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STATES ON BASIC ALGEBRAS

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Abstract. States on commutative basic algebras were considered in the literature as generalizations of states on MV-algebras. It was a natural question if states exist also on basic algebras which are not commutative. We answer this question in the positive and give several examples of such basic algebras and their states. We prove elementary properties of states on basic algebras. Moreover, we introduce the concept of a state-morphism and characterize it among states. For basic algebras which are the certain pastings of Boolean algebras the construction of a state-morphism is shown.

Keywords: basic algebra; commutative basic algebra; symmetric basic algebra; state; homomorphism

MSC 2010: 03G25, 06D35, 06C15

Generalizing the concept of a state on MV-algebras states on commutative basic algebras were considered in [1]. We consider states on arbitrary (not necessarily commutative) basic algebras. First we recall the definition of a basic algebra and the double face of such algebras.

Definition 1. A *basic algebra* is an algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ of type $(2,1,0)$ satisfying the identities

$$x \oplus 0 = x, \quad \neg(\neg x) = x, \quad \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x,$$
$$\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1,$$

where $1 := \neg 0$. In order to avoid too many brackets we agree that $\neg$ binds stronger than the other operation symbols. Two elements x and y of A are said to be *orthogonal* to each other if $x \leqslant \neg y$. On a basic algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ we define

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a binary operation $\ominus$ by $x \ominus y := \neg(\neg x \oplus y)$ for all $x, y \in A$. The algebra $\mathcal{A}$ is called *commutative* if $\oplus$ has this property. An element x of A is called *sharp* if $x \oplus x = x$.

R e m a r k 2. We have $\neg(\neg x \ominus y) = x \oplus y$ for all $x, y \in A$.

R e m a r k 3 (cf. [2]). Every MV-algebra is a commutative basic algebra. A basic algebra is an MV-algebra if and only if $\oplus$ is associative. Every orthomodular lattice is a (in general not commutative) basic algebra satisfying the identity $x \oplus (x \wedge y) = x$ in which every element is sharp. A basic algebra is an orthomodular lattice if and only if it satisfies the identity $x \oplus (x \wedge y) = x$.

Next we define the notion of a bounded lattice with sectionally antitone involutions.

Definition 4. A *bounded lattice with sectionally antitone involutions* is an ordered sextuple $\mathcal{A} = (A, \vee, \wedge, (^x; x \in A), 0, 1)$ such that $(A, \vee, \wedge, 0, 1)$ is a bounded lattice and for every $x \in A$, ^{x} is an antitone involution on $([x, 1], \leqslant)$, i.e., $y \leqslant z$ implies $z^x \leqslant y^x$ and $(y^x)^x = y$ for all $x \in A$ and $y, z \in [x, 1]$.

E x a m p l e 5. The algebra $\mathcal{M} := ([0, 1], \oplus, \neg, 0)$ with

$$x \oplus y := (x + y) \wedge 1, \quad \neg x := 1 - x$$

for all $x, y \in [0, 1]$ is an MV-algebra, called the *standard* MV-*algebra*, where

$$x \ominus y = \neg(\neg x \oplus y) = 1 - (((1 - x) + y) \wedge 1) = (x - y) \vee 0$$

for all $x, y \in [0, 1]$, and $([0, 1], \vee, \wedge, (^x; x \in [0, 1]), 0, 1)$ with

$$x \vee y := \max(x, y), \quad x \wedge y := \min(x, y), \quad x^y := 1 - x + y$$

for all $x, y \in [0, 1]$ is a bounded lattice with sectionally antitone involutions.

Both faces of a basic algebra are in a natural one-to-one correspondence:

Theorem 6 (cf. [2]). *The formulae*

$$x \vee y = \neg(\neg x \oplus y) \oplus y, \quad x \wedge y = \neg(\neg x \vee \neg y), \quad x^y = \neg x \oplus y, \quad 1 = \neg 0$$

and

$$x \oplus y = (x^0 \vee y)^y, \quad \neg x = x^0$$

induce a natural one-to-one correspondence between basic algebras and bounded lattices with sectionally antitone involutions. Moreover,

$$x \leqslant y \quad \textit{if and only if} \quad \neg x \oplus y = 1.$$

R e m a r k 7. The structures mentioned in Example 5 are both faces of the same basic algebra, namely the standard MV-algebra.

Lemma 8. *If $\mathcal{A} = (A, \oplus, \neg, 0)$ is a basic algebra and $a, b, c \in A$, then the following conditions hold:*

(i) $a \leqslant b$ *implies* $a \oplus c \leqslant b \oplus c$ *and* $a \ominus c \leqslant b \ominus c$,
(ii) $(a \oplus b) \ominus b = a \wedge \neg b$,
(iii) $(a \wedge b) \oplus c = (a \oplus c) \wedge (b \oplus c)$,
(iv) $a \oplus b = (a \wedge \neg b) \oplus b$,
(v) $a \leqslant b$ *if and only if* $a \ominus b = 0$.

P r o o f. (i) If $a \leqslant b$ then $\neg a \geqslant \neg b$, thus $\neg a \vee c \geqslant \neg b \vee c$, whence

$$a \oplus c = (\neg a \vee c)^c \leqslant (\neg b \vee c)^c = b \oplus c.$$

Moreover, $\neg a \geqslant \neg b$ implies $\neg a \oplus c \geqslant \neg b \oplus c$, whence

$$a \ominus c = \neg(\neg a \oplus c) \leqslant \neg(\neg b \oplus c) = b \ominus c.$$

(ii) $(a \oplus b) \ominus b = \neg(\neg(a \oplus b) \oplus b) = \neg(\neg a \vee b) = a \wedge \neg b$.

(iii) $(a \wedge b) \oplus c = (a^0 \vee b^0 \vee c)^c = ((a^0 \vee c) \vee (b^0 \vee c))^c = (a^0 \vee c)^c \wedge (b^0 \vee c)^c = (a \oplus c) \wedge (b \oplus c)$.

(iv) According to (iii) we have

$$(a \wedge \neg b) \oplus b = (a \oplus b) \wedge (\neg b \oplus b) = (a \oplus b) \wedge 1 = a \oplus b.$$

(v) The following are equivalent: $a \ominus b = 0$, $\neg(\neg a \oplus b) = 0$, $\neg a \oplus b = 1$ and $a \leqslant b$. □

Next we define three classes of basic algebras.

Definition 9. A *basic algebra* $\mathcal{A} = (A, \oplus, \neg, 0)$ is called *symmetric* if $(x \vee y)^y = x^{x \wedge y}$ for all $x, y \in A$, *monotonous* if $x, y, z \in A$ and $x \leqslant y$ together imply $z \oplus x \leqslant z \oplus y$ and *weakly monotonous* if $x \leqslant x \oplus y$ for all $x, y \in A$. Of course, every monotonous basic algebra is weakly monotonous.

Lemma 10. *If $\mathcal{A} = (A, \oplus, \neg, 0)$ is a basic algebra and $(A, \leqslant)$ is a chain, then $\mathcal{A}$ is symmetric.*

P r o o f. Let $a, b \in A$. If $a \leqslant b$ then $(a \vee b)^b = b^b = 1 = a^a = a^{a \wedge b}$. If $a \geqslant b$ then $(a \vee b)^b = a^b = a^{a \wedge b}$. □

Lemma 11. *Every commutative basic algebra is symmetric and monotonous.*

Proof. If $\mathcal{A} = (A, \oplus, \neg, 0)$ is a commutative basic algebra, then

$$\begin{aligned}(x \vee y)^y &= x^0 \oplus y = y \oplus x^0 = (y^0 \vee x^0)^{x^0} = (x^0 \vee y^0 \vee x^0)^{x^0} = (x \wedge y) \oplus x^0 \\ &= x^0 \oplus (x \wedge y) = (x \vee (x \wedge y))^{x \wedge y} = x^{x \wedge y}\end{aligned}$$

for all $x, y \in A$. The second assertion follows from Lemma 8 (i). $\square$

The following example shows that symmetric basic algebras need not be weakly monotonous and hence monotonous.

Example 12 (cf. [3]). If we define

$$x \vee y := \max(x, y), \quad x \wedge y := \min(x, y), \quad x^0 := 1 - \frac{x}{2} \quad \text{if } x \leqslant \frac{2}{3},$$
$$x^0 := 2 - 2x \quad \text{if } x \geqslant \frac{2}{3}, \quad x^y := 1 - x + y \quad \text{if } y > 0$$

for all $x, y \in [0, 1]$, then $([0, 1], \vee, \wedge, (^x; x \in [0, 1]), 0, 1)$ is a bounded lattice with sectionally antitone involutions. Let $\mathcal{A} := (A, \oplus, \neg, 0)$ denote the corresponding basic algebra. $\mathcal{A}$ is not commutative, since

$$\begin{aligned}\frac{1}{4} \oplus \frac{1}{2} &= \Big(\frac{7}{8} \vee \frac{1}{2}\Big)^{1/2} = \Big(\frac{7}{8}\Big)^{1/2} = 1 - \frac{7}{8} + \frac{1}{2} = \frac{5}{8} \neq \frac{1}{2} \\ &= 1 - \frac{3}{4} + \frac{1}{4} = \Big(\frac{3}{4}\Big)^{1/4} = \Big(\frac{3}{4} \vee \frac{1}{4}\Big)^{1/4} = \frac{1}{2} \oplus \frac{1}{4}.\end{aligned}$$

According to Lemma 10, $\mathcal{A}$ is symmetric. $\mathcal{A}$ is not weakly monotonous since

$$\frac{1}{2} \not\leqslant \frac{9}{20} = 1 - \frac{3}{4} + \frac{1}{5} = \Big(\frac{3}{4}\Big)^{1/5} = \Big(\frac{3}{4} \vee \frac{1}{5}\Big)^{1/5} = \frac{1}{2} \oplus \frac{1}{5}.$$

Now we introduce the notion of a state on a basic algebra.

Definition 13 (cf. [1]). A *state* on a basic algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ is a mapping $s\colon A \to \mathbb{R}$ satisfying conditions (S1)–(S3):

(S1) $s(x) \geqslant 0$ for all $x \in A$,

(S2) $s(1) = 1$,

(S3) $s(x \oplus y) = s(x) + s(y)$ for all $x, y \in A$ with $x \leqslant \neg y$ (i.e., orthogonal elements x, y).

Remark 14. It is easy to see that a convex combination of states is again a state.

Remark 15. This definition coincides with the usual one if the basic algebra is an MV-algebra. On orthomodular lattices $\mathcal{L} = (L, \vee, \wedge, ', 0, 1)$ states are usually defined as mappings $s\colon L \to \mathbb{R}$ satisfying conditions (S1′)–(S3′):

(S1′) $s(x) \geqslant 0$ for all $x \in L$,

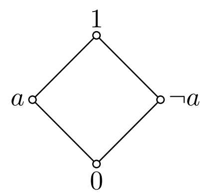
(S2′) $s(1) = 1$,

(S3′) $s(x \vee y) = s(x) + s(y)$ for all $x, y \in A$ with $x \leqslant y'$

(cf. [4]). Since $x \oplus y = x \vee y$ in case $x \leqslant y'$, the notions of a state on $\mathcal{L}$ and that of a state on the corresponding basic algebra coincide.

Lemma 16. *Conditions* (S1)–(S3) *are independent.*

Proof. Let $\mathcal{A} = (A, \oplus, \neg, 0)$ denote the basic algebra with $A = \{0, a, \neg a, 1\}$, the Hasse diagram



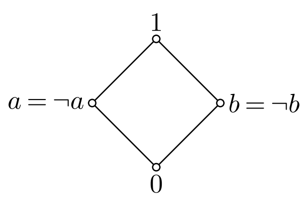
and the next operation table for $\oplus$.

$\oplus$	0	a	$\neg a$	1
0	0	a	$\neg a$	1
a	a	a	1	1
$\neg a$	$\neg a$	1	$\neg a$	1
1	1	1	1	1

Then $s\colon A \to \mathbb{R}$ defined by $s(0) := 0$, $s(a) := -1$, $s(\neg a) := 2$ and $s(1) := 1$ satisfies (S2) and (S3), but not (S1), $s\colon A \to \mathbb{R}$ defined by $s(x) := 0$ for all $x \in A$ satisfies (S1) and (S3), but not (S2), and $s\colon A \to \mathbb{R}$ defined by $s(x) := 1$ for all $x \in A$ satisfies (S1) and (S2), but not (S3). $\square$

There follow examples for states on non-symmetric and hence non-commutative basic algebras.

Example 17. The basic algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ with $A = \{0, a, b, 1\}$, the Hasse diagram



and the operation table

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	1	b	1
b	b	a	1	1
1	1	1	1	1

for $\oplus$ is not symmetric, since

$$(a \vee b)^b = 1^b = b \neq a = a^0 = a^{a \wedge b}$$

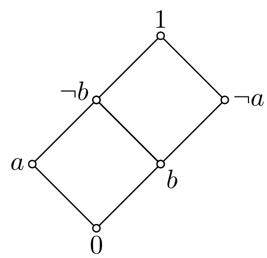
and hence, according to Lemma 11, not commutative. There are the following pairs of orthogonal elements:

$$(0, x), \quad (x, 0) \quad \text{and} \quad (x, \neg x)$$

for $x \in A$. The mapping $s\colon A \to [0, 1]$ defined by $s(0) := 0$, $s(a) = s(b) := 1/2$ and $s(1) := 1$ is a state on $\mathcal{A}$, since

$$\begin{aligned} s(0 \oplus x) &= s(x) = 0 + s(x) = s(0) + s(x), \\ s(x \oplus 0) &= s(x) = s(x) + 0 = s(x) + s(0), \\ s(x \oplus \neg x) &= s(1) = 1 = s(x) + s(\neg x) \quad \text{for all } x \in A. \end{aligned}$$

Example 18. The basic algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ with $A = \{0, a, b, \neg a, \neg b, 1\}$, the Hasse diagram



$(\neg b)^b := \neg b$, $(\neg a)^b := \neg a$ and the operation table

$\oplus$	0	a	b	$\neg a$	$\neg b$	1
0	0	a	b	$\neg a$	$\neg b$	1
a	a	a	$\neg a$	1	$\neg b$	1
b	b	$\neg b$	$\neg b$	$\neg a$	1	1
$\neg a$	$\neg a$	1	$\neg b$	$\neg a$	1	1
$\neg b$	$\neg b$	$\neg b$	1	1	1	1
1	1	1	1	1	1	1

for $\oplus$ is not symmetric, since

$$(a \vee b)^b = (\neg b)^b = \neg b \neq \neg a = a^0 = a^{a \wedge b}$$

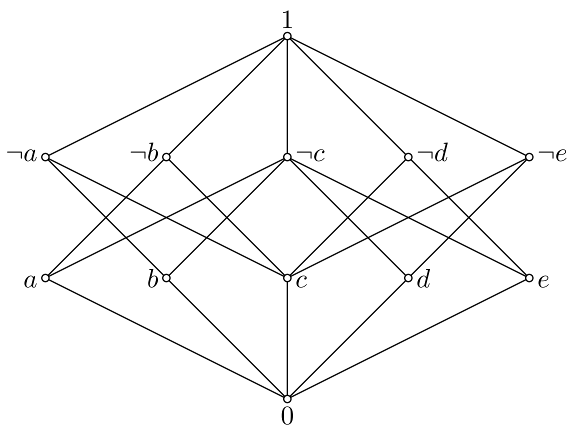
and hence, according to Lemma 11, not commutative. There are the following pairs of orthogonal elements:

$$(0, x), \quad (x, 0), \quad (a, b), \quad (b, a), \quad (b, b) \quad \text{and} \quad (x, \neg x)$$

for $x \in A$. The mapping $s\colon A \to \mathbb{R}$ defined by $s(0) := 0$, $s(a) = s(b) := 1/3$, $s(\neg a) = s(\neg b) := 2/3$ and $s(1) := 1$ is a state on $\mathcal{A}$, since

$$\begin{aligned}
s(0 \oplus x) &= s(x) = 0 + s(x) = s(0) + s(x) \quad \text{for all } x \in A,\\
s(x \oplus 0) &= s(x) = s(x) + 0 = s(x) + s(0) \quad \text{for all } x \in A,\\
s(a \oplus b) &= s(\neg a) = \tfrac{2}{3} = \tfrac{1}{3} + \tfrac{1}{3} = s(a) + s(b),\\
s(b \oplus a) &= s(\neg b) = \tfrac{2}{3} = \tfrac{1}{3} + \tfrac{1}{3} = s(b) + s(a),\\
s(b \oplus b) &= s(\neg b) = \tfrac{2}{3} = \tfrac{1}{3} + \tfrac{1}{3} = s(b) + s(b),\\
s(x \oplus \neg x) &= s(1) = 1 = s(x) + s(\neg x) \quad \text{for all } x \in A.
\end{aligned}$$

Example 19. The basic algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ with $A = \{0, a, b, c, d, e, \neg a, \neg b, \neg c, \neg d, \neg e, 1\}$, the Hasse diagram



$(\neg b)^a := \neg c$, $(\neg c)^a := \neg b$, $(\neg a)^b := \neg c$, $(\neg c)^b := \neg a$, $(\neg a)^c := \neg b$, $(\neg b)^c := \neg a$, $(\neg d)^c := \neg e$, $(\neg e)^c := \neg d$, $(\neg c)^d := \neg e$, $(\neg e)^d := \neg c$, $(\neg c)^e := \neg d$ and $(\neg d)^e := \neg c$

and the operation table

$\oplus$	0	a	b	c	d	e	$\neg a$	$\neg b$	$\neg c$	$\neg d$	$\neg e$	1
0	0	a	b	c	d	e	$\neg a$	$\neg b$	$\neg c$	$\neg d$	$\neg e$	1
a	a	a	$\neg c$	$\neg b$	d	e	1	$\neg b$	$\neg c$	$\neg d$	$\neg e$	1
b	b	$\neg c$	b	$\neg a$	d	e	$\neg a$	1	$\neg c$	$\neg d$	$\neg e$	1
c	c	$\neg b$	$\neg a$	c	$\neg e$	$\neg d$	$\neg a$	$\neg b$	1	$\neg d$	$\neg e$	1
d	d	a	b	$\neg e$	d	$\neg c$	$\neg a$	$\neg b$	$\neg c$	1	$\neg e$	1
e	e	a	b	$\neg d$	$\neg c$	e	$\neg a$	$\neg b$	$\neg c$	$\neg d$	1	1
$\neg a$	$\neg a$	1	$\neg a$	$\neg a$	$\neg e$	$\neg d$	$\neg a$	1	1	$\neg d$	$\neg e$	1
$\neg b$	$\neg b$	$\neg b$	1	$\neg b$	$\neg e$	$\neg d$	1	$\neg b$	1	$\neg d$	$\neg e$	1
$\neg c$	$\neg c$	$\neg c$	$\neg c$	1	$\neg c$	$\neg c$	1	1	$\neg c$	1	1	1
$\neg d$	$\neg d$	$\neg b$	$\neg a$	$\neg d$	1	$\neg d$	$\neg a$	$\neg b$	1	$\neg d$	1	1
$\neg e$	$\neg e$	$\neg b$	$\neg a$	$\neg e$	$\neg e$	1	$\neg a$	$\neg b$	1	1	$\neg e$	1
1	1	1	1	1	1	1	1	1	1	1	1	1

for $\oplus$ is not symmetric, since

$$(a \vee d)^d = (\neg c)^d = \neg e \neq \neg a = a^0 = a^{a \wedge d}$$

and hence, according to Lemma 11, not commutative. There are the following pairs of orthogonal elements:

$$(0,x),\ (x,0),\ (a,b),\ (a,c),\ (b,a),\ (b,c),\ (c,a),\ (c,b),$$
$$(c,d),\ (c,e),\ (d,c),\ (d,e),\ (e,c),\ (e,d) \text{ and } (x,\neg x)$$

for $x \in A$. The mapping $s\colon A \to \mathbb{R}$ defined by $s(0) = s(a) = s(b) = s(d) = s(e) = s(\neg c) := 0$ and $s(c) = s(\neg a) = s(\neg b) = s(\neg d) = s(\neg e) = s(1) := 1$ is a state on $\mathcal{A}$, since

$$\begin{aligned}
&s(0 \oplus x) = s(x) = 0 + s(x) = s(0) + s(x) \quad \text{for all } x \in A,\\
&s(x \oplus 0) = s(x) = s(x) + 0 = s(x) + s(0) \quad \text{for all } x \in A,\\
&s(a \oplus b) = s(\neg c) = 0 = 0 + 0 = s(a) + s(b),\\
&s(a \oplus c) = s(\neg b) = 1 = 0 + 1 = s(a) + s(c),\\
&s(b \oplus a) = s(\neg c) = 0 = 0 + 0 = s(b) + s(a),\\
&s(b \oplus c) = s(\neg a) = 1 = 0 + 1 = s(b) + s(c),\\
&s(c \oplus a) = s(\neg b) = 1 = 1 + 0 = s(c) + s(a),\\
&s(c \oplus b) = s(\neg a) = 1 = 1 + 0 = s(c) + s(b),\\
&s(c \oplus d) = s(\neg e) = 1 = 1 + 0 = s(c) + s(d),\\
&s(c \oplus e) = s(\neg d) = 1 = 1 + 0 = s(c) + s(e),
\end{aligned}$$

$$s(d \oplus c) = s(\neg e) = 1 = 0 + 1 = s(d) + s(c),$$
$$s(d \oplus e) = s(\neg c) = 0 = 0 + 0 = s(d) + s(e),$$
$$s(e \oplus c) = s(\neg d) = 1 = 0 + 1 = s(e) + s(c),$$
$$s(e \oplus d) = s(\neg c) = 0 = 0 + 0 = s(e) + s(d),$$
$$s(x \oplus \neg x) = s(1) = 1 = s(x) + s(\neg x) \quad \text{for all } x \in A.$$

We now consider some properties of states on basic algebras.

Lemma 20. *For a basic algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ and $s\colon A \to \mathbb{R}$, condition* (S3) *is equivalent to any of the following assertions:*

(i) $s(x \ominus y) = s(x \vee y) - s(y)$ *for all* $x, y \in A$,
(ii) $s(x \ominus y) = s(x) - s(y)$ *for all* $x, y \in A$ *with* $y \leqslant x$,
(iii) $s(x \oplus y) = s(x \wedge \neg y) + s(y)$ *for all* $x, y \in A$.

Proof. Let $a, b \in A$.
(S3) $\Rightarrow$ (i): We have

$$a \ominus b = \neg(\neg a \oplus b) = ((a \vee b)^b)^0 \leqslant b^0 = \neg b$$

and hence

$$s(a \ominus b) = s((a \ominus b) \oplus b) - s(b) = s(\neg(\neg a \oplus b) \oplus b) - s(b) = s(a \vee b) - s(b).$$

(i) $\Rightarrow$ (ii): This is clear.
(ii) $\Rightarrow$ (iii): We have $a \oplus b = (a^0 \vee b)^b \geqslant b$ and hence

$$s(a \oplus b) = s((a \oplus b) \ominus b) + s(b) = s(a \wedge \neg b) + s(b),$$

according to Lemma 8.
(iii) $\Rightarrow$ (S3): If $a \leqslant \neg b$ then $s(a \oplus b) = s(a \wedge \neg b) + s(b) = s(a) + s(b)$. $\square$

Lemma 21. *A state s on a basic algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ satisfies the following conditions:*

(i) $s(0) = 0$,
(ii) $s(x) \leqslant s(y)$ *for all* $x, y \in A$ *with* $x \leqslant y$,
(iii) $s(A) \subseteq [0, 1]$,
(iv) $s(\neg x) = 1 - s(x)$ *for all* $x \in A$,
(v) $s(x \oplus y) \leqslant s(x) + s(y)$ *for all* $x, y \in A$.

Proof. (i) $s(0) = s(0 \oplus 0) - s(0) = s(0) - s(0) = 0$.

(ii) Follows from Lemma 20.

(iii) Follows from (ii).

(iv) $s(\neg x) = s(x \oplus \neg x) - s(x) = s(1) - s(x) = 1 - s(x)$ for all $x \in A$.

(v) Follows from Lemma 20 and (ii). $\square$

Lemma 22. *For a state s on a basic algebra $\mathcal{A} = (A, \oplus, \neg, 0)$, the following are equivalent:*

(i) $s(x) > 0$ *for all* $x \in A \setminus \{0\}$,

(ii) $s(x) < s(y)$ *for all* $x, y \in A$ *with* $x < y$.

Proof. (i) $\Rightarrow$ (ii): Let $a, b \in A$ with $a < b$. Then $s(a) \leqslant s(b)$ according to Lemma 21. Now $s(a) = s(b)$ would imply $s(b \ominus a) = 0$ according to Lemma 20, whence

$$0 = b \ominus a = \neg(\neg b \oplus a) = ((b \vee a)^a)^0 = (b^a)^0$$

and hence $b^a = 1$, which shows $b = a$, contradicting $a < b$. Hence $s(a) < s(b)$.

(ii) $\Rightarrow$ (i): This follows from Lemma 21. $\square$

Lemma 23. *Let $\mathcal{A} = (A, \oplus, \neg, 0)$ be a symmetric basic algebra and $s\colon A \to \mathbb{R}$ and consider the following assertions:*

(i) $s(x \ominus y) = s(x) - s(x \wedge y)$ *for all* $x, y \in A$,

(ii) $s(x \vee y) = s(x) + s(y) - s(x \wedge y)$ *for all* $x, y \in A$.

Then (S3) $\Leftrightarrow$ (i) $\Rightarrow$ (ii).

Proof. Let $a, b \in A$.

(S3) $\Rightarrow$ (i): We have

$$a \ominus b = \neg(\neg a \oplus b) = \neg((a \vee b)^b) = \neg(a^{a \wedge b}) \leqslant \neg(a \wedge b)$$

and hence according to (S3)

$$\begin{aligned} s(a \ominus b) &= s((a \ominus b) \oplus (a \wedge b)) - s(a \wedge b) = s(\neg((a \vee b)^b) \oplus (a \wedge b)) - s(a \wedge b) \\ &= s(((a \vee b)^b \vee (a \wedge b))^{a \wedge b}) - s(a \wedge b) = s(((a \vee b)^b)^{a \wedge b}) - s(a \wedge b) \\ &= s((a^{a \wedge b})^{a \wedge b}) - s(a \wedge b) = s(a) - s(a \wedge b). \end{aligned}$$

(i) $\Rightarrow$ (S3): If $a \leqslant \neg b$ then according to (i)

$$\begin{aligned} s(a \oplus b) &= s((a \oplus b) \ominus b) + s((a \oplus b) \wedge b) \\ &= s(\neg(\neg(a \oplus b) \oplus b)) + s((a^0 \vee b)^b \wedge b) = s(\neg(\neg a \vee b)) + s(b) \\ &= s(a \wedge \neg b) + s(b) = s(a) + s(b). \end{aligned}$$

(i) $\Rightarrow$ (ii): According to Lemma 20 and (i) we have

$$s(a \vee b) = s(a \ominus b) + s(b) = s(a) - s(a \wedge b) + s(b) = s(a) + s(b) - s(a \wedge b).$$

□

Now we introduce a special class of states on basic algebras.

Definition 24. A *state-morphism* on a basic algebra $\mathcal{A}$ is a homomorphism from $\mathcal{A}$ to $\mathcal{M}$.

Remark 25. The state considered in Example 17 is not a state-morphism, since

$$s(a \oplus b) = s(b) = \tfrac{1}{2} \neq 1 = \tfrac{1}{2} \oplus \tfrac{1}{2} = s(a) \oplus s(b).$$

The same is true for the state considered in Example 18, since

$$s(a \oplus a) = s(a) = \tfrac{1}{3} \neq \tfrac{2}{3} = \tfrac{1}{3} \oplus \tfrac{1}{3} = s(a) \oplus s(a).$$

Finally, the state considered in Example 19 is a state-morphism as will be proved later.

Lemma 26. *Every state-morphism on a basic algebra $\mathcal{A}$ is a state on $\mathcal{A}$.*

Proof. Let s be a state-morphism on a basic algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ and $a, b \in A$ and assume $a \leqslant \neg b$. Then we have

$$s(a)+s(b) = s(a\wedge\neg b)+s(b) = (s(a)\wedge s(\neg b))+s(b) \leqslant s(\neg b)+s(b) = 1-s(b)+s(b) = 1$$

and hence $s(a \oplus b) = s(a) \oplus s(b) = (s(a) + s(b)) \wedge 1 = s(a) + s(b)$. □

There follow two characterizations of state-morphisms.

Theorem 27. *A state on a basic algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ is a state-morphism on $\mathcal{A}$ if and only if $s(x \vee y) = s(x) \vee s(y)$ for all $x, y \in A$.*

Proof. If $s(x \vee y) = s(x) \vee s(y)$ for all $x, y \in A$, then according to Lemma 20 and Lemma 21 we have

$$\begin{aligned} s(x \oplus y) &= s(\neg(\neg x \ominus y)) = 1 - s(\neg x \ominus y) = 1 - s(\neg x \vee y) + s(y) \\ &= 1 - (s(\neg x) \vee s(y)) + s(y) = (1 - s(\neg x) + s(y)) \wedge 1 \\ &= (s(x) + s(y)) \wedge 1 = s(x) \oplus s(y) \end{aligned}$$

for all $x, y \in A$. The rest follows from Theorem 6. □

Theorem 28. *A state on a symmetric basic algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ is a state-morphism on $\mathcal{A}$ if and only if $s(x \wedge y) = s(x) \wedge s(y)$ for all $x, y \in A$.*

Proof. If $s(x \wedge y) = s(x) \wedge s(y)$ for all $x, y \in A$, then according to Lemma 23

$$\begin{aligned} s(x \oplus y) &= s(\neg(\neg x \ominus y)) = 1 - s(\neg x \ominus y) = 1 - (s(\neg x) - s(\neg x \wedge y)) \\ &= s(x) + (s(\neg x) \wedge s(y)) = 1 \wedge (s(x) + s(y)) = s(x) \oplus s(y) \end{aligned}$$

for all $x, y \in A$. The rest follows from Theorem 6. $\square$

In the last part of the paper we provide a general construction of (in general not symmetric and hence not commutative) basic algebras possessing a state-morphism. We start with an easy lemma for Boolean algebras.

Lemma 29. *If $\mathcal{B} = (B, \vee, \wedge, ', 0, 1)$ is a Boolean algebra, a an atom of $\mathcal{B}$ and $b, c \in B$, then the following conditions hold:*

(i) *$a \nleqslant b$ implies $a \leqslant b'$,*

(ii) *$a \leqslant b \vee c$ if and only if $a \leqslant b$ or $a \leqslant c$.*

Proof. (i): We have

$$a = a \wedge 1 = a \wedge (b \vee b') = (a \wedge b) \vee (a \wedge b') = 0 \vee (a \wedge b') = a \wedge b' \leqslant b'.$$

(ii): If $a \leqslant b \vee c$, $a \nleqslant b$ and $a \nleqslant c$, then

$$a = a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0,$$

a contradiction. $\square$

In the following, let I be a nonempty index set and for every $i \in I$ let $\mathcal{B}_i = (B_i, \vee, \wedge, ', 0, 1)$ be a Boolean algebra such that for all $i, j \in I$ with $i \neq j$ the set $B_i \cap B_j$ equals the four-element set $\{0, a, a', 1\}$ where a is an atom of every Boolean algebra $\mathcal{B}_i$. Put $L = \bigcup\limits_{i \in I} B_i$ and let $b, c \in L$. For $i \in I$ let $\leqslant_i$ denote the partial order relation in $\mathcal{B}_i$. Define $b \leqslant c$ if there exists an $i \in I$ with $b \leqslant_i c$. It is easy to see that then $\leqslant$ is a partial order relation on L. If there exists an $i \in I$ with $b, c \in B_i$, then $b \vee c = b \vee_i c$ and $b \wedge c = b \wedge_i c$. Otherwise we have

$$b \vee c = \begin{cases} a' & \text{if } b, c \leqslant a', \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad b \wedge c = \begin{cases} a & \text{if } a \leqslant b, c, \\ 0 & \text{otherwise.} \end{cases}$$

This shows that $(L, \leqslant, 0, 1)$ is a bounded lattice. Let $'$ be defined on L in an obvious way. If $b \leqslant c$ then there exists some $i \in I$ with $b, c \in B_i$ and $b \leqslant_i c$. Since $\mathcal{B}_i$ is a Boolean algebra, we have $b \vee (c \wedge b') = c$. Hence $(L, \vee, \wedge, ', 0, 1)$ is an orthomodular lattice which can be considered as a basic algebra $\mathcal{L} = (L, \oplus, \neg, 0)$ in the usual way. $(L, \vee, \wedge, ', 0, 1)$ is a Greechie pasting of the Boolean algebras $\mathcal{B}_i$, $i \in I$ (cf. [4]). The basic algebra $\mathcal{L}$ will be called a *pasting of the Boolean algebras* $\mathcal{B}_i$, $i \in I$. This basic algebra need not be commutative in case $|I| > 1$.

Now we state and prove the following result.

Theorem 30. *Let $\mathcal{L} = (L, \oplus, \neg, 0)$ be a basic algebra which is a pasting of the Boolean algebras $\mathcal{B}_i$, $i \in I$, as described above. Then $s\colon L \to [0,1]$ defined by*

$$s(x) := \begin{cases} 1 & \text{if } x \geqslant a, \\ 0 & \text{otherwise} \end{cases}$$

for every $x \in L$ is a state-morphism on $\mathcal{L}$.

Proof. Let $b, c \in L$. If there exists an $i \in I$ with $b, c \in B_i$, then

$$b \oplus c = (b' \vee c)' \vee c = (b \wedge c') \vee c = (b \vee c) \wedge (c' \vee c) = b \vee c.$$

Otherwise

$$b \oplus c = (b \wedge c') \vee c = \begin{cases} a \vee c & \text{if } a \leqslant b \text{ and } a \nleqslant c, \\ 0 \vee c = c & \text{otherwise.} \end{cases}$$

Now we consider two cases.

Case 1. There exists an $i \in I$ with $b, c \in B_i$.

Then $b \oplus c = b \vee c$. Hence the following are equivalent: $s(b \oplus c) = 1$, $a \leqslant b \vee c$, $a \leqslant b$ or $a \leqslant c$, $s(b) = 1$ or $s(c) = 1$, $s(b) \oplus s(c) = 1$. This shows $s(b \oplus c) = s(b) \oplus s(c)$.

Case 2. There does not exist an $i \in I$ with $b, c \in B_i$.

Then $b, c \notin \{0, a, a', 1\}$.

If $a \leqslant b$ and $a \leqslant c$, then $s(b \oplus c) = s(c) = 1 = 1 \oplus 1 = s(b) \oplus s(c)$.

If $a \leqslant b$ and $a \nleqslant c$, then $s(b \oplus c) = s(a \vee c) = 1 = 1 \oplus 0 = s(b) \oplus s(c)$.

If $a \nleqslant b$ and $a \leqslant c$, then $s(b \oplus c) = s(c) = 1 = 0 \oplus 1 = s(b) \oplus s(c)$.

If $a \nleqslant b$ and $a \nleqslant c$, then $s(b \oplus c) = s(c) = 0 = 0 \oplus 0 = s(b) \oplus s(c)$.

Hence in all cases $s(b \oplus c) = s(b) \oplus s(c)$. □

Remark 31. From Theorem 30 it follows that the state s considered in Example 19 is in fact a state-morphism on $\mathcal{A}$.

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