x. \end{aligned}$$

(c3) *For $G = E_6$.*

$$\begin{aligned} \phi^{2,6}_{E_6}\colon I_{E_6,\lambda_2} &\to I_{E_6,\lambda_6} \\ x &\mapsto -K_S - x \end{aligned}$$

and

$$\begin{aligned} \phi^1_{E_6}\colon I_{E_6,\lambda_1} &\to R(E_6) \\ x &\mapsto K_S + x. \end{aligned}$$

(c4) *For* $G = E_5$.

$$\phi_{E_5}^{1,5} \colon I_{E_5,\lambda_1} \to I_{E_5,\lambda_5}$$
$$x \mapsto -K_S - x$$

and

$$\phi_{E_5}^{3} \colon I_{E_5,\lambda_3} \to R(E_5)$$
$$x \mapsto K_S + x.$$

(c5) *For* $G = E_4$.

$$\phi_{E_4}^{1,2} \colon I_{E_4,\lambda_1} \to I_{E_4,\lambda_2}$$
$$x \mapsto -K_S - x$$

and

$$\phi_{E_4}^{3,4} \colon I_{E_4,\lambda_3} \to I_{E_4,\lambda_4}$$
$$x \mapsto -K_S - x.$$

Proof. This follows from a direct computation. For example, we check the bijectivity of $\phi_{D_n}^{n-1}$ in (b2). If $x \in I_{D_n,\lambda_{n-1}}$, then $xf = 0$, $x^2 = -2$, $xK_S = -2$. Thus we have $(x-f)f = 0$, $(x-f)^2 = -2$, $(x-f)K_S = 0$. That is, $x - f \in R(D_n)$. Similarly, if $y \in R(D_n)$, then $y + f \in I_{D_n,\lambda_{n-1}}$. $\square$

These bijective maps reflect the properties of representations of G, for example, the duality, the isomorphisms as representations, and so on.

4. Polytopes in Pic(S) and Weyl group orbits

In this section, we consider representations of Weyl groups on Picard groups $\mathrm{Pic}(S)$ of rational surfaces S. In particular, we consider very special classes λ which correspond to the quasi-minuscule representations after the identification in Proposition 5 and conclude that the corresponding subsets $I_{G,\lambda}$ of $\mathrm{Pic}(S)$ are indeed the orbits of Weyl groups in $\mathrm{Pic}(S)$. As a matter of fact, the Weyl groups that we are considering are given by root lattices defined on subsets $\{d \in \mathrm{Pic}(S) \colon d^2 = -2,\ K_S D = 0\}$ of $\mathrm{Pic}(S)$. In [11], the first author introduces (semi-)regular polytopes in $\mathrm{Pic}(S)$ whose symmetry groups are our Weyl groups and studies the correspondences between their subpolytopes and special divisor classes which turn out to be related to $I_{G,\lambda}$. In this section, we explain the relation between Weyl group orbits $I_{G,\lambda}$ and subpolytopes in those special polytopes. Basically the arguments of those three cases of Weyl groups of types ADE are the same. Therefore, we explain rather in detail the case of E-type Weyl groups which is more complicated, and make brief notes on other rational surfaces we care.

4.1. Polytopes in $\mathrm{Pic}(S)$**.** First we review the general theory on regular polytopes that we use in this section and a family of semiregular polytopes known as Gosset figures (k_{21} according to Coxeter). Here we only present a brief introduction, and for further detail readers can look up [2], [3], [4] and [11].

We consider a convex n-polytope P_n in an n-dimensional Euclidean space. For each vertex of P_n, the set of the midpoints of all the edges emanating from the vertex in P_n is called the *vertex figure* of P_n at the vertex when it forms an $(n-1)$-polytope. As the edges of polytopes considered in this article have the same length, the *vertex figure* of the vertex in P_n is equivalently an $(n-1)$-polytope given by the set of vertices on the other ends of the edges emanating from this vertex.

A *regular* polytope P_n, $n \geqslant 2$, is a polytope whose facets and the vertex figure at each vertex are regular, which is a higher dimensional generalization of regular polygon and regular polyhedron. Naturally, the facets of a regular P_n are all congruent, and the vertex figures are all the same. A polytope P_n is called *semiregular* if its facets are regular and its vertices are equivalent, namely, the symmetry group of P_n acts transitively on the vertices of P_n.

In this article, we consider two classes of regular polytopes and one class of semiregular polytopes.

(1) A *regular simplex* $\mathfrak{a}_n$ is an n-dimensional simplex with equilateral edges. Note that $\mathfrak{a}_n$ is a pyramid based on $\mathfrak{a}_{n-1}$. Thus each facet of a regular simplex $\mathfrak{a}_n$ is a regular simplex $\mathfrak{a}_{n-1}$, and each vertex figure of $\mathfrak{a}_n$ is also $\mathfrak{a}_{n-1}$. For a regular simplex $\mathfrak{a}_n$, only regular simplexes $\mathfrak{a}_k$, $0 \leqslant k \leqslant n-1$ appear as subpolytopes.

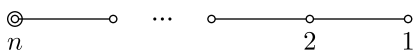

Figure 4. Coxeter-Dynkin diagram of $\mathfrak{a}_n$.

(2) A *crosspolytope* $\mathfrak{b}_n$ is an n-dimensional polytope whose $2n$-vertices are the intersection of an n-dimensional Cartesian coordinate frame and a sphere centered at the origin. Note that $\mathfrak{b}_n$ is a bipyramid based on $\mathfrak{b}_{n-1}$, and the n-vertices in $\mathfrak{b}_n$ form $\mathfrak{a}_{n-1}$ if the choice is made of one vertex from each Cartesian coordinate line. So the vertex figure of a crosspolytope $\mathfrak{b}_n$ is also a crosspolytope $\mathfrak{b}_{n-1}$, and any facet of $\mathfrak{b}_n$ is $\mathfrak{a}_{n-1}$. For a crosspolytope $\mathfrak{b}_n$, only regular simplexes $\mathfrak{a}_k$, $0 \leqslant k \leqslant n-1$ appear as subpolytopes.

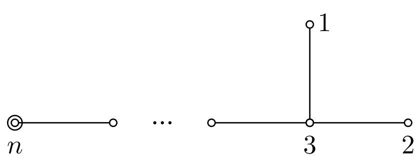

Figure 5. Coxeter-Dynkin diagram of $\mathfrak{b}_n$.

(3) *Gosset polytopes* k_{21}, $k = -1, 0, 1, 2, 3, 4$, are semiregular polytopes discovered by Gosset which are $(k+4)$-dimensional polytopes whose symmetry groups are the Coxeter groups $W(E_{k+4})$. Note that the vertex figure of k_{21} is $(k-1)_{21}$ and the facets of k_{21} are regular simplexes $\mathfrak{a}_{k+3}$ and crosspolytopes $\mathfrak{b}_{k+3}$. For $k \neq -1$, the facets of a k_{21}-polytope are regular simplexes $\mathfrak{a}_{k+3}$ and crosspolytopes $\mathfrak{b}_{k+3}$. But all the lower dimensional subpolytopes are regular simplexes. When $k = -1$, the vertex figure in $(-1)_{21}$ is an isosceles triangle instead of an equilateral triangle, and its facets are regular triangles $\mathfrak{a}_2$ and squares $\mathfrak{b}_2$.

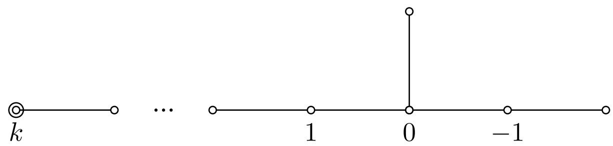

Figure 6. Coxeter-Dynkin diagram of k_{21}, $k \neq -1$.

4.2. Gosset polytopes in Pic(S)**.** Here we present how to construct Gosset polytopes in Pic(S_n). Such a polytope is given by a convex hull of a set of special divisors known as lines in a Del Pezzo surface S_n of degree $9-n$. The constructions of polytopes for the others types of rational surfaces can be obtained in a similar way, and we leave them to readers.

For our case $G = E_n$ and its Weyl group acts on $C^{\perp}$ in Pic(S_{n+1}). Since the divisor classes considered are commonly perpendicular to C, we can apply the blowing down from S_{n+1} to S_n given by a rational curve C with $C^2 = -1$. This blowing down does not change the setups except for forgetting the conditions given by C. Thus, we deal with divisor classes D in Pic(S_n) satisfying $DK_{S_n} = a$, $D^2 = b$ where a and b are integers and K_{S_n} is the canonical class of S_n. Here we observe that the action of $W(E_n)$ on roots in Pic(S_n) can be naturally extended to the whole Pic(S_n), and furthermore it preserves two conditions $DK_{S_n} = a$, $D^2 = b$. In particular, $W(E_n)$ acts as a reflection group on the affine hyperplane section given by the divisor classes D with $DK_{S_n} = a$.

Now, we consider a subset L_n of Pic(S_n) consisting of lines l which are divisor classes with $l^2 = -1$ and $K_{S_n} l = -1$. As the Weyl group $W(E_n)$ acts as a reflection group on the affine hyperplane given by $DK_{S_n} = -1$ and preserves self-intersections, $W(E_n)$ gives an action on L_n which in fact is transitive. Therefrom, a semiregular polytope is constructed as the convex hull of L_n in Pic$(S_n) \otimes \mathbb{Q}$, where the vertices of the polytope are exactly the lines in Pic(S_n). Since the symmetry group of the polytope is $W(E_n)$ which is of E_n-type, the polytope is actually a Gosset polytope $(n-4)_{21}$ (see [11] for details).

For a Gosset polytope $(n-4)_{21}$, subpolytopes are regular simplexes except for the facets which consist of $(n-3)$-simplexes and $(n-3)$-crosspolytopes. Since the subpolytopes in $(n-4)_{21}$ are basically configurations of vertices, we obtain a natural characterization of subpolytopes in $(n-4)_{21}$ as divisor classes in $\mathrm{Pic}(S_n)$. To identify subpolytopes in Gosset polytopes $(n-4)_{21}$ defined in $\mathrm{Pic}(S_n)\otimes\mathbb{Q}$, we want to use the barycenter of the subpolytope. As each vertex of the polytope $(n-4)_{21}$ represents a line in S_n, the honest centers of simplexes (crosspolytopes) are written as $(l_1+\ldots+l_k)/k$ where $l_1,\ldots,l_k$ are disjoint to each other (respectivaly, $(l_1'+l_2')/2$ where $l_1'\cdot l_2'=1$). But these centers in the convex hull of L_n in $\mathrm{Pic}(S_n)\otimes\mathbb{Q}$ may not be elements in $\mathrm{Pic}(S_n)$. Therefore, alternatively, we choose $(l_1+\ldots+l_k)$ as the center of a subpolytope so that it is an element in $\mathrm{Pic}(S_n)$.

For a Gosset polytope $(n-4)_{21}$ in $\mathrm{Pic}(S_n)$, we define

$$L_n^{k-1} := \{l_1+\ldots+l_k \in \mathrm{Pic}(S_n)\colon\ l_1,\ldots,l_k \text{ disjoint lines in } L_n\} \text{ and}$$
$$F_n := \{l_1'+l_2' \in \mathrm{Pic}(S_n)\colon\ l_1',l_2' \text{ lines in } L_n \text{ with } l_1'\cdot l_2'=1\}.$$

In [11], we show that $l_1+\ldots+l_k\in L_n^{k-1}$ represents the center of a $(k-1)$-simplex in $(n-4)_{21}$ in a unique way, and L_n^{k-1} is bijective to the set of $(k-1)$-simplexes in $(n-4)_{21}$. We also show that F_n is bijective to the set of $(n-1)$-crosspolytopes in $(n-4)_{21}$. Here, each center of an $(n-1)$-crosspolytope in $(n-4)_{21}$ can be written as $(n-1)$ different pairs of lines with intersection 1.

4.3. Weyl group orbits. In this subsection, we recover the above study on quasi-minuscule representations in $\mathrm{Pic}(S)$ via the combinatorics of polytopes. We identify Weyl group orbits $W(E_n)\lambda$ for the fundamental dominant weights λ to subpolytopes in a Gosset polytope $(n-4)_{21}$ and compare the orbits with $I_{E_n,\lambda}$. The same can be done for the Weyl groups of A- and D-type, and we leave it to readers.

From Proposition 6, we know that the set of the fundamental dominant weights for E_n is given as $\{\lambda_1,\ldots,\lambda_n\}\subseteq\mathrm{Pic}(S_n)$ where

$$\lambda_1=h,\ \ \lambda_2=h-l_1,\ \ \lambda_3=2h-l_1-l_2,\ \ \lambda_i=l_i+\ldots+l_n,\quad i=4,\ldots,n.$$

It is also useful to note that the reflection group $W(E_n)$ acts transitively on k-subpolytopes of $(n-4)_{21}$ when $k\neq n-1,n-2$. For $(n-1)$-subpolytopes in $(n-4)_{21}$, there are two $W(E_n)$ orbits which are the set of $(n-1)$-simplexes and the set of $(n-1)$-crosspolytopes. For $(n-2)$-subpolytopes in $(n-4)_{21}$ with $5\leqslant n\leqslant 8$, there are also two $W(E_n)$ orbits which are two sets of $(n-2)$-simplexes. In this article, we consider one set of $(n-2)$-simplexes in $(n-4)_{21}$ given by $W(E_n)((h-l_1-l_2)+l_3+\ldots+l_n)$, (see [12]).

Theorem 11. *Let S_n be a Del Pezzo surface of degree $9-n$ and $\{\lambda_1, \ldots, \lambda_n\}$ the set of fundamental dominant weights for E_n as above.*

(1) *For $4 \leqslant i \leqslant n$, $W(E_n)\lambda_i = L_n^{n-i}$, which is the set of $(n-i)$-simplexes in $(n-4)_{21}$.*
(2) *For $i = 3$, $W(E_n)\lambda_3$ is equivariantly equivalent to $W(E_n)((h - l_1 - l_2) + l_3 + \ldots + l_n)$, which is a $W(E_n)$ orbit of $(n-2)$-simplexes in $(n-4)_{21}$.*
(3) *For $i = 2$, $W(E_n)\lambda_2 = F_n$, which is the set of $(n-1)$-crosspolytopes in $(n-4)_{21}$.*
(4) *For $i = 1$, $W(E_n)\lambda_1$ is equivariantly equivalent to L_n^n, which is the set of $(n-1)$-simplexes in $(n-4)_{21}$.*

Proof. (1) Obviously, $W(E_n)\lambda_i \subset L_n^{n-i}$ and we get $|W(E_n)\lambda_i| = |L_n^{n-i}|$ by computing the isotropy subgroup of λ_i.

(2) $2\lambda_3 + K_{S_n} = 2(2h - l_1 - l_2) + K_{S_n} = 2(2h - l_1 - l_2) + (-3h + l_1 + \ldots + l_n) = (h - l_1 - l_2) + l_3 + \ldots + l_n$.

(3) See [11].

(4) $3\lambda_1 + K_{S_n} = 3h + K_{S_n} = l_1 + \ldots + l_n$. □

This theorem leads us to ask if we can extend the results in Theorem 9 and Table 1 given by the quasi-minuscule representations. Indeed we can obtain the following result according to the comparison between combinatorics of Gosset polytopes $(n-4)_{21}$ and special divisor classes in $\mathrm{Pic}(S_n)$. Here I_{E_n,λ_i} is the finite subset of divisor classes D in $\mathrm{Pic}(S_n)$ with $D^2 = \lambda_i^2$, $DK = \lambda_i \cdot K$.

Corollary 12. *Let S_n be a Del Pezzo surface of degree $9-n$ and $\{\lambda_1, \ldots, \lambda_n\}$ the set of fundamental dominant weights for E_n as above.*

(1) *For $n-2 \leqslant i \leqslant n$, $W(E_n)\lambda_i = I_{E_n,\lambda_i}$.*
(2) *For $i = 2$, $W(E_n)\lambda_2 = I_{E_n,\lambda_2}$.*
(3) *For $i = 1$, $n \neq 8$, $W(E_n)\lambda_1 = I_{E_n,\lambda_1}$.*

Proof. For each λ_i, it is easy to see that $W(E_n)\lambda_i \subset I_{E_n,\lambda_i}$. We compute $|I_{E_n,\lambda_i}|$ and compare it with $|W(E_n)\lambda_i|$ given in Theorem 11. To compute $|I_{E_n,\lambda_i}|$, one can use the dual root lattice of E_n and apply the corresponding theta function (see Section 5 in [11] for details). The proofs of (1), (2) and (3) can be found in Theorem 5.2, 5.3 and 5.4 of [11], respectively. □

Remark 13. For λ_3, the proof of this corollary can be applied to show $W(E_6) \cdot \lambda_3 = I_{E_6,\lambda_3}$. But for $n = 8, 7$, $W(E_n)\lambda_3$ is one of the two $W(E_n)$ orbits in I_{E_n,λ_3}. Similarly, for $n = 8$, $W(E_n)\lambda_1$ is also one of the two $W(E_8)$ orbits in I_{E_8,λ_1}.

For A_n- and D_n-surfaces, we have similar results, stated with brief proofs as follows.

Warning: In general, when we consider crosspolytopes $\mathfrak{b}_n$ which are n-dimensional regular polytopes, the symmetry group is a reflection group $W(C_n)$ of order $2^n n!$ (where $W(C_n)$ is the Weyl group of the Lie group C_n), so that it acts transitively on each type of subpolytopes in a crosspolytope $\mathfrak{b}_n$. Thus $(n-1)$-simplexes in $\mathfrak{b}_n$ are in one $W(C_n)$ orbit. In this article, we take a smaller symmetry group $W(D_n)$ of order $2^{n-1}n!$ for $\mathfrak{b}_n$, which is also a reflection group producing crosspolytopes $\mathfrak{b}_n$. But as it is a smaller symmetry group, there are two $W(D_n)$ orbits of $(n-1)$-simplexes in $\mathfrak{b}_n$ corresponding to two spin representations of D_n. Those two $W(D_n)$ orbits in $\mathfrak{b}_n$ are also related to $W(D_n)$ orbits in a D_n lattice in the following proposition. Here the set L_n^{k-1} of divisor classes is defined similarly to the case of Del Pezzo surfaces S_n.

Proposition 14. *Let S be a rational surface of A_n and D_n type. Let $\{\lambda_1, \ldots, \lambda_n\}$ be the set of fundamental dominant weights as above.*

(1) *For A_n, $W(A_n)\lambda_i = L_n^{n-i}$, which is the set of $(n-i)$-simplexes $\mathfrak{a}_{n-i}$ in the n-simplex $\mathfrak{a}_n$.*
(2) *For D_n, $W(D_n)\lambda_n = L_n$, which is the set of vertices in $\mathfrak{b}_n$.*
(3) *For D_n, $W(D_n)\lambda_{n-1} = L_n^1$, which is the set of edges in $\mathfrak{b}_n$.*
(4) *For D_n, $W(D_n)\lambda_2$ is equivariantly equivalent to $W(D_n).((f-l_1)+l_2+\ldots+l_n)$, which is a $W(D_n)$ orbit of $(n-1)$-simplexes in $\mathfrak{b}_n$.*
(5) *For D_n, $W(D_n)\lambda_1$ is equivariantly equivalent to $W(D_n).(l_1+\ldots+l_n)$, which is a $W(D_n)$ orbit of $(n-1)$-simplexes in $\mathfrak{b}_n$.*

P r o o f. (1) Obviously, $W(A_n)\lambda_i \subset L_n^{n-i}$ and $|W(A_n)\lambda_i| = |L_n^{n-i}|$ by computing the isotropy subgroup of λ_i. By Theorem 9, $W(A_n)\lambda_i = I_{A_n,\lambda_i}$. Similarly, (2) and (3) can be proved.

(4) Observe that $W(D_n)$ preserves K and f. And by using

$$2\lambda_2 + K_S + 3f = 2(s-l_1) + (-2f-2s+l_1+\ldots+l_n) + 3f = (f-l_1)+l_2+\ldots+l_n,$$

we conclude that $W(D_n)\cdot\lambda_2$ is equivariantly equivalent to $W(D_n)((f-l_1)+l_2+\ldots+l_n)$, which is a $W(D_n)$ orbit consisting of 2^{n-1} elements of $(n-1)$-simplexes in $\mathfrak{b}_n$. Similarly, (5) can be proved by using

$$2\lambda_1 + K_S + 2f = 2s + (-2f-2s+l_1+\ldots+l_n) + 2f = l_1+\ldots+l_n.$$

□

As a corollary, we obtain the result of Theorem 9 via polytopes, and thus we complete Table 1.

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Authors' addresses: Jae-Hyouk Lee, Department of Mathematics, Ewha Womans University, 11-1 Daehyun-Dong, Seodaemun-Gu, 120-750 Seoul, Korea, e-mail: `jaehyoukl@ewha.ac.kr`; Mang Xu, Department of Mathematics, Southwest Jiaotong University, 2nd Ring Rd., 610031 Chengdu, Sichuan, P. R. China, e-mail: `xumang@home.swjtu.edu.cn`; Jiajin Zhang, Department of Mathematics, Sichuan University, Renmin South Rd 3rd Section, XiaoTianZhu, Wuhou Qu, 610065 Chengdu, Sichuan, P. R. China, e-mail: `jjzhang@scu.edu.cn`.