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ON SOLUTIONS SET OF A MULTIVALUED STOCHASTIC DIFFERENTIAL EQUATION

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Abstract. We analyse multivalued stochastic differential equations driven by semimartingales. Such equations are understood as the corresponding multivalued stochastic integral equations. Under suitable conditions, it is shown that the considered multivalued stochastic differential equation admits at least one solution. Then we prove that the set of all solutions is closed and bounded.

Keywords: multivalued stochastic differential equation; Covitz-Nadler fixed point theorem; multivalued stochastic process

MSC 2010: 93E03, 93C41, 26E25, 60H05, 60H10, 60H20, 60G20

1. Introduction

Multivalued mappings are used in models coming from economics, control theory, optimization, biomathematics, physics, game theory, artificial intelligence (see e.g. [9] and references therein). Also, the dynamical systems with incomplete, uncertain information and dynamical systems with velocities that are not uniquely determined are often formulated by involving multivalued mappings, see e.g. [1], [2], [13], [7], [17], [19], [21], [20], [26]. To model uncertain systems, the multivalued stochastic differential equations (abbreviation MSDEs) [3], [6], [8], [12], [22], [24], [27]–[29] are also applied and they generalize the classical (single-valued) stochastic differential equations. Here the uncertainties, which are incorporated in MSDEs, are a stochastic uncertainty coming from random noises and an uncertainty driven by multivalued mappings.

In this paper we consider MSDEs driven by a large class of integrators—semimartingales, and propose a new formulation of the notion of MSDE. Namely, we consider a more general form of MSDEs in comparison to the classical studies,

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see [27]–[29], where the driving process is the Wiener process and the coefficients are mappings acting from $I \times \mathbb{R}^d$ (I denotes an interval $[0,T]$). In particular, in an integral form of MSDEs we consider multivalued integrands which are mappings acting from the set $I \times \Omega \times L^2$, where $L^2 := L^2(\Omega, \mathcal{A}, P; \mathbb{R}^d)$ and $(\Omega, \mathcal{A}, P)$ is an underlying probability space. Hence in our framework we allow the coefficients to depend on $\omega \in \Omega$, which is also an extension allowing for consideration of truely nonautonomous equations. Moreover, we deal with the infinite dimensional space L^2 instead of the finite dimensional Euclidean space $\mathbb{R}^d$. The studies which we present are more general, since the space $\mathbb{R}^d$ can be emdedded into L^2 in the sense that $x \in \mathbb{R}^d$ can be viewed as a random vector $X \in L^2$ such that $X(\omega) = x$ with probability one.

Under the Lipschitz type condition and the linear growth condition, we prove that the set of solutions to MSDEs driven by semimartingales is nonempty, i.e. there exists at least one solution. In multivalued framework considered in this paper, it is not possible to have unique solutions. Hence we investigate properties of the set of all solutions. We prove that this set is closed and bounded.

2. Preliminaries

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a separable Banach space, $\mathcal{K}(\mathcal{X})$ the family of all nonempty, compact and convex subsets of $\mathcal{X}$. The Hausdorff metric $H_{\mathcal{X}}$ in $\mathcal{K}(\mathcal{X})$ is defined by

$$H_{\mathcal{X}}(A,B) = \max\Big\{ \sup_{a\in A} \operatorname{dist}_{\mathcal{X}}(a,B),\ \sup_{b\in B} \operatorname{dist}_{\mathcal{X}}(b,A) \Big\},$$

where $\operatorname{dist}_{\mathcal{X}}(a,B) = \inf_{b\in B} \|a-b\|_{\mathcal{X}}$. It is known (cf. [16]) that $(\mathcal{K}(\mathcal{X}), H_{\mathcal{X}})$ is a complete and separable metric space. Also, the set $\mathcal{K}(\mathcal{X})$ has a semilinear structure under addition and scalar multiplication defined as

$$A + B = \{a+b\colon\ a\in A,\ b\in B\}, \quad \lambda A = \{\lambda a\colon\ a \in A\}, \quad A, B \in \mathcal{K}(\mathcal{X}),\ \lambda \in \mathbb{R}.$$

Let $(U, \mathcal{U}, \mu)$ be a measure space. A multivalued mapping $F\colon U \to \mathcal{K}(\mathcal{X})$ is said to be measurable if it satisfies

$$\{u \in U\colon\ F(u) \cap O \neq \emptyset\} \in \mathcal{U} \quad \text{for every open set } O \subset \mathcal{X}.$$

A measurable multivalued mapping $F\colon U \to \mathcal{K}(\mathcal{X})$ is said to be $L^p_{\mathcal{U}}(\mu)$-integrally bounded, $p \geqslant 1$, if there exists $h \in L^p(U, \mathcal{U}, \mu; \mathbb{R})$ such that $\|a\|_{\mathcal{X}} \leqslant h(\omega)$ for any a and ω with $a \in F(\omega)$. It is known (see [15]) that F is $L^p_{\mathcal{U}}(\mu)$-integrally bounded if and only if $\omega \mapsto \|F(\omega)\|_{\mathcal{X}}$ is in $L^p(U, \mathcal{U}, \mu; \mathbb{R})$, where

$$\|A\|_{\mathcal{X}} := H_{\mathcal{X}}(A, \{0\}) = \sup_{a\in A} \|a\|_{\mathcal{X}} \quad \text{for } A \in \mathcal{K}(\mathcal{X}).$$

Denote $I = [0,T]$, where $T < \infty$. Let $\mathcal{B}$ denote the Borel sigma-algebra of subsets of I, and $\mathcal{B}_t$ denote the Borel sigma-algebra of subsets of $[0,t]$ for each $t \in I$. Let $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{t\in I}, P)$ be a complete filtered probability space satisfying the usual hypotheses, i.e., $\{\mathcal{A}_t\}_{t\in I}$ is an increasing and right continuous family of sub-σ-algebras of $\mathcal{A}$ and $\mathcal{A}_0$ contains all P-null sets. Later on we will also assume that the σ-algebra $\mathcal{A}$ is separable with respect to the probability measure P.

Let $\mathcal{P}$ denote the σ-algebra of progressive elements in $I \times \Omega$, i.e.

$$\mathcal{P} := \{A \in \mathcal{B} \otimes \mathcal{A}\colon\ A \cap [0,t] \times \Omega \in \mathcal{B}_t \otimes \mathcal{A}_t \ \forall\, t \in I\}.$$

A stochastic process $f\colon\ I \times \Omega \to \mathbb{R}^d$ is called progressive if $f(\cdot,\cdot)$ is $\mathcal{P}$-measurable. A multivalued stochastic process $F\colon\ I \times \Omega \to \mathcal{K}(\mathbb{R}^d)$ is progressive if it is a $\mathcal{P}$-measurable multivalued mapping.

Let $Z\colon\ I \times \Omega \to \mathbb{R}$ be a continuous semimartingale with $Z(0) = 0$. It is known that Z has a unique representation

$$Z = A + M, \quad M(0) = 0, \quad A(0) = 0, \tag{2.1}$$

where $A\colon\ I \times \Omega \to \mathbb{R}$ is an $\{\mathcal{A}\}_t$-adapted continuous stochastic process of finite variation, $M\colon\ I \times \Omega \to \mathbb{R}$ is a local continuous $\{\mathcal{A}_t\}$-martingale.

Since A is of finite variation, almost each (with respect to P) sample path $A(\cdot,\omega)$ generates a measure $\Gamma_{A(\cdot,\omega)}$ with the total variation on the interval $[0,t]$ given by $|A(\omega)|_t = \int_0^t \Gamma_{A(\cdot,\omega)}(\mathrm{d}s)$. For a local martingale M one can define the quadratic variation process $[M]\colon\ I \times \Omega \to \mathbb{R}$ (cf. [10]). Now we denote by $\mathcal{H}^2$ the set of all semimartingales $Z\colon\ I \times \Omega \to \mathbb{R}$ with finite norm $\|\cdot\|_{\mathcal{H}^2}$, where

$$\|Z\|_{\mathcal{H}^2} := \|[M]_T^{1/2}\|_{L^2} + \||A|_T\|_{L^2}$$

and $L^2 := L^2(\Omega, \mathcal{A}, P; \mathbb{R}^d)$.

It is known that for a continuous semimartingale $Z \in \mathcal{H}^2$ the process M in (2.1) is a continuous square integrable martingale (see [25] Chapter II, Section 6, Corollary 4) and $\mathbb{E}|A|_T^2 < \infty$.

The processes A, M from the representation (2.1) of the semimartingale Z induce two measures μ_A, μ_M on $(I \times \Omega, \mathcal{P})$. The measure μ_A is defined similarly to [8], i.e.

$$\mu_A(C) := \int_\Omega \left(\int_I \mathbf{1}_C(t,\omega) |A(\omega)|_T \Gamma_{A(\cdot,\omega)}(\mathrm{d}t) \right) P(\mathrm{d}\omega) \quad \text{for } C \in \mathcal{P}.$$

For $f \in L^2_{\mathcal{P}}(\mu_A)$, where $L^2_{\mathcal{P}}(\mu_A) := L^2(I \times \Omega, \mathcal{P}, \mu_A; \mathbb{R}^d)$ one can define the stochastic Lebesgue-Stieltjes integral $\int_0^t f(s)\,\mathrm{d}A(s)$ sample path by sample path (cf. [25]). Note that

$$
\begin{aligned}
(2.2)\qquad \mathbb{E}\Big\|\int_\eta^t f(s)\,\mathrm{d}A(s)\Big\|^2_{\mathbb{R}^d}
&\leqslant \int_\Omega\Big(\int_\eta^t \|f(s,\omega)\|_{\mathbb{R}^d}\Gamma_{A(\cdot,\omega)}(\mathrm{d}s)\Big)^2 P(\mathrm{d}\omega)\\
&\leqslant \int_\Omega\Big((|A(\omega)|_t-|A(\omega)|_\eta)\int_\eta^t \|f(s,\omega)\|^2_{\mathbb{R}^d}\Gamma_{A(\cdot,\omega)}(\mathrm{d}s)\Big)P(\mathrm{d}\omega)\\
&\leqslant \int_\Omega\Big(|A(\omega)|_T\int_\eta^t \|f(s,\omega)\|^2_{\mathbb{R}^d}\Gamma_{A(\cdot,\omega)}(\mathrm{d}s)\Big)P(\mathrm{d}\omega)=\int_{[\eta,t]\times\Omega}\|f\|^2_{\mathbb{R}^d}\,\mathrm{d}\mu_A.
\end{aligned}
$$

A more expanded insight into the proof of (2.2) yields the following property.

Corollary 2.1. *Assume that $f\in L^2_{\mathcal{P}}(\mu_A)$. Then*

$$
(2.3)\qquad \mathbb{E}\sup_{u\in[\eta,t]}\Big\|\int_\eta^u f(s)\,\mathrm{d}A(s)\Big\|^2_{\mathbb{R}^d}\leqslant\int_{[\eta,t]\times\Omega}\|f\|^2_{\mathbb{R}^d}\,\mathrm{d}\mu_A.
$$

The second measure μ_M is the well-known Doléan-Dade measure (cf. [10]) such that

$$
\mu_M(\{0\}\times A_0)=0,\quad \mu_M((s,t]\times A)=\mathbb{E}\mathbf{1}_A(M(t)-M(s))^2,
$$

where $A_0\in\mathcal{A}_0$, $0\leqslant s<t\leqslant T$, $A\in\mathcal{A}_s$. For $f\in L^2_{\mathcal{P}}(\mu_M)$, where $L^2_{\mathcal{P}}(\mu_M):=L^2(I\times\Omega,\mathcal{P},\mu_M;\mathbb{R}^d)$, and $t\in I$ one can define the stochastic integral $\int_0^t f(s)\,\mathrm{d}M_s$ and we have (cf. [10])

$$
(2.4)\qquad \mathbb{E}\Big\|\int_0^t f(s)\,\mathrm{d}M(s)\Big\|^2_{\mathbb{R}^d}=\int_{[0,t]\times\Omega}\|f\|^2_{\mathbb{R}^d}\,\mathrm{d}\mu_M=\mathbb{E}\int_0^t\|f(s)\|^2_{\mathbb{R}^d}\,\mathrm{d}[M](s).
$$

Moreover, for $f\in L^2_{\mathcal{P}}(\mu_M)$ we have by the Doob inequality

$$
(2.5)\qquad \mathbb{E}\sup_{u\in[0,t]}\Big\|\int_0^u f(s)\,\mathrm{d}M(s)\Big\|^2_{\mathbb{R}^d}\leqslant 4\int_{[0,t]\times\Omega}\|f\|^2_{\mathbb{R}^d}\,\mathrm{d}\mu_M.
$$

For a semimartingale $Z\in\mathcal{H}^2$ with the representation (2.1) one can define a finite measure μ_Z on $(I\times\Omega,\mathcal{P})$ as

$$
\mu_Z(C):=\mu_A(C)+\mu_M(C),\quad C\in\mathcal{P}.
$$

Denote $L^2_{\mathcal{P}}(\mu_Z) := L^2(I \times \Omega, \mathcal{P}, \mu_Z; \mathbb{R}^d)$. For $f \in L^2_{\mathcal{P}}(\mu_Z)$ one can define the single-valued stochastic integral $\int_0^t f(s)\,\mathrm{d}Z(s)$ with respect to a semimartingale Z as follows:

$$\int_0^t f(s)\,\mathrm{d}Z(s) := \int_0^t f(s)\,\mathrm{d}A(s) + \int_0^t f(s)\,\mathrm{d}M(s).$$

Due to (2.2)–(2.4) and (2.3)–(2.5) we claim that the following assertions hold true.

Corollary 2.2. *If $f \in L^2_{\mathcal{P}}(\mu_Z)$ then for every $\eta, t \in I$, $\eta < t$, we have*

$$\begin{aligned}
\mathbb{E}\left\|\int_\eta^t f(s)\,\mathrm{d}Z(s)\right\|^2_{\mathbb{R}^d} &\leqslant 2\int_{[\eta,t]\times\Omega} \|f\|^2_{\mathbb{R}^d}\,\mathrm{d}\mu_Z,\\
\mathbb{E}\sup_{u\in[\eta,t]}\left\|\int_\eta^u f(s)\,\mathrm{d}Z(s)\right\|^2_{\mathbb{R}^d} &\leqslant 2\int_{[\eta,t]\times\Omega} \|f\|^2_{\mathbb{R}^d}\,\mathrm{d}\mu_A + 8\int_{[\eta,t]\times\Omega} \|f\|^2_{\mathbb{R}^d}\,\mathrm{d}\mu_M\\
&\leqslant 8\int_{[\eta,t]\times\Omega} \|f\|^2_{\mathbb{R}^d}\,\mathrm{d}\mu_Z.
\end{aligned}$$

Denote $L^2_t := L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d)$ for $t \in I$. Let $F\colon I \times \Omega \to \mathcal{K}^b_c(\mathbb{R}^d)$ be a progressive set-valued stochastic process which is $L^2_{\mathcal{P}}(\mu_Z)$-integrally bounded. For such a process let us define the set $S^2_{\mathcal{P}}(F, \mu_Z) := \{f \in L^2_{\mathcal{P}}(\mu_Z)\colon\ f \in F,\ \mu_Z\text{-a.e.}\}$. Due to the Kuratowski and Ryll-Nardzewski selection theorem (see [18]) we have $S^2_{\mathcal{P}}(F, \mu_Z) \neq \emptyset$. Therefore we can define the following stochastic integral of Aumann type (cf. [23]).

Definition 2.3. For a progressive and $L^2_{\mathcal{P}}(\mu_Z)$-integrally bounded multivalued stochastic process $F\colon I \times \Omega \to \mathcal{K}^b_c(\mathbb{R}^d)$ and for $\tau, t \in \mathbb{R}_+$, $\tau < t$ the set-valued stochastic trajectory integral (over the interval $[\tau, t]$) of F with respect to the semimartingale Z is the following subset of L^2_t:

$$\int_{[\tau,t]} F(s)\,\mathrm{d}Z(s) := \left\{\int_\tau^t f(s)\,\mathrm{d}Z(s)\colon\ f \in S^2_{\mathcal{P}}(F, \mu_Z)\right\}.$$

Since we consider Z continuous, the integrals $\int_{[\tau,t]} F(s)\,\mathrm{d}Z(s)$, $\int_{(\tau,t]} F(s)\,\mathrm{d}Z(s)$ coincide. For their common value we will write $\int_\tau^t F(s)\,\mathrm{d}Z(s)$. It is known that $\int_\tau^t F(s)\,\mathrm{d}Z(s)$ is a nonempty, bounded, convex, closed and weakly compact subset of L^2_t.

Remark 2.4. If we consider processes A, M from decomposition (2.1) then similarly to the above we can define the multivalued stochastic integral $\int_\tau^t G(s)\,\mathrm{d}A(s)$ for the $L^2_{\mathcal{P}}(\mu_A)$-integrally bounded progressive multivalued stochastic process $G\colon I \times \Omega \to \mathcal{K}(\mathbb{R}^d)$, and the multivalued stochastic integral $\int_\tau^t Q(s)\,\mathrm{d}M(s)$ for the $L^2_{\mathcal{P}}(\mu_M)$-integrally bounded progressive multivalued stochastic process $Q\colon I \times \Omega \to \mathcal{K}(\mathbb{R}^d)$.

Lemma 2.5. *Let $F\colon I \times \Omega \to \mathcal{K}(\mathbb{R}^d)$ be a progressive and $L^2_{\mathcal{P}}(\mu_Z)$-integrally bounded multivalued stochastic process. Then for every $s, a, t \in I$ such that $s \leqslant a \leqslant t$ we have*

$$\int_s^t F(\tau)\,\mathrm{d}Z(\tau) = \int_s^a F(\tau)\,\mathrm{d}Z(\tau) + \int_a^t F(\tau)\,\mathrm{d}Z(\tau).$$

In our investigations we will need some known results which are collected below as lemmata.

Lemma 2.6 ([14], Chapter V.3, Theorem 15). *Let T be a linear mapping of a Banach space $\mathcal{X}$ into a Banach space $\mathcal{Y}$. Then T is continuous with respect to the metric topologies in $\mathcal{X}$ and $\mathcal{Y}$ if and only if it is continuous with respect to the weak topologies.*

Lemma 2.7 ([5], Corollary 8.2.13). *Let $(U, \mathcal{U}, \mu)$ be a complete σ-finite measure space and $(\mathcal{X}, d)$ a complete and separable metric space. Assume that $F\colon U \to \mathcal{K}(\mathcal{X})$ is measurable and $f\colon U \to \mathcal{X}$ is measurable. Then there exists a measurable selection g of F such that $d(f(u), g(u)) = \operatorname{dist}_{\mathcal{X}}(f(u), F(u))$ for every $u \in U$.*

3. Main results

We begin this part of the paper with presenting some motivations to study MSDEs. They should reflect a potential utility of this theory in modeling the dynamics of real-world phenomena. To this end let us suppose that an investigated quantity x at instant t can be described using a stochastic integral equation with control u

$$x(t) = x_0 + \int_0^t f(x(s), u(s))\,\mathrm{d}s + \int_0^t g(x(s), u(s))\,\mathrm{d}W(s), \quad t \in I, \ P\text{-a.e.}, \tag{3.1}$$

where $x_0\colon \Omega \to \mathbb{R}$ is an initial value and $f\colon \mathbb{R}^2 \to \mathbb{R}$ denotes a drift coefficient, $g\colon \mathbb{R}^2 \to \mathbb{R}$ is a diffusion coefficient, u denotes a strategy, $u \in U$, U is a set of controls, W denotes the Wiener process. If it is assumed that $x(t) \in L^2$ for $t \in I$ then we can transform (3.1) to an equation in the space L^2, i.e. to the equation

$$x(t) = x_0 + \int_0^t \tilde{f}(s, x(s), u)\,\mathrm{d}s + \int_0^t \tilde{g}(s, x(s), u)\,\mathrm{d}W(s), \quad t \in I, \tag{3.2}$$

where the mappings $\tilde{f}, \tilde{g}\colon I \times \Omega \times L^2 \times U \to \mathbb{R}$ are defined as

$$\tilde{f}(s, \omega, a, u) := f(a(\omega), u(s, \omega)) \quad \text{and} \quad \tilde{g}(s, \omega, a, u) := g(a(\omega), u(s, \omega)).$$

Now, a quest for a solution to the controlled stochastic integral equation (3.2) in L^2 could be replaced by seeking the solution for MSDE

(3.3)
$$x(t)-x(s)\in\int_s^t F(\tau,x(\tau))\,\mathrm{d}\tau+\int_s^t G(\tau,x(\tau))\,\mathrm{d}W(s),\quad s,t\in I,\ s<t,\ x(0)=x_0,$$

where $F,G\colon I\times\Omega\times L^2\to\mathcal{K}(\mathbb{R})$ are defined as

$$F(s,\omega,a):=\overline{\mathrm{co}}\Big(\bigcup_{u\in U}\tilde{f}(s,\omega,a,u)\Big)\quad\text{and}\quad G(s,\omega,a):=\overline{\mathrm{co}}\Big(\bigcup_{u\in U}\tilde{g}(s,\omega,a,u)\Big).$$

The symbol $\overline{\mathrm{co}}(A)$ denotes the closed and convex hull of the set A. The relation (3.3) is an MSDE driven by the two-dimensional semimartingle $Z(\tau)=(\tau,W(\tau))'$. The problem of existence of a solution to (3.3) is a natural question.

After such a discussion on some motivations we start theoretical examinations of MSDEs driven by a one-dimensional semimartingale. From now on we assume that the σ-algebra $\mathcal{A}$ is separable with respect to the probability measure P. In the paper we consider MSDEs driven by a continuous $\mathcal{H}^2$-semimartigale Z which has the representation $Z=A+M$ described in (2.1), i.e., we consider the relation

$$\mathrm{d}x(t)\in F(t,x(t))\,\mathrm{d}Z(t),\quad t\in I,\ x(0)=x_0,$$

where $F\colon I\times\Omega\times L^2\to\mathcal{K}(\mathbb{R}^d)$, $x_0\in L^2_0$.

In fact, this notation has a symbolic meaning only, because it will be understood as the integral relation

$$x(t)-x(s)\in\int_s^t F(\tau,x(\tau))\,\mathrm{d}Z(\tau)\quad\text{for } 0\leqslant s<t\leqslant T,\qquad x(0)=x_0, \tag{3.4}$$

where the inclusion "$\in$" above is understood in the sense: "a point $x(t)-x(s)$ from L^2 belongs to the subset $\int_s^t F(\tau,x(\tau))\,\mathrm{d}Z(\tau)$ of L^2."

This formulation of MSDE is new and more general than the classical one. In the classical setting the multivalued mapping F acts from $I\times\mathbb{R}^d$ ([28], [29]). Now, we consider F to be a multivalued mapping acting from $I\times\Omega\times L^2$ and extend the studies that way.

Denote by $C(I,L^2)$ the set of all $\|\cdot\|_{L^2}$-continuous mappings $x\colon I\to L^2$. The set $C(I,L^2)$ endowed with the supremum metric becomes a complete metric space.

Definition 3.1. An element $x\in C(I,L^2)$ is said to be a solution to MSDE (3.4) if $x(0)=x_0$ and for any $s,t\in I$, $s<t$, the element $x(t)-x(s)$ of L^2 belongs to the subset $\int_s^t F(\tau,x(\tau))\,\mathrm{d}Z(\tau)$ of L^2.

Remark 3.2. Let $x \in C(I, L^2)$. It can be shown that x is a solution to MSDE (3.4) if and only if there exists $f \in S^2_{\mathcal{P}}(F \circ x, \mu_Z)$ such that for every $t \in I$

$$x(t) = x_0 + \int_0^t f(\tau)\,\mathrm{d}Z(\tau).$$

Let $\mathrm{SOL}(x_0, F, Z)$ denote the set of all solutions to MSDE (3.4). For $x \in \mathrm{SOL}(x_0, F, Z)$, due to the properties of stochastic integrals, we have

$$x(t) \in L^2_t, \quad t \in I.$$

The first aim of this part of the paper is to show that MSDE (3.4) possesses solutions, i.e., $\mathrm{SOL}(x_0, F, Z) \neq \emptyset$.

In the investigations of MSDE (3.4) we will assume that $F\colon I \times \Omega \times L^2 \to \mathcal{K}(\mathbb{R}^d)$ satisfies:

(H1) for every $a \in C(I, L^2)$ such that $a(t) \in L^2_t$ for every $t \in I$, the multivalued stochastic processes $I \times \Omega \ni (t, \omega) \mapsto F(t, \omega, a(t)) \in \mathcal{K}(\mathbb{R}^d)$ is progressive,

(H2) there exists $m \in L^2(I \times \Omega, \mathcal{P}, \mu_Z; \mathbb{R})$ such that for μ_Z-a.a. (t, ω) and for every $a \in L^2_t$

$$\|F(t, \omega, a)\|_{\mathbb{R}^d} \leqslant m(t, \omega)(1 + \|a\|_{L^2}),$$

(H3) there exists $K \in L^2(I \times \Omega, \mathcal{P}, \mu_Z; \mathbb{R})$ such that for μ_Z-a.a. (t, ω) and for every $a_1, a_2 \in L^2_t$

$$H_{\mathbb{R}^d}(F(t, \omega, a_1), F(t, \omega, a_2)) \leqslant K(t, \omega)\|a_1 - a_2\|_{L^2}.$$

In the paper we will also use a stronger condition than (H2). Namely, we will consider the following condition:

(H2′) there exists $m \in L^2(I \times \Omega, \mathcal{P}, \mu_Z; \mathbb{R})$ such that for μ_Z-a.a. (t, ω) and for every $a \in L^2_t$

$$\|F(t, \omega, a)\|_{\mathbb{R}^d} \leqslant m(t, \omega).$$

The assumptions imposed on the multivalued mapping F ensure the existence of a progressive and Lipschitz selection of F. In fact, we can formulate the following assertion which will be used later on.

Lemma 3.3. *Assume that* $F\colon I \times \Omega \times L^2 \to \mathcal{K}(\mathbb{R}^d)$ *satisfies* (H1)–(H3). *Then there exists* $\tilde{f}\colon I \times \Omega \times L^2 \to \mathbb{R}^d$ *such that*

(i) $\tilde{f}(t,\omega,a) \in F(t,\omega,a)$ *for every* $(t,\omega,a) \in I \times \Omega \times L^2$,

(ii) *the mapping* $I \times \Omega \ni (t,\omega) \mapsto \tilde{f}(t,\omega,a(t)) \in \mathbb{R}^d$ *is a progressive stochastic process for every* $a \in C(I, L^2)$ *such that* $a(t) \in L^2_t$ *for every* $t \in I$,

(iii) *there exists* $\widetilde{m} \in L^2(I \times \Omega, \mathcal{P}, \mu_Z; \mathbb{R})$ *such that for* μ_Z-a.a. (t,ω) *and for every* $a \in L^2_t$
$$\|\tilde{f}(t,\omega,a)\|_{\mathbb{R}^d} \leqslant \widetilde{m}(t,\omega)(1 + \|a\|_{L^2}),$$

(iv) *there exists* $\widetilde{K} \in L^2(I \times \Omega, \mathcal{P}, \mu_Z; \mathbb{R})$ *such that for* μ_Z-a.a. (t,ω) *and for every* $a_1, a_2 \in L^2_t$
$$\|\tilde{f}(t,\omega,a_1) - \tilde{f}(t,\omega,a_2)\|_{\mathbb{R}^d} \leqslant \widetilde{K}(t,\omega)\|a_1 - a_2\|_{L^2}.$$

The proof of this lemma will be omitted, since it is immediate. It is enough to define $\tilde{f}$ as $\tilde{f}(t,\omega,a) = s_d(F(t,\omega,a))$, where $s_d(A)$ denotes the Steiner point of the convex compact set $A \subset \mathbb{R}^d$ (see [5], Chapter 9). Then $s_d(F(t,\omega,a)) \in F(t,\omega,a)$ and the Steiner selection $\tilde{f}$ preserves the properties of the multivalued mapping F.

For $\eta, \tau \in I$, $\eta < \tau$ let us denote
$$L^2_{\mathcal{P},\eta,\tau}(\mu_Z) := L^2([\eta,\tau] \times \Omega, \mathcal{P}|_{[\eta,\tau]\times\Omega}, \mu_Z; \mathbb{R}^d).$$
Then for a $\mathcal{P}|_{[\eta,\tau]\times\Omega}$-measurable and $L^2_{\mathcal{P},\eta,\tau}(\mu_Z)$-integrally bounded multivalued stochastic process $F\colon [\eta,\tau] \times \Omega \to \mathcal{K}(\mathbb{R}^d)$ we define
$$S^2_{\mathcal{P}|_{[\eta,\tau]\times\Omega}}(F, \mu_Z) := \{f \in L^2_{\mathcal{P},\eta,\tau}(\mu_Z)\colon\ f \in F\ \mu_Z\text{-a.e.}\}.$$

Remark 3.4. Let $F\colon [\eta,\tau] \times \Omega \to \mathcal{K}(\mathbb{R}^d)$ be $\mathcal{P}|_{[\eta,\tau]\times\Omega}$- measurable and $L^2_{\mathcal{P},\eta,\tau}(\mu_Z)$-integrally bounded. Then these assumptions on F imply immediately that the set $S^2_{\mathcal{P}|_{[\eta,\tau]\times\Omega}}(F, \mu_Z)$ is a nonempty, bounded, convex, closed and weakly compact subset of $L^2_{\mathcal{P},\eta,\tau}(\mu_Z)$.

Let $\eta, \tau \in I$, $\eta < \tau$, $x_\eta \in L^2_\eta$, $F\colon I \times \Omega \times L^2 \to \mathcal{K}(\mathbb{R}^d)$ and $x \in C([\eta,\tau], L^2)$. We define the set $\Lambda_{\eta,\tau,x_\eta}(x)$ as
$$\Lambda_{\eta,\tau,x_\eta}(x) := \Big\{y\colon\ [\eta,\tau] \to L^2 \text{ such that } y(t) = x_\eta + \int_\eta^t f(s)\,\mathrm{d}Z(s) \text{ in } L^2 \text{ for } t \in [\eta,\tau],$$
$$f \in S^2_{\mathcal{P}|_{[\eta,\tau]\times\Omega}}((F|_{[\eta,\tau]\times\Omega\times L^2}) \circ x, \mu_Z)\Big\},$$
where $F|_{[\eta,\tau]\times\Omega\times L^2}\colon [\eta,\tau] \times \Omega \times L^2 \to \mathcal{K}(\mathbb{R}^d)$ is defined as
$$F|_{[\eta,\tau]\times\Omega\times L^2}(t,\omega,a) = F(t,\omega,a) \quad \text{for } (t,\omega,a) \in [\eta,\tau] \times \Omega \times L^2.$$

Lemma 3.5. *Let $F\colon\ I\times\Omega\times L^2\to\mathcal{K}(\mathbb{R}^d)$ satisfy* (H1)–(H3)*. Let $x\in C(I,L^2)$ be such that $x(t)\in L^2_t$ for every $t\in I$ and let $\eta,\tau\in I$, $\eta<\tau$. Then the set $\Lambda_{\eta,\tau,x_\eta}(x)$ is a nonempty subset of $C([\eta,\tau],L^2)$.*

P r o o f. For the mappings F and x we have (due to assumption (H1)) that $I\times\Omega\ni(t,\omega)\mapsto F(t,\omega,x(t))\in\mathcal{K}(\mathbb{R}^d)$ is $\mathcal{P}$-measurable. Hence $[\eta,\tau]\times\Omega\ni(t,\omega)\mapsto F(t,\omega,x(t))\in\mathcal{K}(\mathbb{R}^d)$ is a $\mathcal{P}|_{[\eta,\tau]\times\Omega}$-measurable multivalued stochastic process. From now on we will denote this latter process by $(F\circ x)|_{[\eta,\tau]\times\Omega}$. Note that due to (H2) for almost all (t,ω) with respect to μ_Z we have

$$\|F(t,\omega,x(t)\|_{\mathbb{R}^d}\leqslant m(t,\omega)(1+\|x(t)\|_{L^2})\leqslant m(t,\omega)\Big(1+\sup_{t\in[\eta,\tau]}\|x(t)\|_{L^2}\Big).$$

Thus

$$\begin{aligned}\int_{[\eta,\tau]\times\Omega}&\|F(t,\omega,x(t))\|^2_{\mathbb{R}^d}\mu_Z(\mathrm{d}t,\mathrm{d}\omega)\\&\leqslant 2\Big(1+\sup_{t\in[\eta,\tau]}\|x(t)\|^2_{L^2}\Big)\int_{[\eta,\tau]\times\Omega}m^2(t,\omega)\mu_Z(\mathrm{d}t,\mathrm{d}\omega),\end{aligned}$$

which means that $(F\circ x)\big|_{[\eta,\tau]\times\Omega}\colon\ [\eta,\tau]\times\Omega\to\mathcal{K}(\mathbb{R}^d)$ is $L^2_{\mathcal{P},\eta,\tau}(\mu_Z)$-integrally bounded. Hence the Kuratowski and Ryll-Nardzewski selection theorem [18] allows us to claim that the set $S^2_{\mathcal{P}|_{[\eta,\tau]\times\Omega}}((F\circ x)|_{[\eta,\tau]\times\Omega},\mu_Z)$ is nonempty and we can consider a mapping $[\eta,\tau]\ni t\mapsto x_\eta+\int_\eta^t f(s)\,\mathrm{d}Z(s)\in L^2$, where $f\in S^2_{\mathcal{P}|_{[\eta,\tau]\times\Omega}}((F\circ x)|_{[\eta,\tau]\times\Omega},\mu_Z)$. $\square$

Lemma 3.6. *Under the assumptions of Lemma* 3.5 *the set $\Lambda_{\eta,\tau,x_\eta}(x)$ is a bounded, convex and closed subset of $C([\eta,\tau],L^2)$.*

P r o o f. Observe that for $y\in\Lambda_{\eta,\tau,x_\eta}(x)$ and $t_1,t_2\in[\eta,\tau]$, $t_1<t_2$, we have

$$\|y(t_2)-y(t_1)\|^2_{L^2}=\mathbb{E}\left\|\int_{t_1}^{t_2}f(s)\,\mathrm{d}Z(s)\right\|^2_{\mathbb{R}^d},$$

where $f\in S^2_{\mathcal{P}|_{[\eta,\tau]\times\Omega}}((F\circ x)|_{[\eta,\tau]\times\Omega},\mu_Z)$. By Corollary 2.2

$$\begin{aligned}\|y(t_2)-y(t_1)\|^2_{L^2}&\leqslant 2\int_{[t_1,t_2]\times\Omega}\|f(s,\omega)\|^2_{\mathbb{R}^d}\mu_Z(\mathrm{d}s,\mathrm{d}\omega)\\&\leqslant 2\int_{[t_1,t_2]\times\Omega}\|F(s,\omega,x(s))\|^2_{\mathbb{R}^d}\mu_Z(\mathrm{d}s,\mathrm{d}\omega)\end{aligned}$$

and by assumption (H2)

$$\|y(t_2)-y(t_1)\|^2_{L^2}\leqslant 4\Big(1+\sup_{s\in[\eta,\tau]}\|x(s)\|^2_{L^2}\Big)\int_{[t_1,t_2]\times\Omega}m^2(s,\omega)\mu_Z(\mathrm{d}s,\mathrm{d}\omega).$$

Since $\int_{I\times\Omega} m^2\,\mathrm{d}\mu_Z < \infty$, we can infer that $\int_{[t_1,t_2]\times\Omega} m^2(s,\omega)\mu_Z(\mathrm{d}s,\mathrm{d}\omega) \to 0$ as $|t_1-t_2|\to 0$. This proves that $y\colon I\to L^2$ is uniformly continuous.

B o u n d e d n e s s. Note that for $y\in\Lambda_{\eta,\tau,x_\eta}(x)$ we have

$$\begin{aligned}
\|y\|^2_{C([\eta,\tau],L^2)} &\leqslant 2\|x_\eta\|^2_{L^2} + 2\sup_{t\in[\eta,\tau]}\left\|\int_\eta^t f(s)\,\mathrm{d}Z(s)\right\|^2_{L^2}\\
&\leqslant 2\|x_\eta\|^2_{L^2} + 4\sup_{t\in[\eta,\tau]}\int_{[\eta,t]\times\Omega}\|f(s,\omega)\|^2_{\mathbb{R}^d}\mu_Z(\mathrm{d}s,\mathrm{d}\omega)\\
&\leqslant 2\|x_\eta\|^2_{L^2} + 4\int_{[\eta,\tau]\times\Omega}\|\!|F(s,\omega,x(s))|\!\|^2_{\mathbb{R}^d}\mu_Z(\mathrm{d}s,\mathrm{d}\omega)\\
&\leqslant 2\|x_\eta\|^2_{L^2} + 8(1+\|x\|^2_{C([\eta,\tau],L^2)})\int_{[\eta,\tau]\times\Omega} m^2(s,\omega)\mu_Z(\mathrm{d}s,\mathrm{d}\omega).
\end{aligned}$$

Thus $\Lambda_{\eta,\tau,x_\eta}(x)$ is bounded.

C o n v e x i t y. Since $(F\circ x)|_{[\eta,\tau]\times\Omega}$ has convex values, the set

$$S^2_{\mathcal{P}|_{[\eta,\tau]\times\Omega}}((F\circ x)|_{[\eta,\tau]\times\Omega},\mu_Z)$$

is a convex subset of $L^2_{\mathcal{P},\eta,\tau}(\mu_Z)$ and convexity of $\Lambda_{\eta,\tau,x_\eta}(x)$ follows easily.

C l o s e d n e s s. Let $\{y_n\}_{n=1}^\infty\subset\Lambda_{\eta,\tau,x_\eta}(x)$ be such that

$$y_n \overset{n\to\infty}{\longrightarrow} y \quad\text{in the space } C([\eta,\tau],L^2),$$

$y\in C([\eta,\tau],L^2)$. Thus for every $t\in[\eta,\tau]$ the sequence $\{y_n(t)\}_{n=1}^\infty$ converges to $y(t)$ in the norm topology of the space L^2. Since $y_n\in\Lambda_{\eta,\tau,x_\eta}(x)$ we have

$$y_n(t) = x_\eta + \int_\eta^t f_n(s)\,\mathrm{d}Z(s),\quad \text{where } f_n\in S^2_{\mathcal{P}|_{[\eta,\tau]\times\Omega}}((F\circ x)|_{[\eta,\tau]\times\Omega},\mu_Z)\text{ for } n\in\mathbb{N}.$$

Since the set $S^2_{\mathcal{P}|_{[\eta,\tau]\times\Omega}}((F\circ x)|_{[\eta,\tau]\times\Omega},\mu_Z)$ is weakly compact in the space $L^2_{\mathcal{P},\eta,\tau}(\mu_Z)$, we infer that there exist a subsequence $\{f_{n_k}\}_{k=1}^\infty$ and

$$f\in S^2_{\mathcal{P}|_{[\eta,\tau]\times\Omega}}((F\circ x)|_{[\eta,\tau]\times\Omega},\mu_Z)$$

such that

$$f_{n_k}\rightharpoonup f \text{ in } L^2_{\mathcal{P},\eta,\tau}(\mu_Z),\quad \text{as } k\to\infty,$$

where the symbol $\rightharpoonup$ denotes convergence in weak topology. Obviously

$$x_\eta + \int_\eta^t f_{n_k}(s)\,\mathrm{d}Z(s) \overset{k\to\infty}{\longrightarrow} y(t) \quad\text{in } L^2 \text{ for every } t\in[\eta,\tau]. \tag{3.5}$$

For $t \in [\eta, \tau]$ let us define the linear operator $O_t \colon L^2_{\mathcal{P},\eta,\tau}(\mu_Z) \to L^2$ as follows:

$$O_t(f) := \int_\eta^t f(s)\,\mathrm{d}Z(s).$$

Since for $h, j \in L^2_{\mathcal{P},\eta,\tau}(\mu_Z)$ we have

$$\begin{aligned}\left\|\int_\eta^t h(s)\,\mathrm{d}Z(s) - \int_\eta^t j(s)\,\mathrm{d}Z(s)\right\|_{L^2}^2 &= \mathbb{E}\left\|\int_\eta^t h(s)\,\mathrm{d}Z(s) - \int_\eta^t j(s)\,\mathrm{d}Z(s)\right\|_{\mathbb{R}^d}^2 \\ &\leqslant 2\|h - j\|^2_{L^2_{\mathcal{P},\eta,\tau}(\mu_Z)},\end{aligned}$$

we infer that O_t is norm-to-norm continuous. Now, by Lemma 2.6 we obtain

$$O_t(f_{n_k}) = \int_\eta^t f_{n_k}(s)\,\mathrm{d}Z(s) \rightharpoonup O_t(f) = \int_\eta^t f(s)\,\mathrm{d}Z(s).$$

Hence

$$x_\eta + \int_\eta^t f_{n_k}(s)\,\mathrm{d}Z(s) \rightharpoonup x_\eta + \int_\eta^t f(s)\,\mathrm{d}Z(s) \quad \text{in } L^2,\ t \in [\eta, \tau].$$

Due to this convergence and (3.5) we infer that

$$\left\|y(t) - \left(x_\eta + \int_\eta^t f(s)\,\mathrm{d}Z(s)\right)\right\|_{L^2} = 0, \quad t \in [\eta, \tau].$$

Hence $y \in \Lambda_{\eta,\tau,x_\eta}(x)$. □

Now we are in a position to formulate the first main result on the existence of solutions to (3.4).

Theorem 3.7. *Assume that $F \colon I \times \Omega \times L^2 \to \mathcal{K}(\mathbb{R}^d)$ satisfies* (H1)–(H3). *Then the set* $\mathrm{SOL}(x_0, F, Z)$ *is nonempty.*

P r o o f. Since the semimartingale Z is continuous and $K \in L^2(I \times \Omega, \mathcal{P}, \mu_Z; \mathbb{R})$, we can choose a partition $0 = t_0 < t_1 < t_2 < \ldots < t_{N-1} < t_N = T$ of the interval I such that

$$\max\left\{\int_{[t_k,t_{k+1}]\times\Omega} K^2(s,\omega)\mu_Z(\mathrm{d}s,\mathrm{d}\omega) \colon k = 0, 1, \ldots, N-1\right\} < \frac{1}{2}. \tag{3.6}$$

Now we will show that for every $k \in \{0, 1, \ldots, N-1\}$ and every $x_{t_k} \in L^2_{t_k}$ the mapping

$$x \mapsto \Lambda_{t_k,t_{k+1},x_{t_k}}(x),$$

where $x \in C([t_k, t_{k+1}], L^2)$, is a multivalued contraction.

Denote $\eta = t_k$ and $\tau = t_{k+1}$ for convenience. Let $x_1, x_2 \in C([\eta,\tau], L^2)$. Then for $y^{(1)} \in \Lambda_{\eta,\tau,x_\eta}(x_1)$ there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset \Lambda_{\eta,\tau,x_\eta}(x_1)$ such that

$$\|y_n - y^{(1)}\|_{C([\eta,\tau],L^2)} \xrightarrow{n\to\infty} 0$$

and for every $n \in \mathbb{N}$

$$y_n(t) = x_\eta + \int_\eta^t f_n(s)\,\mathrm{d}Z(s), \quad \text{where } f_n \in S^2_{\mathcal{P}|_{[\eta,\tau]\times\Omega}}((F\circ x_1)|_{[\eta,\tau]\times\Omega}, \mu_Z).$$

Applying Lemma 2.7 we infer that there exists a sequence

$$\{g_n\}_{n=1}^{\infty} \subset S^2_{\mathcal{P}|_{[\eta,\tau]\times\Omega}}((F\circ x_2)|_{[\eta,\tau]\times\Omega}, \mu_Z)$$

such that for every $n \in \mathbb{N}$ and for every $(s,\omega) \in [\eta,\tau]\times\Omega$

$$\|f_n(s,\omega) - g_n(s,\omega)\|_{\mathbb{R}^d} = \operatorname{dist}_{\mathbb{R}^d}(f_n(s,\omega), F(s,\omega,x_2(s))).$$

Hence for every $n \in \mathbb{N}$ we get

$$\|f_n(s,\omega) - g_n(s,\omega)\|_{\mathbb{R}^d} \leqslant H_{\mathbb{R}^d}(F(s,\omega,x_1(s)), F(s,\omega,x_2(s))), \quad (s,\omega) \in [\eta,\tau]\times\Omega.$$

Now for $n \in \mathbb{N}$ we define $y_n^{(2)} \colon [\eta,\tau] \to L^2$ as

$$y_n^{(2)}(t) = x_\eta + \int_\eta^t g_n(s)\,\mathrm{d}Z(s).$$

Then we obtain $y_n^{(2)} \in \Lambda_{\eta,\tau,x_\eta}(x_2)$. Further observe that for $n \in \mathbb{N}$

$$\begin{aligned}
\|y_n - y_n^{(2)}\|^2_{C([\eta,\tau],L^2)} &= \left\|\int_\eta^{\cdot} (f_n(s) - g_n(s))\,\mathrm{d}Z(s)\right\|^2_{C([\eta,\tau],L^2)} \\
&= \sup_{t\in[\eta,\tau]} \mathbb{E}\left\|\int_\eta^t (f_n(s) - g_n(s))\,\mathrm{d}Z(s)\right\|^2_{\mathbb{R}^d} \\
&\leqslant 2\int_{[\eta,\tau]\times\Omega} \|f_n(s,\omega) - g_n(s,\omega)\|^2_{\mathbb{R}^d}\mu_Z(\mathrm{d}s,\mathrm{d}\omega) \\
&\leqslant 2\int_{[\eta,\tau]\times\Omega} H_{\mathbb{R}^d}(F(s,\omega,x_1(s)), F(s,\omega,x_2(s)))\mu_Z(\mathrm{d}s,\mathrm{d}\omega) \\
&\leqslant 2\int_{[\eta,\tau]\times\Omega} K^2(s,\omega)\|x_1(s) - x_2(s)\|^2_{L^2}\mu_Z(\mathrm{d}s,\mathrm{d}\omega) \\
&\leqslant 2\|x_1 - x_2\|^2_{C([\eta,\tau],L^2)} \int_{[\eta,\tau]\times\Omega} K^2(s,\omega)\mu_Z(\mathrm{d}s,\mathrm{d}\omega).
\end{aligned}$$

In view of (3.6) we have

$$\|y_n - y_n^{(2)}\|^2_{C([\eta,\tau],L^2)} < \alpha \|x_1 - x_2\|^2_{C([\eta,\tau],L^2)},$$

where $\alpha \in (0,1)$. Since

$$\begin{aligned}\|y^{(1)} - y_n^{(2)}\|_{C([\eta,\tau],L^2)} &\leqslant \|y^{(1)} - y_n\|_{C([\eta,\tau],L^2)} + \|y_n - y_n^{(2)}\|_{C([\eta,\tau],L^2)} \\ &< \|y^{(1)} - y_n\|_{C([\eta,\tau],L^2)} + \sqrt{\alpha}\, \|x_1 - x_2\|_{C([\eta,\tau],L^2)},\end{aligned}$$

we have

$$\operatorname{dist}_{C([\eta,\tau],L^2)}(y^{(1)}, \Lambda_{\eta,\tau,x_\eta}(x_2)) < \|y^{(1)} - y_n\|_{C([\eta,\tau],L^2)} + \sqrt{\alpha}\, \|x_1 - x_2\|_{C([\eta,\tau],L^2)}.$$

Passing to the limit as $n \to \infty$ we can write

$$\operatorname{dist}_{C([\eta,\tau],L^2)}(y^{(1)}, \Lambda_{\eta,\tau,x_\eta}(x_2)) \leqslant \sqrt{\alpha}\, \|x_1 - x_2\|_{C([\eta,\tau],L^2)}$$

and consequently

$$\sup_{y^{(1)} \in \Lambda_{\eta,\tau,x_\eta}(x_1)} \operatorname{dist}_{C([\eta,\tau],L^2)}(y^{(1)}, \Lambda_{\eta,\tau,x_\eta}(x_2)) \leqslant \sqrt{\alpha}\, \|x_1 - x_2\|_{C([\eta,\tau],L^2)}.$$

Proceeding similarly to the above we obtain

$$\sup_{y^{(2)} \in \Lambda_{\eta,\tau,x_\eta}(x_2)} \operatorname{dist}_{C([\eta,\tau],L^2)}(y^{(2)}, \Lambda_{\eta,\tau,x_\eta}(x_1)) \leqslant \sqrt{\alpha}\, \|x_1 - x_2\|_{C([\eta,\tau],L^2)}.$$

Hence

$$H_{C([\eta,\tau],L^2)}(\Lambda_{\eta,\tau,x_\eta}(x_1), \Lambda_{\eta,\tau,x_\eta}(x_2)) \leqslant \sqrt{\alpha}\, \|x_1 - x_2\|_{C([\eta,\tau],L^2)}.$$

Applying the Covitz-Nadler fixed point theorem, see [11], we can infer that there exists (not necessarily unique) $x^{\eta,\tau} \in C([\eta,\tau],L^2)$ such that $x^{\eta,\tau} \in \Lambda_{\eta,\tau,x_\eta}(x^{\eta,\tau})$. This means

$$x^{\eta,\tau}(t) = x_\eta + \int_\eta^t h_{\eta,\tau}(s)\, \mathrm{d}Z(s) \quad \text{in } L^2 \text{ for } t \in [\eta,\tau],$$

where $h_{\eta,\tau} \in S^2_{\mathcal{P}|_{[\eta,\tau]\times\Omega}}((F \circ x^{\eta,\tau})|_{[\eta,\tau]\times\Omega}, \mu_Z)$. Now we define $x^\star\colon I \to L^2$ as a spline of $x^{t_0,t_1}, x^{t_1,t_2}, \ldots, x^{t_{N-1},t_N}$. Then $x^\star \in C(I,L^2)$. Also it is easy to see that $h\colon I \times \Omega \to \mathbb{R}^d$ defined as

$$\begin{aligned}h(s,\omega) = {}& h_{t_0,t_1}(s,\omega)\mathbf{1}_{[t_0,t_1]\times\Omega}(s,\omega) \\ &+ h_{t_1,t_2}(s,\omega)\mathbf{1}_{[t_1,t_2]\times\Omega}(s,\omega) + \ldots + h_{t_{N-1},t_N}(s,\omega)\mathbf{1}_{[t_{N-1},t_N]\times\Omega}(s,\omega)\end{aligned}$$

satisfies the condition $h \in S^2_{\mathcal{P}}(F \circ x^\star, \mu_Z)$. Moreover

$$x^\star(t) = x_0 + \int_0^t h(s)\, \mathrm{d}Z(s) \quad \text{in } L^2 \text{ for } t \in I,$$

which means (due to Remark 3.2) that $x^\star$ is a solution to MSDE (3.4). $\square$

Since the set $\mathrm{SOL}(x_0, F, Z)$ is nonempty, it makes sense to ask about properties of this set.

Theorem 3.8. *Under assumptions of Theorem* 3.7 *the set* $\mathrm{SOL}(x_0, F, Z)$ *is a closed subset of* $C(I, L^2)$.

P r o o f. Let $\{x_n\} \subset \mathrm{SOL}(x_0, F, Z)$ be such that

$$\|x_n - x\|_{C(I,L^2)} \overset{n\to\infty}{\longrightarrow} 0, \tag{3.7}$$

where $x \in C(I, L^2)$.

Since $x_n(0) = x_0$ for every $n \in \mathbb{N}$, due to (3.7) we get $x(0) = x_0$ easily. In what follows we shall show that for every $s, t \in I$, $s < t$, $x(t) - x(s) \in \int_s^t F(\tau, x(\tau))\,\mathrm{d}Z(\tau)$ holds. To this end let us notice that, due to Corollary 2.2 and (H3), for $s < t$ we have

$$\begin{aligned}
H^2_{L^2}&\left(\int_s^t F(\tau, x_n(\tau))\,\mathrm{d}Z(\tau), \int_s^t F(\tau, x(\tau))\,\mathrm{d}Z(\tau)\right)\\
&\leqslant 2\int_{[s,t]\times\Omega} H^2_{\mathbb{R}^d}(F(\tau,\omega,x_n(\tau,\omega)), F(\tau,\omega,x(\tau,\omega)))\mu_Z(\mathrm{d}\tau,\mathrm{d}\omega)\\
&\leqslant 2\int_{[s,t]\times\Omega} K^2(\tau,\omega)\|x_n(\tau) - x(\tau)\|^2_{L^2}\mu_Z(\mathrm{d}\tau,\mathrm{d}\omega)\\
&\leqslant 2\|x_n - x\|^2_{C(I,L^2)}\int_{I\times\Omega} K^2(\tau,\omega)\mu_Z(\mathrm{d}\tau,\mathrm{d}\omega).
\end{aligned}$$

Hence

$$H_{L^2}\left(\int_s^t F(\tau, x_n(\tau))\,\mathrm{d}Z(\tau), \int_s^t F(\tau, x(\tau))\,\mathrm{d}Z(\tau)\right) \longrightarrow 0 \quad \text{as } n \to \infty. \tag{3.8}$$

Now observe that for $s < t$ and for every $n \in \mathbb{N}$

$$\begin{aligned}
\mathrm{dist}_{L^2}&\left(x(t) - x(s), \int_s^t F(\tau, x(\tau))\,\mathrm{d}Z(\tau)\right)\\
&\leqslant \|x(t) - x_n(t)\|_{L^2} + \|x_n(s) - x(s)\|_{L^2}\\
&\qquad + H_{L^2}\left(\int_s^t F(\tau, x_n(\tau))\,\mathrm{d}Z(\tau), \int_s^t F(\tau, x(\tau))\,\mathrm{d}Z(\tau)\right)\\
&\leqslant 2\|x_n - x\|_{C(I,L^2)}\\
&\qquad + H_{L^2}\left(\int_s^t F(\tau, x_n(\tau))\,\mathrm{d}Z(\tau), \int_s^t F(\tau, x(\tau))\,\mathrm{d}Z(\tau)\right).
\end{aligned}$$

By (3.7) and (3.8) we obtain

$$\operatorname{dist}_{L^2}\left(x(t)-x(s), \int_s^t F(\tau, x(\tau))\,\mathrm{d}Z(\tau)\right) = 0.$$

Since the integral $\int_s^t F(\tau, x(\tau))\,\mathrm{d}Z(\tau)$ is a closed subset of L^2, we get $x(t)-x(s) \in \int_s^t F(\tau, x(\tau))\,\mathrm{d}Z(\tau)$. Thus $x \in \mathrm{SOL}(x_0, F, Z)$. $\square$

Notice that for $x \in \mathrm{SOL}(x_0, F, Z)$ we can write

$$\|x\|^2_{C(I,L^2)} \leqslant 2\|x_0\|^2_{L^2} + 2\sup_{t\in I}\left\|\int_0^t f(s)\,\mathrm{d}Z(s)\right\|^2_{L^2}$$

where $f \in S^2_{\mathcal{P}}(F\circ x, \mu_Z)$. This implies

$$\|x\|^2_{C(I,L^2)} \leqslant 2\|x_0\|^2_{L^2} + 4\int_{I\times\Omega} \|\!|F(s,\omega,x(s))|\!\|^2_{\mathbb{R}^d}\,\mu_Z(\mathrm{d}s,\mathrm{d}\omega).$$

Therefore under assumption (H2′) we have the following additional assertion on boundedness of the set $\mathrm{SOL}(x_0, F, Z)$.

Corollary 3.9. *Assume that $F\colon I\times\Omega\times L^2 \to \mathcal{K}(\mathbb{R}^d)$ satisfies* (H1), (H2′), (H3). *Then $\mathrm{SOL}(x_0, F, Z)$ is a nonempty, bounded and closed subset of $C(I, L^2)$.*

All results of this paper have been established for MSDEs driven by a one-dimensional semimartingale. It is worth mentioning that the presented investigations can be repeated for MSDEs driven by m-dimensional semimartingales $\mathcal{Z} = (Z_1, Z_2, \dots, Z_m)'$, where Z_k's are one-dimensional continuous semimartingales such that $\|Z_k\|_{\mathcal{H}^2} < \infty$, $k = 1, 2, \dots, m$. To be more precise for $\mathcal{Z}$, $x_0 \in L^2_0$ and $\widehat{F}\colon I\times\Omega\times L^2 \to [\mathcal{K}(\mathbb{R}^d)]^{\times m}$ we can consider MSDE

$$x(t)-x(s) \in \int_s^t \widehat{F}(\tau, x(\tau))\,\mathrm{d}\mathcal{Z}(\tau) \quad \text{for } 0 \leqslant s < t \leqslant T,$$
$$x(0) = x_0,$$

which is understood as

$$\begin{gathered} x(t)-x(s) \in \int_s^t F^{(1)}(\tau, x(\tau))\,\mathrm{d}Z_1(\tau) + \int_s^t F^{(2)}(\tau, x(\tau))\,\mathrm{d}Z_2(\tau) \\ + \ldots + \int_s^t F^{(m)}(\tau, x(\tau))\,\mathrm{d}Z_m(\tau) \quad \text{for } 0 \leqslant s < t \leqslant T, \\ x(0) = x_0, \end{gathered}$$

where $\widehat{F}_1 = (F^{(1)}, F^{(2)}, \dots, F^{(m)})$.

Remark 3.10. Although the multivalued results involving the notion of multivalued stochastic integral and established in this paper are considered in the setting of a finite interval $I = [0, T]$, the methods used in the proofs allow to obtain all counterparts of the set-valued results also in the case of infinite interval $I = [0, \infty)$. In this case we consider a continuous semimartingale Z with $Z(0) = 0$ and with decomposition $Z = A + M$. Additionally the processes A and M should satisfy

$$\| |A|_\infty \|_{L^2} < \infty, \quad \| [M]_\infty^{1/2} \|_{L^2} < \infty.$$

Then the notion of a multivalued stochastic integral with respect to the continuous $\mathcal{H}^2$-semimartingale presented in this paper is well-defined.

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