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Archivum Mathematicum, Vol. 52 (2016), No. 3, 141–158

Persistent URL: <http://dml.cz/dmlcz/145829>

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THE G -GRADED IDENTITIES OF THE GRASSMANN ALGEBRA

LUCIO CENTRONE

ABSTRACT. Let G be a finite abelian group with identity element 1_G and $L = \bigoplus_{g \in G} L^g$ be an infinite dimensional G -homogeneous vector space over a field of characteristic 0. Let $E = E(L)$ be the Grassmann algebra generated by L . It follows that E is a G -graded algebra. Let $|G|$ be odd, then we prove that in order to describe any ideal of G -graded identities of E it is sufficient to deal with G' -grading, where $|G'| \leq |G|$, $\dim_F L^{1_{G'}} = \infty$ and $\dim_F L^{g'} < \infty$ if $g' \neq 1_{G'}$. In the same spirit of the case $|G|$ odd, if $|G|$ is even it is sufficient to study only those G -gradings such that $\dim_F L^g = \infty$, where $o(g) = 2$, and all the other components are finite dimensional. We also compute graded cocharacters and codimensions of E in the case $\dim L^{1_G} = \infty$ and $\dim L^g < \infty$ if $g \neq 1_G$.

1. INTRODUCTION

All algebras we refer to are to be considered associative and unitary over a field of characteristic 0 unless explicitly written. Let F be a field and $X = \{x_1, x_2, \dots\}$ be a countable infinite set of variables and let $F\langle X \rangle$ be the free associative algebra freely generated by X . If A is an F -algebra, we say that $f(x_1, \dots, x_n) \in F\langle X \rangle$ is a *polynomial identity* of A if $f(a_1, \dots, a_n) = 0$ for any $a_1, \dots, a_n \in A$. If A has a non-trivial polynomial identity we say that A is a *polynomial identity algebra* or *PI-algebra* and we denote by $T(A)$ the set of all polynomial identities satisfied by A . It is well known that $T(A)$ is an ideal of $F\langle X \rangle$ invariant under all endomorphisms of $F\langle X \rangle$, i.e., it is a T -ideal called the T -ideal of A . We say that the *variety* generated by the algebra A is the class

$$\mathcal{V} = \mathcal{V}(A) = \{B \text{ associative algebra} \mid T(A) \subseteq T(B)\}.$$

The Grassmann algebra E , generated by an infinite dimensional vector space and its identities, plays an important role in the structure theory of Kemer on varieties of associative algebras with polynomial identities [11, 10]. More precisely, Kemer proved that any associative PI-algebra over a field F of characteristic zero satisfies the same identities (is *PI-equivalent*) of the Grassmann envelope of a finite

2010 *Mathematics Subject Classification*: primary 16R10; secondary 16P90, 16S10, 16W50.

Key words and phrases: graded polynomial identities.

Partially supported by FAPESP grant 2013/06752-4 and 2015/08961-5.

Received August 18, 2015, revised June 2016. Editor J. Trlifaj.

DOI: 10.5817/AM2016-3-141

dimensional associative superalgebra, i.e., they have the same T -ideal. Moreover, the matrix algebras $M_n(F)$, $M_n(E)$ with entries in E and its subalgebras $M_{p,q}(E)$ ($p + q = n$), generate the only non-trivial prime varieties.

Here $[a, b] = ab - ba$, and $[a, b, c] = [[a, b], c]$ for every $a, b, c \in F\langle X \rangle$. In [13] Latyshev proved that the T -ideal generated by the triple commutator $[x_1, x_2, x_3]$ is Spechtian, i.e., every proper subvariety of the variety generated by $[x_1, x_2, x_3]$ is finitely based. In [12] Krakowski and Regev proved that the polynomial $[x_1, x_2, x_3]$ forms a basis of the polynomial identities of E . Moreover, they found the codimension sequence of E . Later, Olsson and Regev determined the cocharacter sequence of E (see [14]). In [4] Di Vincenzo gave a different proof of the result of Krakowski and Regev and he also exhibited, for any k , finite bases of the identities of the Grassmann algebra generated by a k -dimensional vector space.

In light of this it seems a natural and interesting problem to investigate more closely the structure of the graded polynomial identities of the Grassmann algebra. For example, the structure of the \mathbb{Z}_2 -graded identities of E with respect to its natural \mathbb{Z}_2 -grading is well known, see for instance [9]. Recently, Di Vincenzo and da Silva gave in [6] a complete description of the \mathbb{Z}_2 -graded polynomial identities of E with respect to any \mathbb{Z}_2 -grading such that the generating space is \mathbb{Z}_2 -homogeneous. This work has been generalized by the author for any infinite field of characteristic $p > 2$ (see [2]). In [1] Anisimov constructed an algorithm to compute the exact value of the graded codimension of E for any \mathbb{Z}_p -grading of E , where p is a prime number.

In this paper we consider a finite abelian group G with identity element 1_G and an infinite dimensional G -homogeneous vector space L over the field F which generates the infinite dimensional Grassmann algebra $E = E(L)$. The latter inherits the structure of a G -graded algebra, hence we are allowed to study its G -graded identities. Let $|G|$ be odd, then we prove that in order to describe any ideal of G -graded identities of E it is sufficient to deal with a G' -grading, where $|G'| \leq |G|$, $\dim_F L^{1_{G'}} = \infty$ and $\dim_F L^{g'} < \infty$ if $g' \neq 1_{G'}$. In the same spirit of the case $|G|$ odd, if $|G|$ is even it is sufficient to study only those G -gradings such that $\dim_F L^g = \infty$ and $o(g) = 2$, where $o(g)$ stands for the order of the group element g . Finally we give a complete description of $T_G(E)$, where $G = \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4$ in some particular cases. We also compute graded cocharacters and codimensions of E in the case $\dim L^{1_G} = \infty$ and $\dim L^g < \infty$ if $g \neq 1_G$.

2. FREE ALGEBRAS, GRADED PI-ALGEBRAS

We introduce the key tools for the study of graded polynomial identities. We start off with the following definition.

Definition 2.1. Let G be a group and A be an algebra over a field F . We say that the algebra A is G -graded if $A = \bigoplus_{g \in G} A^g$ and for all $g, h \in G$, one has $A^g A^h \subseteq A^{gh}$.

It is easy to note that if a is any element of A it can be uniquely written as a finite sum $a = \sum_{g \in G} a_g$, where $a_g \in A^g$. We shall call the subspaces A^g

the G -homogeneous components of A . Accordingly, an element $a \in A$ is called G -homogeneous if exists $g \in G$ such that $a \in A^g$. If $B \subseteq A$ is a subspace of A , B is G -graded if and only if $B = \bigoplus_{g \in G} (B \cap A^g)$. Analogously one can define G -graded algebras, subalgebras, ideals, etc.

Let $\{X^g \mid g \in G\}$ be a family of disjoint countable sets of indeterminates. Set $X = \bigcup_{g \in G} X^g$ and denote by $F\langle X \mid G \rangle$ the free associative algebra freely generated by X . An indeterminate $x \in X$ is said to be of homogeneous G -degree g , written $\|x\| = g$, if $x \in X^g$. We always write x^g if $x \in X^g$. The homogeneous G -degree of a monomial $m = x_{i_1}x_{i_2} \dots x_{i_k}$ is defined to be $\|m\| = \|x_{i_1}\| \cdot \|x_{i_2}\| \cdot \dots \cdot \|x_{i_k}\|$. For every $g \in G$, denote by $F\langle X \mid G \rangle^g$ the subspace of $F\langle X \mid G \rangle$ spanned by all monomials having homogeneous G -degree g . Notice that $F\langle X \mid G \rangle^g F\langle X \mid G \rangle^{g'} \subseteq F\langle X \mid G \rangle^{gg'}$ for all $g, g' \in G$. Thus

$$F\langle X \mid G \rangle = \bigoplus_{g \in G} F\langle X \mid G \rangle^g$$

is a G -graded algebra. The elements of the G -graded algebra $F\langle X \mid G \rangle$ are called G -graded polynomials or, simply, graded polynomials.

Definition 2.2. If A is a G -graded algebra, a G -graded polynomial

$$f(x_1, \dots, x_n)$$

is said to be a graded polynomial identity of A if

$$f(a_1, a_2, \dots, a_n) = 0$$

for all $a_1, a_2, \dots, a_n \in \bigcup_{g \in G} A^g$ such that $a_k \in A^{\|x_k\|}$, $k = 1, \dots, n$. We shall write $f \equiv 0$ in order to say that f is a graded polynomial identity for A .

Given an algebra A graded by a group G , we define

$$T_G(A) := \{f \in F\langle X \mid G \rangle \mid f \equiv 0 \text{ on } A\},$$

the set of G -graded polynomial identities of A .

Definition 2.3. An ideal I of $F\langle X \mid G \rangle$ is said to be a T_G -ideal if it is invariant under all F -endomorphisms $\varphi: F\langle X \mid G \rangle \rightarrow F\langle X \mid G \rangle$ such that $\varphi(F\langle X \mid G \rangle^g) \subseteq F\langle X \mid G \rangle^g$ for all $g \in G$.

Hence $T_G(A)$ is a T_G -ideal of $F\langle X \mid G \rangle$. On the other hand, it is easy to check that all T_G -ideals of $F\langle X \mid G \rangle$ are of this type. We shall denote by $\langle S \rangle^{T_G}$ the T_G -ideal generated by the set S , i.e., the smallest T_G -ideal containing S . In this case we say S is a basis for $\langle S \rangle^{T_G}$ or the elements of $\langle S \rangle^{T_G}$ follow from those of S .

From now on all the groups are assumed to be finite abelian. The theory of G -graded PI-algebras passes through the representation theory of the symmetric group. More precisely we study the following spaces.

Definition 2.4. Let

$$P_n^G = \text{span} \langle x_{\sigma(1)}^{g_1} x_{\sigma(2)}^{g_2} \dots x_{\sigma(n)}^{g_n} \mid g_i \in G, \sigma \in S_n \rangle,$$

then the elements in P_n^G are called multilinear polynomials of degree n of $F\langle X \mid G \rangle$.

It turns out that P_n^G is a left S_n -module under the natural left action of the symmetric group S_n . As a consequence the factor module $P_n^G(A) := P_n^G / (P_n^G \cap T_G(A))$ is an S_n -module, too. We observe that $P_n^G(A)$ affords a representation of the symmetric group S_n which naturally carries on a character of S_n (or S_n -character). Let us denote the S_n -character of $P_n^G(A)$ by $\chi_n^G(A)$, and by $c_n^G(A)$ its dimension over F . We say that

$(\chi_n^G(A))_{n \in \mathbb{N}}$ is the G -graded cocharacter sequence of A

$(c_n^G(A))_{n \in \mathbb{N}}$ is the G -graded codimension sequence of A .

Now, for $l_{g_1}, \dots, l_{g_r} \in \mathbb{N}$ let us consider the blended components of the multilinear polynomials in the indeterminates labeled as follows: $x_{l_{g_1}}^{g_1}, \dots, x_{l_{g_1}}^{g_1}$, then $x_{l_{g_1}+1}^{g_2}, \dots, x_{l_{g_1}+l_{g_2}}^{g_2}$ and so on. We denote this linear space by $P_{l_{g_1}, \dots, l_{g_r}}^G$. Of course, this is a left $S_{l_{g_1}} \times \dots \times S_{l_{g_r}}$ -module. We shall denote by $\chi_{l_{g_1}, \dots, l_{g_r}}^G(A)$ the character of the module $P_{l_{g_1}, \dots, l_{g_r}}^G(A) / (P_{l_{g_1}, \dots, l_{g_r}}^G(A) \cap T_G(A))$ and by $c_{l_{g_1}, \dots, l_{g_r}}^G(A)$ its dimension.

Since the ground field F is infinite, a standard *Vandermonde-argument* yields that a polynomial f is a G -graded polynomial identity for A if and only if its homogeneous components (with respect to the ordinary \mathbb{N} -grading), are identities as well. Moreover, since $\text{char}(F) = 0$, the well known multilinearization process shows that the T_G -ideal of a G -graded algebra A is determined by its multilinear polynomials, i.e. by the various $P_{l_{g_1}, \dots, l_{g_r}}^G(A)$. We remark that, given the cocharacter $\chi_{l_{g_1}, \dots, l_{g_r}}^G(A)$, the graded cocharacter $\chi_n^G(A)$ is known as well. More precisely, the following is due to Di Vincenzo (see [5, Theorem 2]).

Proposition 2.5. *Let A be a G -graded algebra with graded cocharacter sequences $\chi_{l_{g_1}, \dots, l_{g_r}}^G(A)$. Then*

$$\chi_n^G(A) = \sum_{\substack{(l_{g_1}, \dots, l_{g_r}) \\ l_{g_1} + \dots + l_{g_r} = n}} \chi_{l_{g_1}, \dots, l_{g_r}}^G(A) \uparrow^{S_n},$$

where $\chi_{l_{g_1}, \dots, l_{g_r}}^G(A) \uparrow^{S_n}$ stands for the induced S_n -character of the $S_{l_{g_1}} \times \dots \times S_{l_{g_r}}$ -module $P_{l_{g_1}, \dots, l_{g_r}}^G(A)$.

Moreover

$$c_n^G(A) = \sum_{\substack{(l_{g_1}, \dots, l_{g_r}) \\ l_{g_1} + \dots + l_{g_r} = n}} \binom{n}{l_{g_1}, \dots, l_{g_r}} c_{l_{g_1}, \dots, l_{g_r}}^G(A).$$

Let us consider the free algebra $F\langle Y \cup Z \rangle$ (where Y is the set of all indeterminates of G -degree 1_G and Z is the set of all the remaining indeterminates). The Y -proper polynomials (see [7, Section 2]) are the elements of the unitary F -subalgebra B of $F\langle X \rangle$ generated by the elements of Z and by all non-trivial commutators. More precisely, a polynomial $f \in F\langle Y \cup Z \rangle$ is Y -proper if all the $y \in Y$ occurring in f

appear in commutators only. Notice that if $f \in F\langle Z \rangle$, then f is Y -proper. It is well known (see, for instance, Lemma 1 Section 2 in [7]) that all the graded polynomial identities of a superalgebra A follow from the Y -proper ones. This means that the set $T_{\mathbb{Z}_2}(A) \cap B$ generates the whole $T_{\mathbb{Z}_2}(A)$ as a $T_{\mathbb{Z}_2}$ -ideal. Similarly, for any finite abelian group G , all the G -graded polynomial identities of a G -graded algebra A follow from the Y -proper ones. This means that the set $T_G(A) \cap B$ generates the whole $T_G(A)$ as a T_G -ideal. Let us define $B(A) := B/(T_G(A) \cap B)$. We shall refer to $B(A)$ as Y -proper relatively-free algebra of A .

We shall denote by Γ_n^G the set of multilinear Y -proper polynomials of P_n^G . It is not difficult to see that Γ_n^G is a left S_n -submodule of P_n^G and the same holds for $\Gamma_n^G \cap T_G(A)$. Hence the factor module

$$\Gamma_n^G(A) := \Gamma_n^G / (\Gamma_n^G \cap T_G(A))$$

is an S_n -submodule of $P_n^G(A)$. We denote the S_n -character of the factor module $\Gamma_n^G / (\Gamma_n^G \cap T_G(A))$ by $\xi_n^G(A)$, and by $\gamma_n^G(A)$ its dimension over F . We say:

$(\xi_n^G(A))_{n \in \mathbb{N}}$ is the G -graded proper cocharacter sequence of A ;

$(\gamma_n^G(A))_{n \in \mathbb{N}}$ is the G -graded proper codimension sequence of A .

We shall denote by $\Gamma_{m_1, \dots, m_r}^G$ the set of multilinear Y -proper polynomials of P_{m_1, \dots, m_r}^G . We observe that $\Gamma_{m_1, \dots, m_r}^G$ is a left $S_{m_1} \times \dots \times S_{m_r}$ -submodule of P_{m_1, \dots, m_r}^G and the same holds for $\Gamma_{m_1, \dots, m_r}^G \cap T_G(A)$. Hence the factor module

$$\Gamma_{m_1, \dots, m_r}^G(A) := \Gamma_{m_1, \dots, m_r}^G / (\Gamma_{m_1, \dots, m_r}^G \cap T_G(A))$$

is an $S_{m_1} \times \dots \times S_{m_r}$ -submodule of $P_{m_1, \dots, m_r}^G(A)$. We denote the $S_{m_1} \times \dots \times S_{m_r}$ -character of the factor module $\Gamma_{m_1, \dots, m_r}^G / (\Gamma_{m_1, \dots, m_r}^G \cap T_G(A))$ by $\xi_{m_1, \dots, m_r}^G(A)$, and by $\gamma_{m_1, \dots, m_r}(A)$ its dimension over F . When we refer to A without any ambiguity, we shall use γ_{m_1, \dots, m_r} instead of $\gamma_{m_1, \dots, m_r}(A)$.

Let L be an infinite dimensional vector space over F , a field of characteristic zero, then we indicate by $E = E(L)$ the Grassmann algebra generated by L . Let (G, \cdot) be a finite abelian group and suppose E is G -graded. In this section we want to study the G -graded identities of E in the case when L is a G -homogeneous space.

Let $B_L = \{e_1, e_2, \dots\}$ be a linear basis of L , where for any $i \in \mathbb{N}$, e_i is a G -homogeneous element, so $B_E = \{e_{i_1} e_{i_2} \dots e_{i_n} \mid n \in \mathbb{N}, i_1 < i_2 < \dots < i_n\}$ is a basis of E as a vector space over F . Notice that the existence of a homogeneous G -grading is equivalent to the existence of a map

$$\| \cdot \| : B_L \rightarrow G.$$

We have that the G -degree of the element $e_{i_1} e_{i_2} \dots e_{i_n}$ is

$$\|e_{i_1} e_{i_2} \dots e_{i_n}\| = \|e_{i_1}\| \|e_{i_2}\| \dots \|e_{i_n}\|.$$

In this case we say that the set

$$\{e_{i_1}, e_{i_2}, \dots, e_{i_n}\}$$

is the *support* of $e_{i_1} e_{i_2} \dots e_{i_n}$ and the non-negative integer n is its *length*.

For our purposes we shall pass from a fixed G -grading of E to the grading associated to the quotient group G/H , for some special subgroups H of G . More generally, let $A = \bigoplus_{g \in G} A^g$ be a G -graded algebra and let $H < G$, then for every coset $gH \in G/H$, we define $A^{gH} = \bigoplus_{f \in gH} A^f$. In particular, if T is a transversal set for H in G , then

$$A = \bigoplus_{t \in T} A^{tH}.$$

We observe that for every $g, g' \in G$ $A^{gH} A^{g'H} \subseteq A^{gg'H}$, so A inherits a structure of G/H -graded algebra and we shall call it *quotient grading of A* .

3. GRADED IDENTITIES OF E

In what follows we shall denote by $Z(A)$ the center of the algebra A . Recall that in the case E is the infinite dimensional Grassmann algebra, then $Z(E) = E^0$, where E^0 is the \mathbb{Z}_2 -component of degree 0 in the canonical \mathbb{Z}_2 -grading of E . In what follows we shall use the following notation: if H is a normal subgroup of G and no confusion occurs, we shall denote by \bar{g} the coset gH .

In order to investigate the relations between the graded identities of E with respect to G -gradings and to its quotient G/H -gradings, it is reasonable to consider the following homomorphism between free graded algebras

$$\pi: F\langle X \mid G \rangle \rightarrow F\langle Y \mid G/H \rangle,$$

where Y is an infinite set of G/H -graded variables, such that for every $g \in G$ and for every $i \in \mathbb{N}$, $\pi(x_i^g) = y_i^{gH}$. For any G -graded algebra A and for any subgroup H of G we have the next result.

Lemma 3.1. *Let $f(x_1, \dots, x_n) \in F\langle X \mid G \rangle$ be a multilinear polynomial. If $\pi(f) \in T_{G/H}(A)$, then $f \in T_G(A)$.*

Proof. Let $\varphi: x_i \mapsto a_i$ be a G -graded substitution, so $\|a_i\| = \|x_i\| = g_i \in G$ for some $g_i \in G$. Now we have that $a_i \in A^{g_i}$ and a_i is homogeneous of degree g_iH in the quotient grading. Then φ is a G/H -graded substitution too. Due to the fact that $\pi(f) = f(y_1, \dots, y_n) \in T_{G/H}(A)$, we have $0 = f(a_1, \dots, a_n)$ and $f \in T_G(A)$. □

Under opportune hypothesis, it is possible to invert this result. Above all, we give the following definition.

Definition 3.2. Let G be a finite abelian group and suppose E is G -graded. We say that the subgroup H of G has the *property \mathcal{P}* if for any $h \in H$, E^h has infinite elements of even length with pairwise disjoint support.

The interest of this property is given by the following proposition.

Proposition 3.3. *Let $H < G$ having the property \mathcal{P} and let $f \in F\langle X \mid G \rangle$ be a multilinear polynomial. Then $f \in T_G(E)$ if and only if $\pi(f) \in T_{G/H}(E)$.*

Proof. We have to prove just the only if part. Let

$$f = f(x_1^{g_1}, x_2^{g_1}, \dots, x_{l_{g_1}}^{g_1}, \dots, x_{\sum_{i=1}^{r-1} l_{g_i} + 1}^{g_r}, \dots, x_{\sum_{i=1}^r l_{g_i}}^{g_r}) \in T_G(E)$$

and let $F = \pi(f)$. Let φ be any G/H -graded substitution, hence $\varphi(y_j^{gH}) = \sum_{h \in H} a_j^{gh}$, and by the multilinearity of f , we can consider only substitutions φ such that $y_j^{gH} \mapsto a_j^{gh}$, for some $h \in H$ and for any j . Now observe that if every homogeneous component E^h has infinite elements of even length, then for every j and for every $h \in H$ there exists $b_j^{h^{-1}}$ of even length such that $\|b_j^{h^{-1}}\| = h^{-1}$. For every $h \in H$, $w_j^g = a_j^{gh} b_j^{h^{-1}}$ is a homogeneous element of degree g in the G -grading of E . Let us consider a new substitution ψ such that $x_j^g \mapsto w_j^g$. This is a G -graded substitution. Now, since $f \in T_G(E)$, $0 = f(w_1^{g_1}, \dots, w_{l_{g_r}}^{g_r}) = \prod_{h \in H, j} b_j^{h^{-1}} \cdot F(a_j^{gh})$ because the $b_j^{h^{-1}}$'s are in $Z(E)$ and this implies $F(a_j^{gh}) = 0$. \square

We consider now some subgroups of G having the property \mathcal{P} .

Lemma 3.4. *Let $H = \langle g \rangle$ for some $g \in G$. If L^g is infinite dimensional and $|H| = n$ is odd, then H has the property \mathcal{P} .*

Proof. Let $\{v_1, v_2, \dots\}$ be a linear basis of L^g and let $h = g^t$ for some $t \in \mathbb{N}$. Notice that

$$\|v_{i_1} \dots v_{i_l}\| = g^l,$$

hence $v_{i_1} \dots v_{i_l} \in E^h$ if and only if $g^l = g^t$ that is if and only if $l \equiv t \pmod n$. Now, if t is even, then the elements of E^h , $v_1 \dots v_t, v_{t+1} \dots v_{2t}, \dots, v_{kt+1} \dots v_{(k+1)t}, \dots$ have pairwise disjoint supports of even length. Similarly, if t is odd, an infinite subset of elements of E^h having pairwise disjoint supports is given by $v_1 \dots v_{t+n}, v_{t+n+1} \dots v_{2(t+n)}, \dots, v_{k(t+n)+1} \dots v_{(k+1)(t+n)}, \dots$ and we are done. \square

Proposition 3.5. *Let G be a finite abelian group and*

$$H = \langle g \mid \dim_F L^g = \infty \text{ and } o(g) \text{ is odd} \rangle.$$

Then H has the property \mathcal{P} .

Proof. For any $h \in H$ there exist distinct elements $b_1, \dots, b_s \in H$ such that $h = b_1^{t_1} \dots b_s^{t_s}$ for some positive integers t_1, \dots, t_s . Then by Lemma 3.4 and its proof, for every $i = 1, \dots, s$, $E^{b_i^{t_i}}$ has infinite elements

$$w_1^i, w_2^i, \dots, w_m^i, \dots$$

of even length with pairwise disjoint supports, moreover these elements belong to the Grassmann algebra E_i generated by the subspace L^{b_i} . We set

$$u_m = w_m^1 w_m^2 \dots w_m^s,$$

for $m \geq 1$ and clearly $\{u_1, u_2, \dots, u_m, \dots\}$ is the required subset of E^h . \square

As a consequence of Propositions 3.5 and 3.3, we have the following.

Theorem 3.6. *Let G be a finite abelian group of odd order and let*

$$H = \langle g \mid \dim_F L^g = \infty \rangle.$$

Then the following properties hold:

(1) *for any multilinear polynomial $f(x_1, \dots, x_n) \in F\langle X \rangle$ one has*

$$f \in T_G(E) \quad \text{if and only if} \quad \pi(f) \in T_{G/H}(E).$$

(2) *In the quotient grading of E , $L^{\bar{g}}$ is infinite dimensional if and only if $\bar{g} = 1_{G/H}$.*

If G is any finite abelian group, we have the following result.

Proposition 3.7. *Let G be a finite abelian group and $g \in G$ such that $\dim_F L^g = \infty$. Let $H = \langle g \rangle$ and $|H| = n$, an even number. Then $K = \langle g^2 \rangle$ has the property \mathcal{P} .*

Proof. Let $\{e_1, e_2, \dots\}$ be a linear basis of L^g and let $k = g^{2t} \in K$, then the elements of E^k $e_1 \dots e_{2t}, e_{2t+1} \dots e_{4t}, \dots, e_{s(2t)+1} \dots e_{(s+1)(2t)}, \dots$ have pairwise disjoint supports of even length. \square

Now let us consider the following subsets of G :

$$\mathcal{I} = \{g \in G \mid \dim_F L^g = \infty\},$$

$$\mathcal{I}_1 = \{g \in \mathcal{I} \mid o(g) \text{ is odd}\},$$

$$\mathcal{I}_2 = \mathcal{I} - \mathcal{I}_1 \quad \text{and}$$

$$\mathcal{I}_3 = \{g^2 \mid g \in \mathcal{I}_2\} - \mathcal{I}_1.$$

We have the following.

Theorem 3.8. *Let G be a finite abelian group and let $H = \langle g \mid g \in \mathcal{I}_1 \cup \mathcal{I}_3 \rangle$. Then the following properties hold:*

(1) *for any multilinear polynomial $f = f(x_1, \dots, x_n) \in F\langle X \rangle$ one has*

$$f \in T_G(E) \quad \text{if and only if} \quad \pi(f) \in T_{G/H}(E).$$

(2) *In the quotient grading of E , if $L^{\bar{g}}$ is infinite dimensional, then $\bar{g}^2 = \bar{1} \in G/H$.*

Proof. (1) Let $h \in H$, then there exist $a_1, \dots, a_r \in \mathcal{I}_1, b_1, \dots, b_s \in \mathcal{I}_3$ and positive integers be such that $h = a_1^{m_1} \dots a_r^{m_r} b_1^{m_{r+1}} \dots b_s^{m_{r+s}}$. Let $a_{r+1}, \dots, a_{r+s} \in \mathcal{I}_2$ such that $b_i = a_{r+i}^2$, then $\dim_F L^{a_i} = \infty$ for any $i = 1, \dots, r+s$. Let us denote by E_i the Grassmann algebra generated by the subspace $L^{a_i^{m_i}}$. As in the proof of Proposition 3.5, for any $i = 1, \dots, r+s$, E_i contains infinitely many elements

$$w_1^i, w_2^i, \dots, w_m^i, \dots$$

of even length with pairwise disjoint supports. Moreover, for all $m \geq 1$ we have that $\|w_m^i\| = a_i^{m_i}$ if $i = 1, \dots, r$ and $\|w_m^i\| = b_i^{m_i}$ for $i = r+1, \dots, r+s$. We consider in E^h the elements $u_m = w_m^1 \dots w_m^{r+s}$, $m \geq 1$; clearly the elements $\{u_m \mid m \geq 1\}$ have pairwise disjoint supports and they have even length. Now H has the property \mathcal{P} and the assertion comes by Proposition 3.3.

(2) Let $\bar{g} = gH \in G/H$ be such that $L^{\bar{g}} = \bigoplus_{h \in H} L^{g^h}$ is infinite dimensional. Since G is finite there exists $g' \in gH$ such that $L^{g'}$ is infinite dimensional. If $o(g')$ is odd, then $g' \in H$ and so $gH = g'H = 1_{G/H}$. If $o(g')$ is even, then $g'^2 \in H$ and so $(gH)^2 = (g'H)^2 = 1_{G/H}$. \square

4. GRADED CODIMENSIONS AND COCHARACTERS OF E

We shall study graded codimensions and graded cocharacters for E in the case $\dim_F L^{1_G}$ is infinite and all the other homogeneous components of L have finite dimension. We shall use the language of the representation theory of symmetric groups (see the book [15] by Sagan for more details).

Theorem 4.1. *Let $G = \{g_1, \dots, g_r\}$ be a finite abelian group with $g_1 = 1_G$. Suppose that L^{g_1} has infinite dimension. Let*

$$l_{g_1}, l_{g_2}, \dots, l_{g_r} \in \mathbb{N}$$

such that

$$l_{g_1} + l_{g_2} + \dots + l_{g_r} = m.$$

Then $P_{l_{g_1}, \dots, l_{g_r}} \subseteq T_G(E)$ or for any $f \in P_{l_{g_1}, l_{g_2}, \dots, l_{g_r}}$ one has

$$f(x_1^{g_1}, \dots, x_{l_{g_1}}^{g_1}, \dots, x_{\sum_{i=1}^{r-1} l_{g_i} + 1}^{g_r}, \dots, x_{\sum_{i=1}^r l_{g_i}}^{g_r}) \in T_G(E)$$

if and only if $f(x_1, \dots, x_m) \in T(E)$.

Proof. It is sufficient to prove that if $P_{l_{g_1}, \dots, l_{g_r}} \not\subseteq T_G(E)$, then any element of $P_{l_{g_1}, \dots, l_{g_r}} \cap T_G(E)$ is an ordinary polynomial identity for E . Let us suppose

$$P_{l_{g_1}, \dots, l_{g_r}} \not\subseteq T_G(E),$$

then there exists a graded monomial with a non-zero graded evaluation of elements a_1, \dots, a_m of the basis B_L of E . Any other monomial of $P_{l_{g_1}, \dots, l_{g_r}}$ is non-zero with respect to the same evaluation. Since L^{g_1} is infinite dimensional, we can always suppose a_1, \dots, a_m are of even length multiplying them by some e_i 's of degree g_1 . Now let us consider the elements of the basis of L^{g_1} which are not involved in the expression of the given elements a_1, \dots, a_m , to say v_i 's. Clearly, the latter generate an infinite dimensional Grassmann algebra E' , hence $T(E) = T(E')$. Let $f = f(x_1^{g_1}, \dots, x_{l_{g_1}}^{g_1}, \dots, x_{\sum_{i=1}^{r-1} l_{g_i} + 1}^{g_r}, \dots, x_{\sum_{i=1}^r l_{g_i}}^{g_r}) \in T_G(E)$ and let φ be any substitution such that $x_i \mapsto v_i \in E'$ for any i . Let us consider a new substitution ψ such that $x_i^{g_j} \mapsto v_i a_i$. This is a G -graded substitution on E . Now, since $f \in T_G(E)$, $0 = f(v_1 a_1, \dots, v_m a_m) = a_1 \cdots a_m f(v_1, \dots, v_m)$ because the a_i 's are in $Z(E)$ and this implies $f(v_1, \dots, v_m) = 0$ because the supports of v_1, \dots, v_m are distinct from those of a_1, \dots, a_m and $a_1 \cdots a_m \neq 0$ by hypothesis, then $f \in T(E') = T(E)$ and we are done. \square

Theorem 4.2. *Let $G = \{g_1, \dots, g_r\}$ be a finite abelian group with $g_1 = 1_G$. Let L be a G -homogeneous vector space over L such that $\dim_F L^{g_1} = \infty$ and $\dim_F L^{g_i} = k_i < \infty$, if $i \neq 1$. If $E = E(L)$ is the Grassmann algebra generated by L , then $T_G(E)$ is generated as a T_G -ideal by the following polynomials:*

- (1) $[u_1, u_2, u_3]$ for any choice of the G -degree of the variables u_1, u_2, u_3 ,
- (2) monomials of P_{0,t_2,\dots,t_r} such that $\sum_{i=2}^r t_i = 1 + \sum_{i=2}^r k_i$,
- (3) monomials of P_{0,t_2,\dots,t_r} such that $\sum_{i=2}^r t_i < 1 + \sum_{i=2}^r k_i$ and $P_{0,t_2,\dots,t_r} \subseteq T_G(E)$.

Proof. In light of Theorem 4.1, we have that $T_G(E)$ is generated by the polynomials from (1) of the claim and by all monomials of P_{l_1,\dots,l_r} such that $P_{l_1,\dots,l_r} \subseteq T_G(E)$. Notice that a graded monomial w is surely a graded polynomial identity when the sum of the numbers of its indeterminates of G -degree different from g_1 is strictly greater than $\sum_{i=2}^r k_i$. Moreover, for any $l_1, \dots, l_r \in \mathbb{N}$, any monomial in P_{l_1,\dots,l_r} is in the T_G -ideal generated by the monomials in P_{0,l_2,\dots,l_r} . Now we have just to observe that the monomials in P_{0,l_2,\dots,l_r} follow from the monomials in $P_{0,l_2-1,\dots,l_r}, \dots, P_{0,l_2,\dots,l_i-1,\dots,l_r}$ for $l_i \geq 1$ due to the Young rule. Hence if $l_2, \dots, l_r \in \mathbb{N}$ are such that $l_2 + \dots + l_r > 1 + \sum_{i=2}^r k_i$, then P_{0,l_2,\dots,l_r} is in the T_G -ideal generated by the monomials of P_{0,t_2,\dots,t_r} such that $t_2 + \dots + t_r = 1 + \sum_{i=2}^r k_i$ and the claim follows. \square

We have the following corollary which proof repeats verbatim the one of Proposition 5 of [6].

Corollary 4.3. *Let $G = \{g_1, \dots, g_r\}$ be a finite abelian group with $g_1 = 1_G$. If L^{g_1} has infinite dimension and $l_{g_1}, l_{g_2}, \dots, l_{g_r} \in \mathbb{N}$ are such that $l_{g_1} + l_{g_2} + \dots + l_{g_r} = m$, then*

$$c_{l_{g_1}, \dots, l_{g_r}}(E) = 0 \quad \text{or} \quad c_{l_{g_1}, \dots, l_{g_r}}(E) = 2^{m-1}$$

and in the latter case, $P_{l_{g_1}, \dots, l_{g_r}}(E)$ and $P_m(E)$ are isomorphic $S_{l_{g_1}} \times \dots \times S_{l_{g_r}}$ -modules.

If G is a finite abelian group and L is a vector space with basis $B_L = \{e_1, e_2, \dots\}$, let

$$\varphi: B_L \rightarrow G$$

be any map. As we said before, φ induces a G -grading on E . Let us consider now the set

$$S(\varphi) = \{(l_{g_1}, l_{g_2}, \dots, l_{g_r}) \in \mathbb{N}^r \mid P_{l_{g_1}, l_{g_2}, \dots, l_{g_r}} \subseteq T_G(E)\}.$$

We note that if L^{1_G} is the only homogeneous subspace of L such that $\dim_F L^{1_G} = \infty$, then $S(\varphi) \neq \emptyset$.

$S(\varphi)$ allows us to give the complete description of the sequence of the graded cocharacters and codimensions of E . In fact, we have the following proposition.

Proposition 4.4. *Let $G = \{g_1, \dots, g_r\}$ be a finite abelian group and L be a G -homogeneous vector space with linear basis $\{e_1, e_2, \dots\}$. Let $\varphi: B_L \rightarrow G$ be a map such that $|\varphi^{-1}(1_G)| = \infty$ and consider E , the G -graded Grassmann algebra obtained by φ . Then*

$$\chi_{l_{g_1}, \dots, l_{g_r}}^G(E) = 2^{|G|-1} \sum_{a_1=0}^{l_{g_1}-1} \sum_{a_2=0}^{l_{g_2}-1} \dots \sum_{a_r=0}^{l_{g_r}-1} \lambda_{a_1} \otimes \lambda_{a_2} \otimes \dots \otimes \lambda_{a_r}$$

if $(l_{g_1}, \dots, l_{g_r}) \notin S(\varphi)$, where λ_{a_i} is the hook partition of leg a_i and arm $l_{g_i} - a_i + 1$.

Moreover

$$c_n^G(E) = 2^{n-1} \sum_{\substack{(l_{g_1}, \dots, l_{g_r}) \notin S(\varphi) \\ l_{g_1} + \dots + l_{g_r} = n}} \binom{n}{l_{g_1}, \dots, l_{g_r}}.$$

Proof. By Corollary 4.3, the spaces $P_n(E)$ and $P_{l_{g_1}, \dots, l_{g_r}}^G(E)$ are $S_{l_{g_1}} \times \dots \times S_{l_{g_r}}$ -isomorphic modules. Hence the result follows using the decomposition of $\chi_n(E) = \sum_{i=0}^{n-1} (n-i, 1^i)$ and the representation theory of symmetric groups. More precisely, it follows by Branching Rule that when we restrict the irreducible representation $\nu_i = (n-i, 1^i)$ of S_n to its subgroup $S_{l_{g_1}} \times \dots \times S_{l_{g_r}}$ then its $S_{l_{g_1}} \times \dots \times S_{l_{g_r}}$ -irreducible components are $\lambda_{a_1} \otimes \lambda_{a_2} \otimes \dots \otimes \lambda_{a_r}$ for some $\lambda_{a_i} = (l_{g_i} - a_i, 1^{a_i})$. By Frobenius Reciprocity Law the multiplicity of $\lambda_{a_1} \otimes \lambda_{a_2} \otimes \dots \otimes \lambda_{a_r}$ in the decomposition of ν_i equals the multiplicity of ν_i in the induced representation $(\lambda_{a_1} \otimes \lambda_{a_2} \otimes \dots \otimes \lambda_{a_r})^{\uparrow S_n}$. We argue only for $r = 2$ because the other cases are treated similarly. By the Littlewood-Richardson Rule, if $c_{a,b}^i$ is the multiplicity of ν_i in the induced representation $(\lambda_a \otimes \lambda_b)^{\uparrow S_n}$, $c_{a,b}^i$ is the number of semistandard tableau T such that T has shape ν_i/λ_a , content λ_b and the row word of T is a reverse lattice permutation. Since ν_i and λ_a are both hook partitions, then the skew shape ν_i/λ_a has at most two connected components. The first one is a row of length $n-i-(l_{g_1}-a) = l_{g_2}-(i-a)$, the second is a column of height $i-a$. By the previous conditions on the semistandard tableau T , we obtain that the entries in the column constitute a standard tableau T' . If 1 does not appear in T' then $1+b = l_{g_2}-(i-a)$, on the other hand if one entry of T' is 1 then $b = l_{g_2}-(i-a)$. Therefore $c_{a,b}^i$ is non-zero if and only if either $i-a+b = l_{g_2}-1$ or $i-a+b = l_{g_2}$, in both cases one has $c_{a,b}^i = 1$ since the semistandard tableau T is uniquely determined. Then there exist exactly two hook partitions in the decomposition of $(\lambda_a \otimes \lambda_b)^{\uparrow S_n}$. Repeating this process, we have that the total multiplicity of the hook partitions appearing in the decomposition of $(\lambda_{a_1} \otimes \lambda_{a_2} \otimes \dots \otimes \lambda_{a_r})^{\uparrow S_n}$ is $2^{r-1} = 2^{|G|-1}$. Due to the fact that all of these partitions are components of $\chi_n(E)$, we have that the multiplicity of $\lambda_{a_1} \otimes \lambda_{a_2} \otimes \dots \otimes \lambda_{a_r}$ in $\chi_{l_{g_1}, \dots, l_{g_r}}$ is exactly $2^{|G|-1}$.

Finally we have just to use Proposition 2.5, while Corollary 4.3 says that

$$c_{l_{g_1}, \dots, l_{g_r}}^G(A) = c_n(E) = 2^{n-1}$$

and the assertion follows. □

In light of the previous results, we can give a new proof of an Anisimov's result (see [1]). Let p be a prime odd number and let $G = \mathbb{Z}_p$, then we have the following:

Proposition 4.5. *If there exists $k \in G$, $k \neq 0$ such that $\dim_F L^k = \infty$, then for any $m \in \mathbb{N}$,*

$$c_m(E) = p^m 2^{m-1}.$$

If for any $k \in \mathbb{Z}_p - \{0\}$ $\dim_F L^k < \infty$, then for any $m \in \mathbb{N}$,

$$c_m(E) = 2^{m-1} \sum_{\substack{(m_0, \dots, m_{p-1}) \notin S(\varphi) \\ \sum_{i=0}^{p-1} m_i = m}} \binom{m}{m_0, \dots, m_{p-1}}.$$

Proof. If exists $k \in \mathbb{Z}_p - \{0\}$ such that $\dim_F L^k = \infty$, then $\langle k \rangle$ has the property \mathcal{P} . In particular, \mathbb{Z}_p has this property. The quotient grading on E is the trivial one and in light of Proposition 3.3, every G -graded polynomial identity of E is an ordinary polynomial identity of E . Then, for any $m \in \mathbb{N}$, we have

$$\begin{aligned} c_m(E) &= \sum_{m_0 + \dots + m_{p-1} = m} \binom{m}{m_0, \dots, m_{p-1}} c_{m_0, \dots, m_{p-1}}(E) \\ &= 2^{m-1} \sum_{m_0 + \dots + m_{p-1} = m} \binom{m}{m_0, \dots, m_{p-1}} = p^m 2^{m-1}. \end{aligned}$$

If for any $k \in \mathbb{Z}_p - \{0\}$ $\dim_F L^k < \infty$, then $\dim_F L^0 = \infty$. By Corollary 4.3 and Proposition 4.4, for any $m \in \mathbb{N}$, we have

$$\begin{aligned} c_m(E) &= \sum_{\substack{(m_0, \dots, m_{p-1}) \notin S(\varphi) \\ \sum_{i=0}^{p-1} m_i = m}} \binom{m}{m_0, \dots, m_{p-1}} c_{m_0, \dots, m_{p-1}}(E) \\ &= 2^{m-1} \sum_{\substack{(m_0, \dots, m_{p-1}) \notin S(\varphi) \\ \sum_{i=0}^{p-1} m_i = m}} \binom{m}{m_0, \dots, m_{p-1}}. \end{aligned}$$

and we are done. □

Notice that the case $p = 2$ has been completely solved in [3] and in [6].

5. TWO EXAMPLES OF GRADINGS BY GROUPS OF ORDER 4

5.1. **\mathbb{Z}_4 -grading on E .** The group $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ is the first cyclic group such that its order is not prime. In light of Theorem 3.8 we have that $T_G(E)$ “behaves” as $T_{\mathbb{Z}_2}(E)$ in the quotient grading if $\dim_F L^1 = \infty$ or $\dim_F L^3 = \infty$. Moreover, because of Theorem 4.1 the only cases to be studied are the ones for which $\dim_F L^2 = \infty$. We study a particular case of G -grading when $\dim_F L^2 = \infty$.

Let L be a vector space with basis $B_L = \{e_1, e_2, \dots\}$ and let us consider the following map:

$$\varphi: B_L \rightarrow G$$

such that $\varphi(e_1) = 1$, $\varphi(e_2) = 3$ and $\varphi(e_i) = 2$ for any $i \neq 1, 2$. Then φ induces a G -grading on E such that $\dim L^2 = \infty$. In particular, it is easy to see that:

- $E^0 = \text{span}\langle e_1^k e_2^t e_{i_1} \dots e_{i_s} \mid s \equiv 0 \pmod 2 \text{ and } (k, t) \in \{(1, 1), (0, 0)\} \rangle$;
- $E^1 = \langle e_1^k e_2^t e_{i_1} \dots e_{i_s} \mid s \equiv 0 \pmod 2 \text{ and } (k, t) = (1, 0) \text{ or } s \equiv 1 \pmod 2 \text{ and } (k, t) = (0, 1) \rangle$;

- $E^2 = \text{span}\langle e_1^k e_2^t e_{i_1} \dots e_{i_s} \mid s \equiv 1 \pmod 2 \text{ and } (k, t) \in \{(1, 1), (0, 0)\} \rangle$;
- $E^3 = \text{span}\langle e_1^k e_2^t e_{i_1} \dots e_{i_s} \mid s \equiv 0 \pmod 2 \text{ and } (k, t) = (0, 1) \text{ or } s \equiv 1 \pmod 2 \text{ and } (k, t) = (1, 0) \rangle$.

From the previous description of the G -graded homogeneous components of E one easily has the following.

Proposition 5.1. *The following monomials are G -graded polynomial identities of E :*

$$x_1 x_2 x_3^1, x_1^3 x_2^3 x_3^3, x_1^1 x_2^1 x_3^3, x_1^1 x_2^3 x_3^1, x_1^3 x_2^1 x_3^1, x_1^1 x_2^3 x_3^3, x_1^3 x_2^1 x_3^3, x_1^3 x_2^3 x_3^1.$$

Proof. We argue only for the monomial $x_1^1 x_2^1 x_3^1$ because the other cases are treated similarly. From the previous observations it follows that if we want to evaluate one variable of G -homogeneous degree 1, we shall deal with a word which contains at least one of the basis elements e_1, e_2 . Now the proposition follows because any evaluation of three variables of G -degree 1 repeats twice one between e_1 or e_2 and we are done. \square

We have not only monomial graded identities.

Proposition 5.2. *The following polynomials are G -graded polynomial identities of E :*

$$x_1^2 x_2^2 + x_2^2 x_1^2, [x_1^1, x_2^1], [x_1^3, x_2^3], [x_1^0, x_1^g],$$

for any $g \in G$.

Proof. The fact that $x_1^2 x_2^2 + x_2^2 x_1^2$ and $[x_1^0, x_1^g]$ are graded identities follows directly from the description of E^0, E^2 . For, the elements of E^0 have even length so they are in the center of E . On the other hand, the elements of E^2 have odd length.

Let us argue for $[x_1^1, x_2^1]$. If we evaluate the variable x_1^1 with a G -degree 1 element of E of odd length, we are dealing with a word containing e_1 . Hence the evaluation of x_2^1 lies in the center of E and the commutator vanishes, otherwise e_1 appears twice. We argue analogously for $[x_1^3, x_2^3]$ and we are done. \square

We are now ready to compute $T_G(E)$. For this purpose, let

$$I_1 = \langle [u_1, u_2, u_3], x_1^2 x_2^2 + x_2^2 x_1^2, [x_1^1, x_2^1], [x_1^3, x_2^3], x_1^2 x_2^2 + x_2^2 x_1^2, [x_1^0, x_1^g], x_1^1 x_2^1 x_3^1, x_1^3 x_2^3 x_3^3, x_1^1 x_2^1 x_3^3, x_1^1 x_2^3 x_3^3 \rangle^{T_G},$$

for any $g \in G$. Observe that modulo I the identity $[x_1^1 x_2^3, x_3^g]$ equals the polynomial $x_1^1 [x_2^3, x^g] + x_2^3 [x_1^1, x^g]$ that is

$$(1) \quad x_1^1 [x_2^3, x^g] \equiv -x_2^3 [x_1^1, x^g] \pmod{I}.$$

Analogously we have

$$(2) \quad x_1^1 [x_2^2, x_3^3] \equiv -x_2^2 [x_1^1, x_3^3] \pmod{I},$$

$$(3) \quad x_1^3 [x_2^2, x_3^1] \equiv +x_2^2 [x_1^3, x_3^1] \pmod{I},$$

$$(4) \quad [x_1^1, x_2^2] x_3^3 \equiv -[x_1^1, x_3^3] x_2^2 \pmod{I}.$$

Then we have the following.

Theorem 5.3. $I_1 = T_G(E)$.

Proof. The Propositions 5.1 and 5.2 give the inclusion $I_1 \subseteq T_G(E)$. We shall use the method of Y -proper polynomials. In light of Proposition 5.1, we have that the only non-trivial subspaces of multilinear Y -proper polynomials are: $\Gamma_{0,1,l,1}$, $\Gamma_{0,1,l,0}$, $\Gamma_{0,0,l,1}$, $\Gamma_{0,0,l,0}$, $\Gamma_{0,2,l,0}$, and $\Gamma_{0,0,l,2}$ for any $l \in \mathbb{N}$. Let us argue only for $\Gamma_{0,1,l,1}$ because the other cases are treated similarly. Let w be any non-zero element in $\Gamma_{0,1,l,1}$, then w can be written as a linear combination of the following polynomials

$$\begin{aligned} & x^1 x_1^2 \dots x_l^2 x^3, \\ & x_1^2 \dots x_l^2 [x^1, x^3], \\ & x^1 x_1^2 \dots \widehat{x_i^2} \dots x_l^2 [x_i^2, x^3], \\ & x_1^2 \dots \widehat{x_i^2} \dots x_l^2 x^3 [x^1, x_i^2]. \end{aligned}$$

The Equations (1) and (2) give us

$$x^1 x_1^2 \dots \widehat{x_i^2} \dots x_l^2 [x_i^2, x^3] + \alpha x_1^2 \dots x_l^2 [x^1, x^3] \equiv x_1^2 \dots \widehat{x_i^2} \dots x_l^2 x^3 [x_i^2, x^1].$$

Analogously it can be shown $x^1 x_1^2 \dots \widehat{x_i^2} \dots x_l^2 [x_i^2, x^3]$ is a linear combination of $x_1^2 \dots x_l^2 [x^1, x^3]$ and $x^1 x_1^2 \dots x_l^2 x^3$. Finally, any non-trivial polynomial of $\Gamma_{0,1,l,1}$ is a linear combination of the following polynomials:

$$\begin{aligned} w_1 &= x^1 x_1^2 \dots x_l^2 x^3, \\ w_2 &= x_1^2 \dots x_l^2 [x^1, x^3]. \end{aligned}$$

Now it suffices to show that w_1, w_2 are linearly independent modulo $T_G(E)$. Suppose by contradiction they are linearly dependent, then there exist $\alpha_1, \alpha_2 \in F$ such that $\sum_{i=1}^2 \alpha_i w_i \in T_G(E)$. Let us consider the following substitution φ :

$$\begin{aligned} \varphi(x^1) &= e_2 e_3, \\ \varphi(x^3) &= e_1 e_4, \\ \varphi(x_i^2) &= e_{i+4} \quad \text{for any } i = 1, \dots, l. \end{aligned}$$

Then

$$\varphi(w_2) = 0$$

but

$$\varphi(w_1) = e_2 e_3 e_5 \dots e_{l+4} e_1 e_4 \neq 0,$$

a contradiction and the proof is complete. \square

According to [6], it seems that f is a multilinear \mathbb{Z}_4 -graded identity of E if and only if $\gamma(f)$ is a \mathbb{Z}_2 -graded identity of E_{2^*} for some special function γ .

5.2. $\mathbb{Z}_2 \times \mathbb{Z}_2$ -gradings on E . Theorem 3.8 is useful in order to reduce the order of the grading group if G has non-trivial squares. This is not the case of finite powers of \mathbb{Z}_2 . In this section we shall deal with some special cases of $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading of E .

Let us suppose firstly that L is a G -homogeneous vector space over F such that

$$\dim_F L^{(0,1)} = \dim_F L^{(1,0)} = \dim_F L^{(1,1)} = \infty, \dim_F L^{(0,0)} < \infty,$$

and let $E = E(L)$ be the Grassmann algebra generated by L . Let $B_1 = \{e_1, e_2, \dots\}$ be a basis of $L^{(1,0)}$, $B_2 = \{e'_1, e'_2, \dots\}$ be a basis of $L^{(0,1)}$ and $B_3 = \{e''_1, e''_2, \dots\}$ be a basis of $L^{(1,1)}$ as vector spaces. Let us consider the map

$$\varphi: B_1 \cup B_2 \cup B_3: \rightarrow G$$

associated to the G -grading over E . It is such that $\varphi(e_i) = (1, 0)$ for any $i = 1, 2, \dots$, $\varphi(e'_j) = (0, 1)$ for any j and $\varphi(e''_s) = (1, 1)$ for any s . We have the following.

Lemma 5.4. G has the property \mathcal{P} .

Proof. The pairwise disjoint sets of elements $\{e_{2k+1}e'_{2k+1} \mid k \geq 0\}$, $\{e_{2k}e''_{2k} \mid k \geq 1\}$, $\{e'_{2k}e''_{6k+1} \mid k \geq 1\}$, and $\{e''_{6k+3}e''_{6k+5} \mid k \geq 1\}$ belong respectively to $E^{(1,1)}$, $E^{(0,1)}$, $E^{(1,0)}$, $E^{(0,0)}$ and the proof is complete. \square

In light of the Proposition 3.3, we have the following result.

Theorem 5.5. Let L be a G -homogeneous vector space over F such that

$$\dim_F L^{(0,1)} = \dim_F L^{(1,0)} = \dim_F L^{(1,1)} = \infty, \dim_F L^{(0,0)} < \infty,$$

and let $E = E(L)$ the Grassmann algebra generated by L . Let f be a multilinear polynomial in $F\langle X \mid G \rangle$. Then $f \in T_G(E)$ if and only if $\pi(f) \in T(E)$.

Proof. By Lemma 5.4 G has the property \mathcal{P} and we are done because of Proposition 3.3. \square

Suppose now that L is a G -homogeneous vector space over F such that

$$\dim_F L^{(0,1)} = \dim_F L^{(1,0)} = \infty, \dim_F L^{(0,0)} < \infty, \dim_F L^{(1,1)} < \infty$$

and let $E = E(L)$ the Grassmann algebra generated by L . Let $B_1 = \{e_1, e_2, \dots\}$ be a basis of $L^{(1,0)}$, $B_2 = \{f_1, f_2, \dots\}$ be a basis of $L^{(0,1)}$ as vector spaces. Let us consider the map

$$\varphi: B_1 \cup B_2: \rightarrow G$$

associated to the G -grading over E . It is such that $\varphi(e_i) = (1, 0)$ for any $i = 1, 2, \dots$, $\varphi(f_j) = (0, 1)$ for any j . We have the analog of Lemma 5.4. Let $H = \langle g \rangle$, where $g = (1, 1)$. Notice that $H \equiv \mathbb{Z}_2$. Then we obtain the next result.

Lemma 5.6. H has the property \mathcal{P} .

Theorem 5.7. Let L be a G -homogeneous vector space over F such that

$$\dim_F L^{(0,1)} = \dim_F L^{(1,0)} = \infty, \dim_F L^{(0,0)} < \infty, \dim_F L^{(1,1)} < \infty$$

and let $E = E(L)$ the Grassmann algebra generated by L . Let f be a multilinear polynomial in $F\langle X \mid G \rangle$. Then $f \in T_G(E)$ if and only if $\pi(f) \in T_{\mathbb{Z}_2}(E)$, where in the \mathbb{Z}_2 -grading, the dimension of the G -homogeneous underlying vector space L^0 is finite.

Proof. By Lemma 5.6, we have that H has the property \mathcal{P} . In light of Proposition 3.3, we have that $f \in T_G(E)$ if and only if $\pi(f) \in T_{G/H}(E)$. It is easy to see that $G/H \cong \mathbb{Z}_2$. Now, by the definition of quotient grading, we have that the new \mathbb{Z}_2 -grading is such that

$$E^0 = E^{(0,0)} \oplus E^{(1,1)}$$

$$E^1 = E^{(0,1)} \oplus E^{(1,0)} .$$

By hypothesis we have that L^0 is finite dimensional and we are done. □

We shall deal now with G -gradings such that there exists one and only one $g \in G$ $g \neq (0,0)$ such that $\dim_F L^g = \infty$.

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and L a G -homogeneous vector space over F such that $\dim_F L^{(0,1)} = \infty$, and $\dim_F L^{(1,0)} = k$. Let $E = E(L)$ be the Grassmann algebra generated by L . Let $B_1 = \{e_1, e_2, \dots, e_k\}$ be a basis of $L^{(1,0)}$ as a vector space and let $B_2 = \{f_1, f_2, \dots\}$ be a basis of $L^{(0,1)}$ as a vector space. Let us consider the map:

$$\varphi: B_L \rightarrow G,$$

associated to the G -grading of E . It is such that $\varphi(e_i) = (1,0)$ for any $i = 1, 2, \dots, k$ and $\varphi(f_j) = (0,1)$ for any j . It is easy to see that:

- $E^{(0,0)} = \text{span}\langle e_{i_1} e_{i_2} \dots e_{i_t} f_{j_1} \dots f_{j_s} \mid s \equiv t \equiv 0 \pmod 2 \rangle$;
- $E^{(0,1)} = \text{span}\langle e_{i_1} e_{i_2} \dots e_{i_t} f_{j_1} \dots f_{j_s} \mid s \equiv 1 \pmod 2 \text{ and } t \equiv 0 \pmod 2 \rangle$;
- $E^{(1,0)} = \text{span}\langle e_{i_1} e_{i_2} \dots e_{i_t} f_{j_1} \dots f_{j_s} \mid s \equiv 0 \pmod 2 \text{ and } t \equiv 1 \pmod 2 \rangle$;
- $E^{(1,1)} = \text{span}\langle e_{i_1} e_{i_2} \dots e_{i_t} f_{j_1} \dots f_{j_s} \mid s \equiv t \equiv 1 \pmod 2 \rangle$.

Let $r, s \in \mathbb{N}$ and w be a monomial in the variables of G -degree $(1,0)$ and $(1,1)$ only. We say that $w \in W_{r,s}$ if the number of variables appearing in w having G -degree $(1,0)$ is exactly r and the number of variables appearing in w having G -degree $(1,1)$ is s . From the previous description of the G -graded homogeneous components of E one easily has the following:

Proposition 5.8. *The following monomials are G -graded polynomial identities of E :*

$$\bigcup_{r+s \geq k+1} W_{r,s} .$$

Proof. In light of the fact that any element of G -degree $(1,0)$ and $(1,1)$ may contain at least one element among $\{e_1, \dots, e_k\}$, we have that if φ is any graded substitution of $w_{r,s}$, one of the k basis elements repeats at least twice and the Proposition follows. □

We have not only monomial graded identities.

Proposition 5.9. *The following polynomials are G -graded polynomial identities of E :*

$$x_1^{(0,1)} x_2^{(0,1)} + x_2^{(0,1)} x_1^{(0,1)}, \quad x_1^{(0,1)} x_2^{(1,0)} + x_2^{(1,0)} x_1^{(0,1)}, \quad x_1^{(1,0)} x_2^{(1,0)} + x_2^{(1,0)} x_1^{(1,0)},$$

$$[x_1^{(0,0)}, x_2^g], \quad [x_1^{(1,1)}, x_2^g], \quad \text{for any } g \in G .$$

Proof. It follows directly from the description of the various E^g . □

We are now ready to compute $T_G(E)$. For this purpose, let I_2 the T_G ideal generated by

$$[u_1, u_2, u_3], \bigcup_{r+s=k+1} W_{r,s}, x_1^{(0,1)} x_2^{(0,1)} + x_2^{(0,1)} x_1^{(0,1)}, x_1^{(0,1)} x_2^{(1,0)} + x_2^{(1,0)} x_1^{(0,1)}, \\ x_1^{(1,0)} x_2^{(0,1)} + x_2^{(1,0)} x_1^{(1,0)}, [x_1^{(0,0)}, x_2^g], [x_1^{(1,1)}, x_2^g], \text{ for any } g \in G.$$

We have the following:

Theorem 5.10. $I_2 = T_G(E)$.

Proof. The Propositions 5.8 and 5.9 give the inclusion $I_2 \subseteq T_G(E)$. We shall use the method of Y -proper polynomials once again. The only non-trivial subspaces of Y -proper polynomials are $\Gamma_{0,t,r,s}$, such that $r + s \leq k$. Due to the anticommutativity of the variables of G -degree $(0, 1)$, $(1, 0)$ and the commutativity of the variables of G -degree $(1, 1)$, as in the previous proposition, we can write any polynomial in $\Gamma_{0,t,r,s}$ as linear combination of polynomials

$$x_1^{(0,1)} \dots x_t^{(0,1)} x_{t+1}^{(1,0)} \dots x_{t+r}^{(1,0)} x_{t+r+1}^{(1,1)} \dots x_{t+r+s}^{(1,1)},$$

such that $r + s \leq k$ which are clearly linearly independent modulo $T_G(E)$. The conclusion follows as in the proof of Theorem 5.3. \square

Again, according to [6], it seems that f is a multilinear $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded identity of E if and only if $\gamma(f)$ is a \mathbb{Z}_2 -graded identity of E_{2^*} for some special function γ .

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