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A. Mohammadzadeh; Ali Reza Moghaddamfar

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# Several quantitative characterizations of some specific groups

A. Mohammadzadeh, A.R. Moghaddamfar

*Abstract.* Let $G$ be a finite group and let $\pi(G) = \{p_1, p_2, \dots, p_k\}$ be the set of prime divisors of $|G|$ for which $p_1 < p_2 < \dots < p_k$. The Gruenberg-Kegel graph of $G$, denoted $\mathrm{GK}(G)$, is defined as follows: its vertex set is $\pi(G)$ and two different vertices $p_i$ and $p_j$ are adjacent by an edge if and only if $G$ contains an element of order $p_ip_j$. The degree of a vertex $p_i$ in $\mathrm{GK}(G)$ is denoted by $d_G(p_i)$ and the $k$-tuple $D(G) = (d_G(p_1), d_G(p_2), \dots, d_G(p_k))$ is said to be the degree pattern of $G$. Moreover, if $\omega \subseteq \pi(G)$ is the vertex set of a connected component of $\mathrm{GK}(G)$, then the largest $\omega$-number which divides $|G|$, is said to be an order component of $\mathrm{GK}(G)$. We will say that the problem of OD-characterization is solved for a finite group if we find the number of pairwise non-isomorphic finite groups with the same order and degree pattern as the group under study. The purpose of this article is twofold. First, we completely solve the problem of OD-characterization for every finite non-abelian simple group with orders having prime divisors at most 29. In particular, we show that there are exactly two non-isomorphic finite groups with the same order and degree pattern as $U_4(2)$. Second, we prove that there are exactly two non-isomorphic finite groups with the same order components as $U_5(2)$.

*Keywords:* OD-characterization of finite group; prime graph; degree pattern; simple group; 2-Frobenius group

*Classification:* 20D05, 20D06, 20D08

## 1. Introduction

Throughout this article, all the groups under consideration are *finite*, and simple groups are *non-abelian*. Given a group $G$, the spectrum $\omega(G)$ of $G$ is the set of orders of elements in $G$. Clearly, the spectrum $\omega(G)$ is closed and partially ordered by the divisibility relation, and hence is uniquely determined by the set $\mu(G)$ of its elements which are maximal under the divisibility relation. If $n$ is a natural number, then $\pi(n)$ denotes the set of all prime divisors of $n$, in particular, we set $\pi(G) = \pi(|G|)$.

One of the most well-known graphs associated with $G$ is the *Gruenberg-Kegel graph* (or *prime graph*) denoted by $\mathrm{GK}(G)$. The vertex set of this graph is $\pi(G)$ and two distinct vertices $p$ and $q$ are joined by an edge (abbreviated $p \sim q$) if and only if $pq \in \omega(G)$. The number of connected components of $\mathrm{GK}(G)$ is denoted

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by $s(G)$ and the sets of vertices of connected components of $\mathrm{GK}(G)$ are denoted as $\pi_i = \pi_i(G)(i = 1, 2, \ldots, s(G))$. If $G$ is a group of even order, then we put $2 \in \pi_1(G)$. The vertex sets of connected components of all finite simple groups are obtained in [16] and [36].

Given a group $G$, suppose that $\pi(G) = \{p_1, p_2, \ldots, p_k\}$ in which $p_1 < p_2 < \cdots < p_k$. The *degree* $d_G(p_i)$ of a vertex $p_i$ in the prime graph $\mathrm{GK}(G)$ is the number of edges incident on $p_i$. We define $D(G) = (d_G(p_1), d_G(p_2), \ldots, d_G(p_k))$, and we call this $k$-tuple the *degree pattern of* $G$. In addition, we denote by $\mathcal{OD}(G)$ the set of pairwise non-isomorphic finite groups with the same order and degree pattern as $G$, and we put $h_{\mathrm{OD}}(G) = |\mathcal{OD}(G)|$. Clearly, there are only finitely many isomorphism types of groups of order $|G|$, because there are just finitely many ways that an $|G| \times |G|$ multiplication table can be filled in. Finally, for each group $G$, it is clear that $1 \leq h_{\mathrm{OD}}(G) < \infty$.

**Definition 1.1.** A group $G$ is called *k-fold OD-characterizable* if $h_{\mathrm{OD}}(G) = k$. Usually, a 1-fold OD-characterizable group is simply called *OD-characterizable*, and it is called *quasi OD-characterizable* if it is $k$-fold OD-characterizable for some $k > 1$.

We will say that the OD-characterization problem is solved for a group $G$ if the value of $h_{\mathrm{OD}}(G)$ is known. Studies in recent years by several researchers have shown that many simple groups are OD-characterizable. Some of these results are summarized in Table 1.

In connection with the simple groups which are quasi OD-characterizable, it was shown in [4], [29] and [30] that:

$$\begin{aligned}
\mathcal{OD}(\mathbb{A}_{10}) &= \{\mathbb{A}_{10},\ \mathbb{Z}_3 \times J_2\},\\
\mathcal{OD}(S_6(5)) &= \{S_6(5), O_7(5)\},\\
\mathcal{OD}(S_{2m}(q)) &= \{S_{2m}(q), O_{2m+1}(q)\}, \quad m = 2^f \geq 2, \quad \left|\pi\left(\frac{q^m+1}{2}\right)\right| = 1,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad q \text{ odd prime power},\\
\mathcal{OD}(S_{2p}(3)) &= \{S_{2p}(3), O_{2p+1}(3)\}, \quad \left|\pi\left(\frac{3^p-1}{2}\right)\right| = 1,\ p \text{ odd prime.}
\end{aligned}$$

In addition to the above results, it has been shown that in [22] there exist many infinite families of alternating and symmetric groups, $\{\mathbb{A}_n\}$ and $\{\mathbb{S}_n\}$, which are quasi OD-characterizable, with $h_{\mathrm{OD}}(G) \geq 3$.

Here we consider the simple groups $S$ such that $\pi(S) \subseteq \pi(29!)$, and we denote the set of all these simple groups by $\mathcal{S}_{\leq 29}$. Using the classification of finite simple groups it is not hard to obtain a full list of all groups in $\mathcal{S}_{\leq 29}$. Actually, there are 110 such groups (see [24, Table 4] or [40, Table 1]). For convenience, the groups $S$ in $\mathcal{S}_{\leq 29}$ and their orders are listed in Table 2. The comparison between simple groups listed in Table 1 and the simple groups in $\mathcal{S}_{\leq 29}$, shows that there are only 5 groups in $\mathcal{S}_{\leq 29}$, that is $L_3(11)$, $U_4(2^3)$, ${}^2E_6(2)$, $S_4(17)$ and $U_4(17)$, for which

**Table 1.** Some OD-characterizable groups.

| $G$ | Conditions on $G$ | $h_{\mathrm{OD}}(G)$ | References |
|---|---|---|---|
| $\mathbb{A}_n$ | $n = p, p+1, p+2$ ($p$ a prime, $p \geq 5$) | 1 | [27], [30] |
| | $5 \leq n \leq 100, n \neq 10$ | 1 | [9], [15], [24], [28], [31] |
| | $n = 106, 112, 116, 134$ | 1 | [37], [38] |
| $L_2(q)$ | $q \neq 2, 3$ | 1 | [30], [43] |
| $L_3(q)$ | $\left\|\pi\left(\frac{q^2+q+1}{d}\right)\right\| = 1,\ d = (3, q-1)$ | 1 | [30] |
| $L_4(q)$ | $q \leq 17$ and $q = 19, 23, 27, 29, 31, 32, 37$ | 1 | [1], [3], [5] |
| $L_n(2)$ | $n = p$ or $p+1$, $2^p - 1$ is Mersenne prime | 1 | [5] |
| $L_n(2)$ | $n = 9, 10, 11$ | 1 | [13], [26] |
| $L_3(9)$ | | 1 | [32] |
| $L_6(3)$ | | 1 | [2] |
| $U_3(q)$ | $\left\|\pi\left(\frac{q^2-q+1}{d}\right)\right\| = 1,\ d = (3, q+1),\ q > 5$ | 1 | [30] |
| $U_4(q)$ | $q = 5, 7$ | 1 | [2], [5] |
| $U_6(2)$ | | 1 | [42] |
| $R(q)$ | $\|\pi(q \pm \sqrt{3q} + 1)\| = 1, q = 3^{2m+1}, m \geq 1$ | 1 | [30] |
| $\mathrm{Sz}(q)$ | $q = 2^{2n+1} \geq 8$ | 1 | [30] |
| $O_5(q) \cong S_4(q)$ | $\|\pi((q^2+1)/2)\| = 1,\ q \neq 3$ | 1 | [4] |
| $O_{2n+1}(q) \cong S_{2n}(q)$ | $n = 2^m \geq 2,\ 2 \mid q,\ \|\pi(q^n+1)\| = 1,\ (n,q) \neq (2,2)$ | 1 | [4] |
| $S_6(4)$ | | 1 | [21] |
| $G$ | $G$ is a sporadic group | 1 | [30] |
| $G$ | $\|G\| \leq 10^8, G \neq \mathbb{A}_{10}, U_4(2)$ | 1 | [33] |
| $G$ | $\|\pi(G)\| = 4, G \neq \mathbb{A}_{10}$ | 1 | [41] |
| $G$ | $G$ is a simple with $\pi_1(G) = \{2\}$ | 1 | [27] |
| $G$ | $G$ is a simple with $\pi(G) \subseteq \pi(17!), G \neq \mathbb{A}_{10}, U_4(2)$ | 1 | [25] |

the OD-characterization problem has not been solved. Therefore, one of the aims of this article is to prove these groups are OD-characterizable.

**Theorem 1.2.** *The simple groups $L_3(11)$, $U_4(2^3)$, ${}^2E_6(2)$, $S_4(17)$ and $U_4(17)$ are OD-characterizable.*

We recall that a group $G$ is called a 2-Frobenius group if $G = ABC$, where $A$ and $AB$ are normal subgroups of $G$, $B$ is a normal subgroup of $BC$, and $AB$ and $BC$ are Frobenius groups. Zinov'eva and V.D. Mazurov observed that the prime graph of a 2-Frobenius group is always disconnected, more precisely, it is the union of two connected components each of which is a complete graph [45, Lemma 3(a)]. On the other hand, Mazurov constructed a 2-Frobenius group of the same order as the simple group $U_4(2)$ ([20], [44]). In particular, this shows that $h_{\mathrm{OD}}(U_4(2)) \geq 2$ (see also [33]). In this article we also prove the following result.

**Theorem 1.3.** *The simple group $U_4(2)$ is 2-fold OD-characterizable. In fact, there exists a unique 2-Frobenius group $F = (2^4 \times 3^4) : 5 : 4$ with the same order and degree pattern as $U_4(2)$, and so $\mathcal{OD}(U_4(2)) = \{U_4(2), F\}$.*

As an immediate consequence of Theorems 1.2, 1.3 and the results in [20], [29], [30], we have the following corollary.

**Corollary 1.4.** *All simple groups in the class $\mathcal{S}_{\leq 29}$, other than $\mathbb{A}_{10}$, $S_6(3)$, $O_7(3)$ and $U_4(2)$, are OD-characterizable. In addition, each of these groups is 2-fold OD-characterizable.*

Given a group $G$, the order of $G$ can be expressed as a product of some coprime natural numbers $m_i = m_i(G)$, $i = 1, 2, \ldots, s(G)$, with $\pi(m_i) = \pi_i$. The numbers $m_1, m_2, \ldots, m_{s(G)}$ are called the *order components of* $G$. We set

$$\mathrm{OC}(G) = \left\{m_1, m_2, \ldots, m_{s(G)}\right\}.$$

In the similar manner, we denote by $\mathcal{OC}(G)$ the set of isomorphism classes of finite groups with the same set $\mathrm{OC}(G)$ of order components, and we put $h_{\mathrm{OC}}(G) = |\mathcal{OC}(G)|$. Again, in terms of function $h_{\mathrm{OC}}(\cdot)$, the groups $G$ are classified as follows:

**Definition 1.5.** A group $G$ is called *$k$-fold OC-characterizable*, if $h_{\mathrm{OC}}(G) = k$. Usually, a 1-fold OC-characterizable group is simply called *OC-characterizable*, and it is called *quasi OC-characterizable* if it is $k$-fold OC-characterizable for some $k > 1$.

Obviously, if $p$ is a prime number, then $h_{\mathrm{OC}}(\mathbb{Z}_p) = 1$ while $h_{\mathrm{OC}}(\mathbb{Z}_{p^2}) = h_{\mathrm{OC}}(\mathbb{Z}_p \times \mathbb{Z}_p) = 2$. Examples of OC-characterizable groups are abundant (see for instance, [10], [11], [12] and [14]). Also, one family examples of simple groups $S$ with $h_{\mathrm{OC}}(S) = 2$ is given in [12], namely

$$\mathcal{OC}(O_{2n+1}(q)) = \mathcal{OC}(S_{2n}(q)) = \{O_{2n+1}(q), S_{2n}(q)\} \quad (q \text{ odd}, \ n = 2^m \geq 4).$$

As the reader might have noticed, the values of the functions $h_{\mathrm{OD}}$ and $h_{\mathrm{OC}}$ may be different. For example, there are exactly two non-isomorphic groups of order $1814400 = 2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ and degree pattern $(2, 3, 2, 1)$, they are $\mathbb{A}_{10}$ and $\mathbb{Z}_3 \times J_2$, and hence $h_{\mathrm{OD}}(\mathbb{A}_{10}) = 2$. However, since the prime graph $\mathrm{GK}(\mathbb{A}_{10})$ is connected, $\mathrm{OC}(\mathbb{A}_{10}) = \{|\mathbb{A}_{10}|\}$, and so we obtain $h_{\mathrm{OC}}(\mathbb{A}_{10}) > \nu_{\mathrm{a}}(|\mathbb{A}_{10}|) = 150$, where $\nu_{\mathrm{a}}(m)$ denotes the number of types of abelian groups of order $m$. Therefore, we have $h_{\mathrm{OD}}(\mathbb{A}_{10}) \neq h_{\mathrm{OC}}(\mathbb{A}_{10})$. The simple group $U_5(2)$ is another example of this type. On the one hand, we have $h_{\mathrm{OD}}(U_5(2)) = 1$ by Theorem 3.3 in [41]. On the other hand, there exists a 2-Frobenius group $F$ such that $|F| = |U_5(2)|$ (see [20]) which implies that $h_{\mathrm{OC}}(U_5(2)) \geq 2$. Hence, $h_{\mathrm{OD}}(U_5(2)) < h_{\mathrm{OC}}(U_5(2))$. Finally, we show the following.

**Theorem 1.6.** *The simple group $U_5(2)$ is 2-fold OC-characterizable. In fact, there exists a unique 2-Frobenius group $F = (2^{10} \times 3^5) : 11 : 5$ with the same order components as $U_5(2)$, and so we have $\mathcal{OC}(U_5(2)) = \{U_5(2), F\}$.*

It is worth noting that the pair $\{U_5(2), (2^{10} \times 3^5) : 11 : 5\}$ is the first pair of a finite simple group and a solvable group with the same order components. Note that these groups have different prime graphs: the first connected component of $\mathrm{GK}(U_5(2))$ is the path $2 \sim 3 \sim 5$, while the first connected component of $(2^{10} \times 3^5) : 11 : 5$ is the complete subgraph $2 \sim 3 \sim 5 \sim 2$.

**Table 2**. Simple groups with orders having prime divisors at most 29 except alternating ones.

| $S$ | $\lvert S\rvert$ | $S$ | $\lvert S\rvert$ |
|---|---|---|---|
| $U_4(2)$ | $2^6 \cdot 3^4 \cdot 5$ | $Fi_{22}$ | $2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ |
| $L_2(7)$ | $2^3 \cdot 3 \cdot 7$ | $L_2(17)$ | $2^4 \cdot 3^2 \cdot 17$ |
| $L_2(2^3)$ | $2^3 \cdot 3^2 \cdot 7$ | $L_2(2^4)$ | $2^4 \cdot 3 \cdot 5 \cdot 17$ |
| $U_3(3)$ | $2^5 \cdot 3^3 \cdot 7$ | $S_4(2^2)$ | $2^8 \cdot 3^2 \cdot 5^2 \cdot 17$ |
| $L_2(7^2)$ | $2^4 \cdot 3 \cdot 5^2 \cdot 7^2$ | $He$ | $2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$ |
| $U_3(5)$ | $2^4 \cdot 3^2 \cdot 5^3 \cdot 7$ | $O_8^-(2)$ | $2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$ |
| $L_3(2^2)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7$ | $L_4(2^2)$ | $2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$ |
| $J_2$ | $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ | $S_8(2)$ | $2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$ |
| $U_4(3)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 7$ | $U_4(2^2)$ | $2^{12} \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17$ |
| $S_4(7)$ | $2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$ | $U_3(17)$ | $2^6 \cdot 3^4 \cdot 7 \cdot 13 \cdot 17^3$ |
| $S_6(2)$ | $2^9 \cdot 3^4 \cdot 5 \cdot 7$ | $O_{10}^-(2)$ | $2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$ |
| $O_8^+(2)$ | $2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$ | $L_2(13^2)$ | $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 17$ |
| $L_2(11)$ | $2^2 \cdot 3 \cdot 5 \cdot 11$ | $S_4(13)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^4 \cdot 17$ |
| $M_{11}$ | $2^4 \cdot 3^2 \cdot 5 \cdot 11$ | $L_3(2^4)$ | $2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$ |
| $M_{12}$ | $2^6 \cdot 3^3 \cdot 5 \cdot 11$ | $S_6(2^2)$ | $2^{18} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17$ |
| $U_5(2)$ | $2^{10} \cdot 3^5 \cdot 5 \cdot 11$ | $O_8^+(2^2)$ | $2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ |
| $M_{22}$ | $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ | $F_4(2)$ | $2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$ |
| $M^cL$ | $2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$ | ${}^2E_6(2)$ | $2^{36} \cdot 3^9 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$ |
| $HS$ | $2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$ | $U_3(2^3)$ | $2^9 \cdot 3^4 \cdot 7 \cdot 19$ |
| $U_6(2)$ | $2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$ | $U_4(2^3)$ | $2^{18} \cdot 3^7 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$ |
| $L_3(3)$ | $2^4 \cdot 3^3 \cdot 13$ | $L_3(7)$ | $2^5 \cdot 3^2 \cdot 7^3 \cdot 19$ |
| $L_2(5^2)$ | $2^3 \cdot 3 \cdot 5^2 \cdot 13$ | $L_4(7)$ | $2^9 \cdot 3^4 \cdot 5^2 \cdot 7^6 \cdot 19$ |
| $U_3(2^2)$ | $2^6 \cdot 3 \cdot 5^2 \cdot 13$ | $L_3(11)$ | $2^4 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11^3 \cdot 19$ |
| $S_4(5)$ | $2^6 \cdot 3^2 \cdot 5^4 \cdot 13$ | $L_2(19)$ | $2^2 \cdot 3^2 \cdot 5 \cdot 19$ |
| $L_4(3)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 13$ | $U_3(19)$ | $2^5 \cdot 3^2 \cdot 5^2 \cdot 7^3 \cdot 19^3$ |
| ${}^2F_4(2)'$ | $2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$ | $J_1$ | $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ |
| $L_2(13)$ | $2^2 \cdot 3 \cdot 7 \cdot 13$ | $J_3$ | $2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$ |
| $L_2(3^3)$ | $2^2 \cdot 3^3 \cdot 7 \cdot 13$ | $F_5$ | $2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$ |
| $G_2(3)$ | $2^6 \cdot 3^6 \cdot 7 \cdot 13$ | $L_2(23)$ | $2^3 \cdot 3 \cdot 11 \cdot 23$ |
| ${}^3D_4(2)$ | $2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$ | $U_3(23)$ | $2^7 \cdot 3^2 \cdot 11 \cdot 13^2 \cdot 23^3$ |
| $\mathrm{Sz}(2^3)$ | $2^6 \cdot 5 \cdot 7 \cdot 13$ | $M_{23}$ | $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ |
| $L_2(2^6)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ | $M_{24}$ | $2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ |
| $U_4(5)$ | $2^7 \cdot 3^4 \cdot 5^6 \cdot 7 \cdot 13$ | $Co_1$ | $2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$ |
| $L_3(3^2)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$ | $Co_2$ | $2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$ |
| $S_6(3)$ | $2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$ | $Co_3$ | $2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$ |
| $O_7(3)$ | $2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$ | $Fi_{23}$ | $2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$ |
| $G_2(2^2)$ | $2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$ | $U_4(17)$ | $2^{11} \cdot 3^7 \cdot 5 \cdot 7 \cdot 13 \cdot 17^6 \cdot 29$ |
| $S_4(2^3)$ | $2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$ | $S_4(17)$ | $2^{10} \cdot 3^4 \cdot 5 \cdot 17^4 \cdot 29$ |
| $O_8^+(3)$ | $2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$ | $L_2(17^2)$ | $2^5 \cdot 3^2 \cdot 5 \cdot 17^2 \cdot 29$ |
| $L_5(3)$ | $2^9 \cdot 3^{10} \cdot 5 \cdot 11^2 \cdot 13$ | $L_2(29)$ | $2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$ |
| $L_6(3)$ | $2^{11} \cdot 3^{15} \cdot 5 \cdot 7 \cdot 11^2 \cdot 13^2$ | $Ru$ | $2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$ |
| $Suz$ | $2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ | $Fi_{24}'$ | $2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$ |

We introduce some more notation. Let $\Gamma$ be a simple graph. An *independent set* of vertices in $\Gamma$ is a set of vertices that are pairwise non-adjacent to each other in $\Gamma$. We denote by $\alpha(\Gamma)$ the maximal number of vertices in independent sets of $\Gamma$. Given a group $G$, we put $t(G) = \alpha(\mathrm{GK}(G))$. Moreover, for each prime $r \in \pi(G)$, $t(r, G)$ denotes the maximal number of vertices in independent sets of GK($G$) containing $r$. Generally, our notation for simple groups follows [8]. Especially, the alternating and symmetric group on $n$ letters are denoted by $\mathbb{A}_n$ and $\mathbb{S}_n$, respectively. We also denote by $\mathrm{Syl}_p(G)$ the set of all Sylow $p$-subgroups of $G$, where $p \in \pi(G)$.

The sequel of this article is organized as follows: In Section 2, we recall some basic results, especially, on the spectra of certain finite simple groups. Section 3 is devoted to the proofs of main results (Theorems 1.2, 1.3, 1.6). We conclude our article with some open problems in Section 4.

## 2. Preliminaries

In this section we consider some results which will be needed for our further investigations.

**Lemma 2.1** ([35]). *Let $G$ be a finite group with $t(G) \geq 3$ and $t(2, G) \geq 2$, and let $K$ be the maximal normal solvable subgroup of $G$. Then the quotient group $G/K$ is an almost simple group, i.e., there exists a non-abelian simple group $P$ such that $P \leq G/K \leq \mathrm{Aut}(P)$.*

**Lemma 2.2** ([17, Lemma 8]). *Let $G$ be a finite group with $|\pi(G)| \geq 3$. If there exist prime numbers $r, s, t \in \pi(G)$ such that $\{tr, ts, rs\} \cap \omega(G) = \emptyset$, then $G$ is non-solvable.*

According to Table 4 in [24], we have the following result:

**Lemma 2.3.** *If $S \in \mathcal{S}_{\leq 29}$, then either $\mathrm{Out}(S) = 1$ or $\pi(\mathrm{Out}(S)) \subseteq \{2, 3\}$.*

**Lemma 2.4** ([34]). *Suppose that $q = p^n$, where $p$ is an odd prime. Then we have*

$$\mu(L_2(q)) = \left\{p, \ \frac{q-1}{2}, \ \frac{q+1}{2}\right\}.$$

**Lemma 2.5** ([23]). *Suppose that $q = p^n$, where $p$ is an odd prime. Then there holds:*

$$\mu(L_3(q)) = \begin{cases} \left\{q^2+q+1, \ q^2-1, \ p(q-1)\right\} & \text{if } \ q \not\equiv 1 \pmod 3, \\ \left\{\frac{q^2+q+1}{3}, \ \frac{q^2-1}{3}, \ \frac{p(q-1)}{3}, \ q-1\right\} & \text{if } \ q \equiv 1 \pmod 3. \end{cases}$$

**Lemma 2.6** ([19]). *Let $q$ be a power of prime 2. Then there holds:*

$$\mu(U_4(q)) = \left\{(q^2+1)(q-1), \ q^3+1, \ 2(q^2-1), \ 4(q+1)\right\}.$$

**Lemma 2.7** ([39]). *Let $q$ be a power of an odd prime $p$. Denote $d = \gcd(4, q+1)$. Then $\mu(U_4(q))$ contains the following (and only the following) numbers:*

(i) $\frac{q^4-1}{d(q+1)}$, $\frac{q^3+1}{d}$, $\frac{p(q^2-1)}{d}$, $q^2-1$;
(ii) $p(q+1)$, *if and only if* $d = 4$;
(iii) 9, *if and only if* $p = 3$.

**Lemma 2.8** ([18])**.** *Let $q = p^n$, where $p > 3$ is an odd prime. Then there holds:*

$$\mu(S_4(q)) = \left\{ \frac{q^2+1}{2},\ \frac{q^2-1}{2},\ p(q+1),\ p(q-1) \right\}.$$

Using Corollaries 2.5, 2.6, 2.7, 2.8, [24, Table 4] and [8] some results are summarized in Table 3. In this table we assume that $s = |\mathrm{Out}(S)|$.

**Table 3.** Some simple groups in $\mathcal{S}_{\leq 29}$.

| $S$ | $\lvert S\rvert$ | $\mu(S)$ | $\mathrm{D}(S)$ | $s$ |
|---|---|---|---|---|
| $L_3(11)$ | $2^4 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11^3 \cdot 19$ | 110, 120, 133 | (3, 2, 3, 1, 2, 1) | 2 |
| $U_4(2^3)$ | $2^{18} \cdot 3^7 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$ | 36, 126, 455, 513 | (2, 3, 2, 4, 2, 1) | 6 |
| $^2E_6(2)$ | $2^{36} \cdot 3^9 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$ | 13, 16, ..., 22, 24, 28, 30, 33, 35 | (4, 4, 3, 3, 2, 0, 0, 0) | 6 |
| $S_4(17)$ | $2^{10} \cdot 3^4 \cdot 5 \cdot 17^4 \cdot 29$ | 144, 145, 272, 306 | (2, 2, 1, 2, 1) | 2 |
| $U_4(17)$ | $2^{11} \cdot 3^7 \cdot 5 \cdot 7 \cdot 13 \cdot 17^6 \cdot 29$ | 288, 2320, 2448, 2457 | (4, 4, 2, 2, 2, 2, 2) | 4 |

The following proposition is taken from [33].

**Proposition 2.9** ([33])**.** *Let $M$ be a simple group whose order is less than $10^8$. If $G$ is a finite group with the same order and degree pattern as $M$, then the following statements hold:*

(a) *If $M \neq A_{10}, U_4(2)$, then $G \cong M$;*
(b) *If $M = A_{10}$, then $\mathcal{OD}(M) = \{A_{10}, J_2 \times \mathbb{Z}_3\}$;*
(c) *If $M = U_4(2)$, then $G$ is isomorphic to $M$ or a 2-Frobenius group.*

In particular, item (c) of Proposition 2.9 shows that $h_{\mathrm{OD}}(U_4(2)) \geq 2$. As we mentioned in the Introduction, in fact, there is such a 2-Frobenius group (see [20]). Indeed, when we have a Frobenius group, say, $F = K : C$ with abelian kernel $K$, and a faithful irreducible $\mathbb{Z}_pF$-module $V$, then the semidirect product $VF$ is a 2-Frobenius group. Now, we consider the general linear groups $\mathrm{GL}(4,2)$ and $\mathrm{GL}(4,3)$. In $\mathrm{GL}(4,2)$ and also in $\mathrm{GL}(4,3)$ there exists a Frobenius group $F = K : C$ of order 20 such that $K$ acts fixed-point-freely on corresponding natural modules $V_1$ of dimension 4 over $\mathbb{F}_2$ and $V_2$ of dimension 4 over $\mathbb{F}_3$. Now, we take $(V_1 \times V_2) \cdot F$ with the natural action of $F$ on direct factors. Then we obtain a 2-Frobenius group $(2^4 \times 3^4) : 5 : 4$ with the same order as $U_4(2)$. Note that the prime graphs of $U_4(2)$ and $(2^4 \times 3^4) : 5 : 4$ coincide.

## 3. Main results

In this section we will prove Theorems 1.2, 1.3 and 1.6. Before beginning the proof of Theorem 1.2, we draw the prime graphs of the groups $L_3(11)$, $U_4(2^3)$, $^2E_6(2)$, $S_4(17)$ and $U_4(17)$ in Figure 1.

Proof of Theorem 1.2: Let $S$ be one of the following simple groups $L_3(11)$, $U_4(2^3)$, $^2E_6(2)$ or $U_4(17)$. Suppose that $G$ is a finite group such that $|G| = |S|$ and $D(G) = D(S)$. We have to prove that $G \cong S$. In all cases we will prove that $t(G) \geq 3$ and $t(2, G) \geq 2$. Therefore, it follows from Lemma 2.1 that there exists a simple group $P$ such that $P \leq G/K \leq \mathrm{Aut}(P)$, where $K$ is the maximal normal

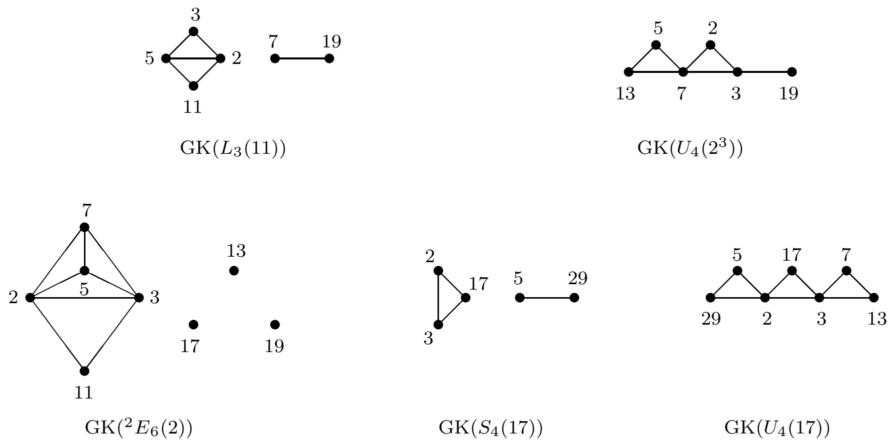



**Figure 1.** The prime graph of some simple groups.

solvable subgroup of $G$. In addition, we will prove that $P \cong S$, which implies that $K = 1$ and since $|G| = |S|$, $G$ is isomorphic to $S$, as required. We handle every case singly.

(a) $S = L_3(11)$. Let $G$ be a finite group such that

$$|G| = |S| = 2^4 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11^3 \cdot 19 \quad \text{and} \quad D(G) = D(S) = (3, 2, 3, 1, 2, 1).$$

According to our hypothesis there are five possibilities for the prime graph of $G$, as shown in Figure 2. Here, $p_1, p_2 \in \{2, 5\}$, $p_3, p_4 \in \{3, 11\}$, $p_5, p_6 \in \{7, 19\}$.

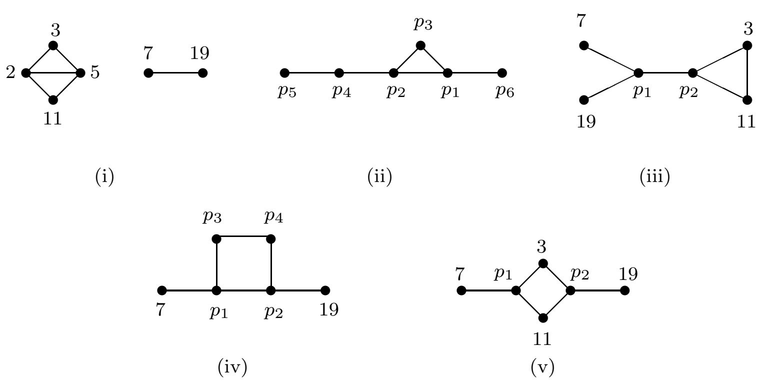



**Figure 2.** All possibilities for the prime graph of $G$.

We now consider two subcases separately.

(a.1) *Assume first that* GK($G$) *is disconnected.* In this case we immediately imply that GK($G$) = GK($L_3(11)$), and the hypothesis that $|G| = |L_3(11)|$

yields $\mathrm{OC}(G) = \mathrm{OC}(L_3(11))$. Now, by the Main Theorem in [10], $G$ is isomorphic to $L_3(11)$, as required.

(a.2) *Assume next that* $\mathrm{GK}(G)$ *is connected.* In this case $7 \nsim 19$ in $\mathrm{GK}(G)$. Since $\{7, 19, p_3\}$ is an independent set, $t(G) \geq 3$, and so by Lemma 2.2, $G$ is a non-solvable group. Moreover, since $d_G(2) = 3$ and $|\pi(G)| = 6$, $t(2, G) \geq 2$. Thus by Lemma 2.1 there exists a simple group $P$ such that $P \leq G/K \leq \mathrm{Aut}(P)$, where $K$ is the maximal normal solvable subgroup of $G$. We claim that $K$ is a $\{7, 11, 19\}'$-group. We first show that $K$ is a $\{7, 19\}'$-group. If $\{7, 19\} \subseteq \pi(K)$, then a Hall $\{7, 19\}$-subgroup of $K$ is an abelian group. Hence $7 \sim 19$ in $\mathrm{GK}(K)$, and so in $\mathrm{GK}(G)$, which is a contradiction. Let $\{r, s\} = \{7, 19\}$. Now assume that $r \in \pi(K)$ and $s \notin \pi(K)$. Let $T \in \mathrm{Syl}_r(K)$. By Frattini argument $G = KN_G(T)$. Therefore, the normalizer $N_G(T)$ contains an element of order $s$, say $x$. Now, $T\langle x\rangle$ is an abelian subgroup of $G$, so it leads to a contradiction as before.

Finally, suppose that $11 \in \pi(K)$ and $T \in \mathrm{Syl}_{11}(K)$. Then $G = KN_G(T)$ by Frattini argument. Evidently, $N_G(T)$ contains some elements of order 7 and 19, that we respectively denote by $u$ and $v$. Now, $T\langle u\rangle$ and $T\langle v\rangle$ are nilpotent subgroups of $G$, of orders $11^3 \cdot 7$ and $11^3 \cdot 19$, respectively, which implies that $7 \sim 11 \sim 19$, a contradiction. Since $K$ and $\mathrm{Out}(P)$ are $\{7, 11, 19\}'$-groups, $|P|$ is divisible by $7 \cdot 11^3 \cdot 19$. Considering the orders of simple groups in $\mathcal{S}_{\leq 29}$, we conclude that $P$ is isomorphic to $L_3(11)$, and so $K = 1$ and since $|G| = |L_3(11)|$, $G$ is isomorphic to $L_3(11)$. But then $\mathrm{GK}(G) = \mathrm{GK}(L_3(11))$ is disconnected, which is impossible.

(b) $S = U_4(2^3)$. Assume that $G$ is a finite group such that

$$|G| = |S| = 2^{18} \cdot 3^7 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19 \quad \text{and} \quad D(G) = D(S) = (2, 3, 2, 4, 2, 1).$$

So, the prime graph of $G$ is one of the following graphs as shown in Figure 3. Here $p_1, p_2, p_3 \in \{2, 5, 13\}$.

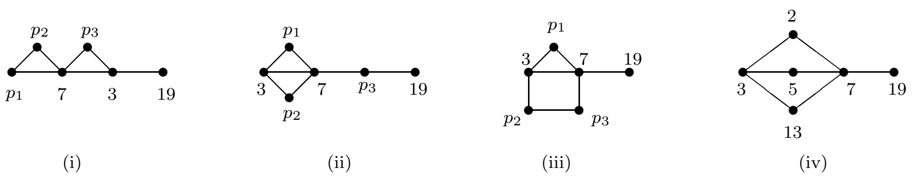



**Figure 3.** All possibilities for the prime graph of $G$.

In what follows, we will consider two subcases separately.

(b.1) *First, suppose that* $\mathrm{GK}(G)$ *is one of the graphs* (i), (iii) *or* (iv). Note that in each case $13 \nsim 19$ in $\mathrm{GK}(G)$ and $t(G) \geq 3$. Now, it follows from Lemma 2.2 that $G$ is a non-solvable group. Moreover, since $d_G(2) = 2$ and $|\pi(G)| = 6$, $t(2, G) \geq 2$. Thus by Lemma 2.1 there exists a simple group $P$ such that $P \leq G/K \leq \mathrm{Aut}(P)$, where $K$ is the maximal normal solvable subgroup of $G$. As in the previous case, one can show that $K$ is a $\{13, 19\}'$-group. Since $K$ and $\mathrm{Out}(P)$ are $\{13, 19\}'$-groups, thus $|P|$ is divisible by $13 \cdot 19$. Considering the orders of simple groups in $\mathcal{S}_{\leq 29}$ yields $P$ is isomorphic to $U_4(8)$, and so $K = 1$ and $G$ is isomorphic to $U_4(8)$, because $|G| = |U_4(8)|$. Therefore, the prime graph of $G$ and the graph (i) coincide, and in other cases we get a contradiction.

(b.2) *Next, suppose that* $\mathrm{GK}(G)$ *is the graph* (ii). In this case, 7 is not adjacent to 19 in $\mathrm{GK}(G)$. Since $\{p_1, p_2, 19\}$ is an independent set, $t(G) \geq 3$ and by Lemma 2.2, $G$ is a non-solvable group. Moreover, since $d_G(2) = 2$ and $|\pi(G)| = 6$, $t(2, G) \geq 2$. Thus by Lemma 2.1 there exists a simple group $P$ such that $P \leq G/K \leq \mathrm{Aut}(P)$, where $K$ is the maximal normal solvable subgroup of $G$. Using similar arguments to those in the previous case, one can show that $K$ is a $\{7, 19\}'$-group and $G$ is isomorphic to $U_4(8)$. But then 3 is adjacent to 19 in $\mathrm{GK}(G)$, which is a contradiction.

(c) $S = {}^2E_6(2)$. Assume that $G$ is a finite group such that

$$|G| = |S| = 2^{36} \cdot 3^9 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \ \text{ and } \ D(G) = D(S) = (4, 4, 3, 3, 2, 0, 0, 0).$$

Then, the prime graphs of $G$ and ${}^2E_6(2)$ coincide, and the hypothesis that $|G| = |{}^2E_6(2)|$ yields $\mathrm{OC}(G) = \mathrm{OC}({}^2E_6(2))$. Now, by [14], $G$ is isomorphic to ${}^2E_6(2)$, as required.

(d) $S = U_4(17)$. Assume that $G$ is a finite group such that

$$|G| = |S| = 2^{11} \cdot 3^7 \cdot 5 \cdot 7 \cdot 13 \cdot 17^6 \cdot 29 \ \text{ and } D(G) = D(S) = (4, 4, 2, 2, 2, 2, 2).$$

According to our hypothesis there are four possibilities for the prime graph of $G$, as shown in Figure 4. Here $p_1, p_2, p_3, p_4, p_5 \in \{5, 7, 13, 17, 29\}$.

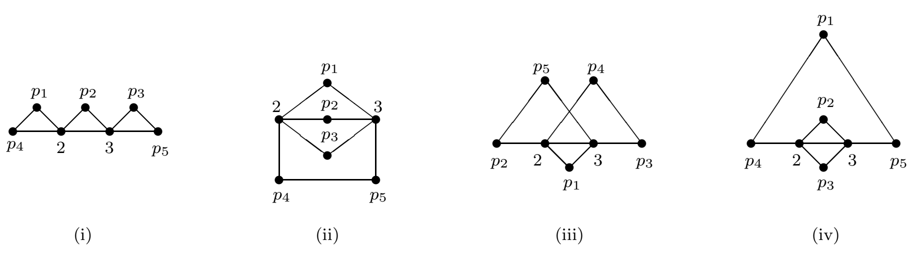



**Figure 4.** All possibilities for the prime graph of $G$.

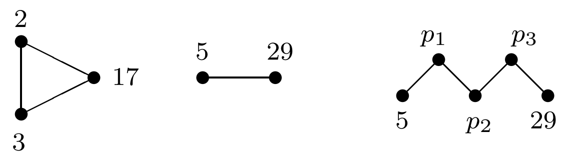
In all cases $\{p_1, p_2, p_3\}$ is an independent set, and hence $t(G) \geq 3$. Moreover, since $d_G(2) = 4$ and $|\pi(G)| = 7$, $t(2, G) \geq 2$. Now, from Lemma 2.1 there exists a simple group $P$ such that $P \leq G/K \leq \mathrm{Aut}(P)$. We claim now that $K$ is a $\{2, 3\}$-group. In fact, if there exists $p_i \in \pi(K)$, for some $i$, then with similar arguments as before, we can verify that for each $j \neq i$, $p_i \sim p_j$ in $\mathrm{GK}(G)$, except $\{p_i, p_j\} = \{7, 29\}$, and this contradicts the fact that $d_G(p_i) = 2$. Hence $K$ and

$\mathrm{Out}(P)$ are $\{2,3\}$-groups, thus $|P|$ is divisible by $5 \cdot 7 \cdot 13 \cdot 17^6 \cdot 29$. Again, considering the orders of simple groups in $\mathcal{S}_{\leq 29}$ yields $P$ is isomorphic to $U_4(17)$, and so $K = 1$ and $G$ is isomorphic to $U_4(17)$, because $|G| = |U_4(17)|$.

(e) $S = S_4(17)$. Assume that $G$ is a finite group such that

$$|G| = |S| = 2^{10} \cdot 3^4 \cdot 5 \cdot 17^4 \cdot 29 \quad \text{and} \quad D(G) = D(S) = (2, 2, 1, 2, 1).$$

Under these conditions, there are two possibilities for the prime graph of $G$, as shown in Figure 5. Here $p_1, p_2, p_3 \in \{2, 3, 17\}$.




(i) (ii)

**Figure 5.** All possibilities for the prime graph of $G$.

We now consider two cases separately, depending on $\mathrm{GK}(G)$ is connected or disconnected.

(2.1) *Assume first that* $\mathrm{GK}(G)$ *is connected*. Since $\{5, p_2, 29\}$ is an independent set, $t(G) \geq 3$. Moreover, since $d_G(2) = 2$ and $|\pi(G)| = 5$, $t(2, G) \geq 2$. Thus by Lemma 2.1 there exists a simple group $P$ such that $P \leq G/K \leq \mathrm{Aut}(P)$. We shall treat the cases $17 \nsim 29$ and $17 \sim 29$ in $\mathrm{GK}(G)$, separately.

(2.1.a) First we consider the case where $17 \nsim 29$ in $\mathrm{GK}(G)$. In this case as before, one can show that $K$ is a $\{17, 29\}'$-group. Since $K$ and $\mathrm{Out}(P)$ are $\{17, 29\}'$-groups, thus $|P|$ is divisible by $17^4 \cdot 29$. Considering the orders of simple groups in $\mathcal{S}_{\leq 29}$ yields $P$ is isomorphic to $S_4(17)$, and so $K = 1$ and $G$ is isomorphic to $S_4(17)$, because $|G| = |S_4(17)|$. But then $\mathrm{GK}(G) = \mathrm{GK}(S_4(17))$ is disconnected, which is impossible.

(2.1.b) Next we discuss the case where $17 \sim 29$ in $\mathrm{GK}(G)$. An argument similar to that in the above paragraphs shows that $K$ is a $\{3, 29\}'$-group. Since $K$ and $\mathrm{Out}(P)$ are $\{3, 29\}'$-groups, thus $|P|$ is divisible by $3^4 \cdot 29$. Considering the orders of simple groups in $\mathcal{S}_{\leq 29}$ yields $P$ is isomorphic to $S_4(17)$, and so $K = 1$ and $G$ is isomorphic to $S_4(17)$, because $|G| = |S_4(17)|$. But then $\mathrm{GK}(G) = \mathrm{GK}(S_4(17))$ is disconnected, which is impossible.

(2.2) *Assume next that* $\mathrm{GK}(G)$ *is disconnected*. In this case, it is easy to see that the prime graphs of $G$ and $S_4(17)$ coincide. Now, by the main theorem in [11], $G$ is isomorphic to $S_4(17)$.

This completes the proof of Theorem 1.2. $\square$

Proof of Theorem 1.3: Let $G$ be a finite group satisfying

$$(1)\ |G| = |U_4(2)| = 2^6 \cdot 3^4 \cdot 5, \quad \text{and} \quad (2)\ D(G) = D(U_4(2)) = (1, 1, 0).$$

By Proposition 2.9, $G$ is isomorphic to $U_4(2)$ or a 2-Frobenius group. First of all, it should be noted that the existence of a 2-Frobenius group satisfying conditions (1) and (2) is guaranteed by [20], [44]. To prove uniqueness, we note that any such group will be a subdirect product of 2-Frobenius groups of orders $2^4 \cdot 5 \cdot 4$ and $3^4 \cdot 5 \cdot 4$. As a matter of fact, since 4 is the order of 2 modulo 5, 4 is the smallest dimension of an irreducible module for $\mathbb{Z}_5$ over $\mathbb{F}_2$, so there is a unique Frobenius group of order $2^4 \cdot 5$ and its kernel is elementary abelian. Actually, this is a subgroup of the 1-dimensional affine group over $\mathbb{F}_{2^4}$ which is denoted by $\mathrm{AGL}(1, \mathbb{F}_{2^4})$. We can now extend this subgroup by an element of order 4 acting as a field automorphism of $\mathbb{F}_{2^4}$, giving a unique isomorphism class of 2-Frobenius groups of order $2^4 \cdot 5 \cdot 4$. Another way of looking at it is that the normalizer of a subgroup of order 5 in $\mathrm{GL}(4, 2)$ is the semilinear group, which is metacyclic with structure $\mathbb{Z}_{15} : \mathbb{Z}_4$, and this has the Frobenius group $\mathbb{Z}_5 : \mathbb{Z}_4$ as a subgroup. Reasoning exactly as before, we can show that there is a unique 2-Frobenius group of order $3^4 \cdot 5 \cdot 4$, and it has elementary abelian normal subgroup of order $3^4$. Now, taking the subdirect product of these gives a unique isomorphism class of 2-Frobenius groups of order $2^6 \cdot 3^4 \cdot 5$. This completes the proof. $\square$

Proof of Theorem 1.6: Let $G$ be a finite group satisfying

$$\mathrm{OC}(G) = \mathrm{OC}(U_5(2)) = \{2^{10} \cdot 3^5 \cdot 5, 11\}.$$

Clearly $|G| = |U_5(2)| = 2^{10} \cdot 3^5 \cdot 5 \cdot 11$ and $s(G) = 2$, in fact, we have $\pi_1(G) = \{2, 3, 5\}$ and $\pi_2(G) = \{11\}$. Then, by Theorem A in [36], one of the following statements holds:

(1) $G$ is a Frobenius group,
(2) $G$ is a 2-Frobenius group, or
(3) $G$ has a normal series $1 \trianglelefteq H \triangleleft K \trianglelefteq G$ such that $H$ is a nilpotent $\pi_1$-group, $K/H$ is a non-abelian simple group, $G/K$ is a $\pi_1$-group, and any odd order component of $G$ is equal to one of the odd order components of $K/H$.

If $G$ is a Frobenius group with kernel $K$ and complement $C$, then $\mathrm{OC}(G) = \{|K|, |C|\}$, and since $|C| < |K|$, the only possibility is $|K| = 2^{10} \cdot 3^5 \cdot 5$ and $|C| = 11$. However, this is a contradiction because $|C| \nmid |K| - 1$.

If $G$ is a 2-Frobenius group of order $2^{10} \cdot 3^5 \cdot 5 \cdot 11$, then, by the definition, $G = ABC$, where $A$ and $AB$ are normal subgroups of $G$ and $AB$ and $BC$ are Frobenius groups with kernels $A$ and $B$, respectively. Reasoning as in the proof of Theorem 1.3, we observe that there are unique 2-Frobenius groups $A_1BC$ and $A_2BC$ of orders $2^{10} \cdot 11 \cdot 5$ and $3^5 \cdot 11 \cdot 5$, respectively. Note that $A_1$ and $A_2$ are elementary abelian normal subgroups of orders $2^{10}$ and $3^5$, respectively. Therefore,

$G$ is a subdirect product $(A_1 \times A_2)BC = (2^{10} \times 3^5) : 11 : 5$ of $A_1BC$ and $A_2BC$. So there is a unique 2-Frobenius group $G = ABC$ of order $|U_5(2)| = 2^{10} \cdot 3^5 \cdot 5 \cdot 11$.

Finally, we suppose that $G$ satisfies condition (3). Then, by Table 2, $K/H$ is isomorphic to one of the simple groups $L_2(11)$, $M_{11}$, $M_{12}$, or $U_5(2)$. We see that, in general, $K/H \leq G/H \leq \mathrm{Aut}(K/H)$. Let $K/H \cong L_2(11)$. Since $|\mathrm{Aut}(K/H)| = 2^3 \cdot 3 \cdot 5 \cdot 11$ is not divisible by $3^2$, it follows that $3^4$ divides $|H|$. Let $P$ be a Sylow 3-subgroup of $H$ and let $Q$ be a Sylow 11-subgroup of $G$ . Then, $P$ is a normal subgroup of $G$, because $H$ is nilpotent. It now follows that $PQ$ is a subgroup of $G$ of order $3^4 \cdot 11$. Since all groups of order $3^4 \cdot 11$ are nilpotent, we conclude that 3 is adjacent to 11 in $\mathrm{GK}(G)$, which is a contradiction.

Reasoning exactly as above, we conclude that $K/H \ncong M_{11}, M_{12}$. Therefore, we deduce that $K/H \cong U_5(2)$, and since $|G| = |U_5(2)|$ it follows that $|H| = 1$ and $G = K \cong U_5(2)$. This completes the proof. $\square$

## 4. Some open problems

We conclude this article with some open problems. Actually, in this section, we restrict our attention to the relationship between degree patterns and prime graphs. A natural question is:

**Question 4.1.** Let $G$ and $M$ be two finite groups with $|G| = |M|$. Clearly $\mathrm{GK}(G) = \mathrm{GK}(M)$ implies $D(G) = D(M)$. Does the converse hold?

Assuming the converse is *true*, under these hypotheses we conclude that $\mathrm{OC}(G) = \mathrm{OC}(M)$, and so $h_{\mathrm{OD}}(M) \leq h_{\mathrm{OC}}(M)$. In particular, if $M$ is OC-characterizable, then $M$ is also OD-characterizable. In [15, Lemma 2.15] it was shown that if $G$ is a finite group with $\pi(G) = \pi(M)$ and $D(G) = D(M)$, where $M$ is an arbitrary alternating or symmetric group, then the prime graphs of $G$ and $M$ coincide. Therefore, we have the following consequence.

**Corollary 4.2.** *The symmetric and alternating groups which are OC-characterizable are also OD-characterizable.*

On the other hand, in view of the Main Theorem in [6], the symmetric groups $S_p$ and $S_{p+1}$, and the alternating groups $A_p$, $A_{p+1}$ and $A_{p+2}$, where $p \geq 3$ is a prime number, are OC-characterizable. Therefore, by Corollary 4.2, they are also OD-characterizable (see also [27, Theorem 1.5]). We notice that other alternating and symmetric groups are *not* OC-characterizable. In fact, for all alternating groups $A_n$ $(n \geq 5)$, except $A_p$, $A_{p+1}$ and $A_{p+2}$, where $p$ is a prime, the vertex 3 is adjacent to all other vertices in $\mathrm{GK}(A_n)$. Similarly, for all symmetric groups $S_n$ $(n \geq 5)$, except $S_p$ and $S_{p+1}$, where $p$ is a prime, the vertex 2 is adjacent to all other vertices in $\mathrm{GK}(S_n)$. Therefore, the prime graphs associated with these groups are connected. Assume now that $G$ is the alternating group (resp. the symmetric group) on $n \geq 5$ letters, except $A_p$, $A_{p+1}$, $A_{p+2}$ (resp. $S_p$, $S_{p+1}$) where $p$ is a prime. Let $H$ be a nilpotent group of order $|G|$ (for instance, consider a cyclic group of order $|G|$). Clearly, $\mathrm{GK}(H)$ is complete. Now, by the definition of order components, we have $\mathrm{OC}(H) = \mathrm{OC}(G) = \{|G|\}$, while $H$ is not isomorphic to $G$. But the situation of OD-characterizability of alternating and symmetric groups looks a little differently. As pointed out in the Introduction, there are infinitely many alternating groups $\mathbb{A}_n$ (resp. symmetric groups $\mathbb{S}_n$) which satisfy $h_{\mathrm{OD}}(\mathbb{A}_n) \geq 3$ (resp. $h_{\mathrm{OD}}(\mathbb{S}_n) \geq 3$), in particular, neither $h_{\mathrm{OD}}(\mathbb{A}_n)$ nor $h_{\mathrm{OD}}(\mathbb{S}_n)$ is bounded above (see [22]).

We now focus our attention on the sporadic simple groups. By Table 1 in [30], it is easy to see that if $G$ is a finite group with $\pi(G) = \pi(M)$ and $D(G) = D(M)$, where $M$ is a sporadic simple group, then the prime graphs of $G$ and $M$ coincide. Moreover, it is proved in [7] that all sporadic simple groups are OC-characterizable, hence we conclude that they are also OD-characterizable (see [30, Proposition 3.1.]).

Finally, we consider the OD-characterizability of simple groups of Lie type. Studies show that between simple groups of Lie type there are many simple orthogonal and symplectic groups which are 2-fold OD-characterizable (see [4]). Moreover, by Theorem 1.3, we have $h_{\mathrm{OD}}(U_4(2)) = 2$. So far we have not found a simple group of Lie type $S$ satisfying $h_{\mathrm{OD}}(S) > 2$. So it seems natural to ask the following question.

**Question 4.3.** Does there exist a finite simple group $S$ of Lie type such that $h_{\mathrm{OD}}(S) \geq 3$?

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Faculty of Mathematics, K.N. Toosi University of Technology, P.O. Box 16315–1618, Tehran, Iran

*E-mail:* moghadam@kntu.ac.ir
moghadam@ipm.ir

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