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ANNIHILATING AND POWER-COMMUTING GENERALIZED SKEW DERIVATIONS ON LIE IDEALS IN PRIME RINGS

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Abstract. Let R be a prime ring of characteristic different from 2 and 3, Q_r its right Martindale quotient ring, C its extended centroid, L a non-central Lie ideal of R and $n \geqslant 1$ a fixed positive integer. Let α be an automorphism of the ring R. An additive map $D\colon R \to R$ is called an α-derivation (or a skew derivation) on R if $D(xy) = D(x)y + \alpha(x)D(y)$ for all $x, y \in R$. An additive mapping $F\colon R \to R$ is called a generalized α-derivation (or a generalized skew derivation) on R if there exists a skew derivation D on R such that $F(xy) = F(x)y + \alpha(x)D(y)$ for all $x, y \in R$.

We prove that, if F is a nonzero generalized skew derivation of R such that $F(x) \times [F(x), x]^n = 0$ for any $x \in L$, then either there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$, or $R \subseteq M_2(C)$ and there exist $a \in Q_r$ and $\lambda \in C$ such that $F(x) = ax + xa + \lambda x$ for any $x \in R$.

Keywords: generalized skew derivation; Lie ideal; prime ring

MSC 2010: 16W25, 16N60

1. Introduction

Let R be a prime ring with center $Z(R)$, extended centroid C, right Martindale quotient ring Q_r and symmetric Martindale quotient ring Q. An additive mapping $d\colon R \to R$ is a *derivation* on R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Let $a \in R$ be a fixed element. Many results in literature indicate how the global structure of a ring R is often tightly connected to the behaviour of additive mappings defined on R. A well known result of Posner [22] states that if d is a derivation of R such that $[d(x), x] \in Z(R)$ for any $x \in R$, then either $d = 0$ or R is commutative. In [17] Lanski generalized Posner's theorem to a Lie ideal. Later in [2] the following result was proved:

Theorem 1.1. *Let R be a prime ring of characteristic different from 2, L a Lie ideal of R, d a nonzero derivation of R such that $[d(u), u]^n \in Z(R)$ for any $u \in L$. Then R satisfies s_4, the standard identity of degree 4.*

In particular, if d satisfies $[d(u), u]^n = 0$ for any $u \in L$, then $L \subseteq Z(R)$.

More recently in [9] the author considered a similar situation in the case the derivation d is replaced by a generalized derivation. More specifically, an additive map $G\colon R \to R$ is said to be a generalized derivation if there exists a derivation d of R such that for all $x, y \in R$, $G(xy) = G(x)y + xd(y)$. More precisely, the main result in [9] is the following:

Theorem 1.2. *Let R be a prime ring of characteristic different from 2 with right Martindale quotient ring U and extended centroid C, $G \neq 0$ a generalized derivation of R, L a non-central Lie ideal of R and $n \geqslant 1$ such that $[G(u), u]^n = 0$ for all $u \in L$. Then there exists an element $a \in C$ such that $G(x) = ax$ for all $x \in R$, unless when R satisfies s_4 and there exist $b \in U$, $\beta \in C$ such that $G(x) = bx + xb + \beta x$ for all $x \in R$.*

In particular, if $[G(x), x]^n = 0$ for all $x \in R$, then there exists an element $a \in C$ such that $G(x) = ax$ for all $x \in R$.

In [24], Wang considered a similar situation in the case the derivation d is replaced by a nontrivial automorphism σ of R and proved the following:

Theorem 1.3. *Let R be a prime ring with center Z, L a noncentral Lie ideal of R, and σ a nontrivial automorphism of R such that $[u^\sigma, u]^n \in Z$ for all $u \in L$. If either $\mathrm{char}(R) > n$ or $\mathrm{char}(R) = 0$, then R satisfies s_4.*

More recently, in [12] Dhara and Mondal extended the results contained in [22], [17], [2] and [9], by studying an annihilating condition on commutators and proved the following:

Theorem 1.4 ([12], Theorem 1.2)**.** *Let R be a prime ring with right Martindale quotient ring Q_r and extended centroid C, $F \neq 0$ a generalized derivation of R and $n \geqslant 1$ such that $F(x)[F(x), x]^n = 0$ for all $x \in R$. Then there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$, unless when $R \subseteq M_2(C)$ and $\mathrm{char}(R) = 2$.*

Theorem 1.5 ([12], Theorem 1.1)**.** *Let R be a prime ring with right Martindale quotient ring Q_r and extended centroid C, $F \neq 0$ a generalized derivation of R, L a noncentral Lie ideal of R and $n \geqslant 1$ such that $F(x)[F(x), x]^n = 0$ for all $x \in L$. Then either there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$, or $R \subseteq M_2(C)$ and there exist $a \in Q_r$ and $\lambda \in C$ such that $F(x) = ax + xa + \lambda x$ for any $x \in R$, unless when $R \subseteq M_2(C)$ and $\mathrm{char}(R) = 2$.*

Here we continue this line of investigation and examine what happens in case $F \neq 0$ is a generalized skew derivation of R such that $F(x)[F(x),x]^n = 0$ for all $x \in S$, where S is an appropriate subset of R and $n \geqslant 1$ is a fixed integer. More specifically, let α be an automorphism of a ring R. An additive map $D\colon R \to R$ is called an *α-derivation* (or a *skew derivation*) on R if $D(xy) = D(x)y + \alpha(x)D(y)$ for all $x, y \in R$. In this case α is called an *associated automorphism* of D. Basic examples of α-derivations are the usual derivations and the map α-id, where "id" denotes the identity map. Let $b \in Q$ be a fixed element. Then a map $D\colon R \to R$ defined by $D(x) = bx - \alpha(x)b$, $x \in R$, is an α-derivation on R and it is called an *inner α-derivation* (an *inner skew derivation*) defined by b. If a skew derivation D is not inner, then it is called *outer*.

An additive mapping $F\colon R \to R$ is called a *generalized α-derivation* (or a *generalized skew derivation*) on R if there exists an additive mapping D on R such that $F(xy) = F(x)y + \alpha(x)D(y)$ for all $x, y \in R$. The map D is uniquely determined by F and it is called an *associated additive map* of F. Moreover, it turns out that D is always an α-derivation (see [19], [20] for more details).

Let us also mention that an automorphism $\alpha\colon R \to R$ is *inner* if there exists an invertible $q \in Q$ such that $\alpha(x) = qxq^{-1}$ for all $x \in R$. If an automorphism $\alpha \in \mathrm{Aut}(R)$ is not inner, then it is called *outer*.

The first step in the study of power commuting condition on generalized skew derivation was done in [3], where the following result is proved:

Theorem 1.6. *Let R be a non-commutative prime ring of characteristic different from 2 with extended centroid C, $F \neq 0$ a generalized skew derivation of R, and $n \geqslant 1$ such that $[F(x),x]^n = 0$ for all $x \in R$. Then there exists an element $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$.*

In this paper we would like to extend all the previously cited results to the case of prime rings of characteristic different from 2 and 3.

The result we obtain is the following:

Theorem 1.7. *Let R be a prime ring of characteristic different from 2 and 3, Q_r its right Martindale quotient ring, C its extended centroid, F a nonzero generalized skew derivation of R, L a non-central Lie ideal of R and $n \geqslant 1$ a fixed positive integer. If $F(x)[F(x),x]^n = 0$ for any $x \in L$, then either there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$, or $R \subseteq M_2(C)$ and there exist $a \in Q_r$ and $\lambda \in C$ such that $F(x) = ax + xa + \lambda x$ for any $x \in R$.*

In order to prove our result, we need to recall the following known facts:

Fact 1.8. *Let R be a prime ring and I a two-sided ideal of R. Then I, R and Q satisfy the same generalized polynomial identities with coefficients in Q (see* [7]*). Furthermore, I, R and Q_r satisfy the same generalized polynomial identities with automorphisms (Theorem* 1 *in* [5]*).*

Fact 1.9. *If R is a prime ring satisfying a nontrivial generalized polynomial identity and α an automorphism of R such that $\alpha(x) = x$ for all $x \in C$, then α is an inner automorphism of R (*[1]*, Theorem* 4.7.4*).*

2. The inner case

Let $a, b \in Q_r$ and $F\colon R \to R$, such that $F(x) = ax + \alpha(x)b$ for all $x \in R$. In this section we study the case when $(ar + \alpha(r)b)[ar + \alpha(r)b, r]^n = 0$ for all $r \in [R, R]$. Under this assumption, we prove that F is a generalized derivation of R, so that the conclusions of Theorem 1.5 hold.

The starting point is the case when there exists an invertible element $q \in Q$ such that $\alpha(x) = qxq^{-1}$ for all $x \in R$.

In the sequel we make a frequent use of the following:

Fact 2.1 ([10])**.** *Let $\mathcal{K}$ be an infinite field and $n \geqslant 2$. If $A_1, \ldots, A_k$ are not scalar matrices in $M_n(\mathcal{K})$ then there exists an invertible matrix $P \in M_n(\mathcal{K})$ such that each of the matrices $PA_1P^{-1}, \ldots, PA_kP^{-1}$ has all nonzero entries.*

Fact 2.2 ([11], Proposition 1)**.** *Let H be a field of characteristic different from* 2*, $R = M_t(H)$ the matrix ring over H and $t \geqslant 3$. Let a, b be elements of R, with $a = \sum\limits_{r,s=1}^{t} a_{rs}e_{rs}$ and $b = \sum\limits_{r,s=1}^{t} b_{rs}e_{rs}$, with $a_{rs}, b_{rs} \in H$. For any automorphism φ of R, we denote $\varphi(a) = \sum\limits_{r,s=1}^{t} \varphi(a)_{rs}e_{rs}$, $\varphi(b) = \sum\limits_{r,s=1}^{t} \varphi(b)_{rs}e_{rs}$, with $\varphi(a)_{rs}$, $\varphi(b)_{rs} \in H$.*

If $a_{ij}b_{ij} = 0$ for any $i \neq j$ and $\varphi(a)_{ij}\varphi(b)_{ij}$ for any $i \neq j$ and for any $\varphi \in \mathrm{Aut}(R)$, then $a \in Z(R)$ or $b \in Z(R)$.

Lemma 2.3. *Let $R = M_k(C)$ be the ring of $k \times k$ matrices over C, with $k \geqslant 3$. If $\mathrm{char}(R) \neq 2$ and $(ar + qrq^{-1}b)[ar + qrq^{-1}b, r]^n = 0$ for all $r \in [R, R]$, then either $q \in Z(R)$ or $q^{-1}b \in Z(R)$. In any case F is an inner generalized derivation of R.*

P r o o f. The symbol e_{ij} will always denote the usual matrix unit with 1 at the (i, j)-entry and zero elsewhere.

By our assumption R satisfies

$$(2.1)\qquad (a[x_1,x_2]+q[x_1,x_2]q^{-1}b)\big[a[x_1,x_2]+q[x_1,x_2]q^{-1}b,[x_1,x_2]\big]^n.$$

Say $q=\sum_{hl} q_{hl}e_{hl}$ and $q^{-1}b=\sum_{hl} v_{hl}e_{hl}$ for $q_{hl},v_{hl}\in C$. For $i\neq j$, $[x_1,x_2]=e_{ij}$ in (3.1) and right multiplying by e_{ij} we have that $(-1)^n qe_{ij}q^{-1}b(e_{ij}qe_{ij}q^{-1}b)^n e_{ij}=0$, that is $q_{ji}v_{ji}=0$ for any $i\neq j$. Moreover, for any automorphism φ of R one has that

$$(\varphi(a)[x_1,x_2]+\varphi(q)[x_1,x_2]\varphi(q^{-1}b))\big[\varphi(a)[x_1,x_2]+\varphi(q)[x_1,x_2]\varphi(q^{-1}b),[x_1,x_2]\big]^n$$

is still an identity for R. Thus, in light of Fact 2.2, it follows that either $q\in Z(R)$ or $q^{-1}b\in Z(R)$, as required. □

Lemma 2.4. *Let $R=M_2(C)$ be the ring of 2×2 matrices over C. If $(ar+qrq^{-1}b)[ar+qrq^{-1}b,r]^n=0$ for all $r\in[R,R]$, then either $q\in Z(R)$ or $q^{-1}b\in Z(R)$. In any case F is an inner generalized derivation of R.*

Proof. First we recall that for any $x,y\in M_2(C)$, either $[x,y]^2=0$ or $0\neq[x,y]^2\in Z(R)$.

Assume that there exists $r\in[R,R]$ such that $0\neq[ar+qrq^{-1}b,r]^2\in Z(R)$. Thus, by our assumption and since $[ar+qrq^{-1}b,r]$ is an invertible matrix, it follows that $ar+qrq^{-1}b=0$, which is a contradiction.

Therefore we may assume that

$$(2.2)\qquad [ar+qrq^{-1}b,r]^2=0$$

for all $r\in[R,R]$. Suppose that $q\notin Z(R)$ and $q^{-1}b\notin Z(R)$, that is neither q nor $q^{-1}b$ is a scalar matrix.

Assume first that C is infinite, then, by Fact 2.1, there exists an invertible matrix $T\in M_m(C)$ such that each of the matrices $TqT^{-1}, Tq^{-1}bT^{-1}$ has all nonzero entries. Denote by $\chi(x)=TxT^{-1}$ the inner automorphism induced by T. Say $\chi(q)=\sum_{hl} q'_{hl}e_{hl}$ and $\chi(q^{-1}b)=\sum_{hl} v'_{hl}e_{hl}$ for $0\neq q'_{hl}$, $0\neq v'_{hl}\in C$. Without loss of generality, we may replace q, $q^{-1}b$ by $\chi(q)$ and $\chi(q^{-1}b)$, respectively. As above in the relation (2.2), let $i\neq j$, $r=e_{ij}$ and multiply on the left by e_{ij}. Thus it follows $e_{ij}(qe_{ij}q^{-1}be_{ij})^2$, which means $q'_{ji}v'_{ji}=0$, a contradiction.

Now let E be an infinite field which is an extension of the field C and let $\overline{R}=M_t(E)\cong R\otimes_C E$. Consider the generalized polynomial

$$\Phi(x_1,x_2)=\big[a[x_1,x_2]+q[x_1,x_2]q^{-1}b,[x_1,x_2]\big]^2$$

which is a generalized polynomial identity for R. Moreover, $\Phi(x_1, x_2)$ is homogeneous in both x_1 and x_2 of degree 4. Hence the complete linearization of $\Phi(x_1, x_2)$ is a multilinear generalized polynomial $\Theta(x_1, x_2, y_1, y_2)$, and

$$\Theta(x_1, x_2, x_1, x_2) = 4^2\Phi(x).$$

Clearly, the multilinear polynomial $\Theta(x, y)$ is a generalized polynomial identity for R and $\overline{R}$ too. Since $\operatorname{char}(C) \neq 2$, we obtain $\Phi(r_1, r_2) = 0$ for all $r_1, r_2 \in \overline{R}$, and the conclusion follows from the first part of the present Lemma 2.4. □

Application of Theorem 1.5 to Lemmas 2.3 and 2.4 leads to the following:

Lemma 2.5. *Let $R = M_k(C)$ be the ring of $k \times k$ matrices over C, with $k \geqslant 2$ and $F(x) = ax + qxq^{-1}b$ for any $x \in R$, where a, b, q are fixed elements of R and q is invertible. If $\operatorname{char}(R) \neq 2$ and $(ar + qrq^{-1}b)[ar + qrq^{-1}b, r]^n = 0$ for all $r \in [R, R]$, then either there exists $\lambda \in Z(R)$ such that $F(x) = \lambda x$ for all $x \in R$, or $k = 2$ and there exist $a' \in R$ and $\lambda \in Z(R)$ such that $F(x) = a'x + xa' + \lambda x$ for any $x \in R$.*

As a consequence we also have:

Corollary 2.6. *Let $R = M_k(C)$ be the ring of $k \times k$ matrices over C with $k \geqslant 2$ and $F(x) = ax + qxq^{-1}b$ for any $x \in R$, where a, b, q are fixed elements of R and q is invertible. If $\operatorname{char}(R) \neq 2$ and $(ar + qrq^{-1}b)[ar + qrq^{-1}b, r]^n = 0$ for all $r \in R$, then there exists $\lambda \in Z(R)$ such that $F(x) = \lambda x$ for all $x \in R$.*

Proof. By using the same argument as in Lemmas 2.3 and 2.4, we have that either $q \in Z(R)$ or $q^{-1}b \in Z(R)$. In any case F is an inner generalized derivation of R and the conclusion follows from Theorem 1.4. □

Proposition 2.7. *Let R be a prime ring of characteristic different from 2, $a, b, q \in Q_r$, where q is an invertible element, and $n \geqslant 1$ a fixed integer such that $F(x) = ax + qxq^{-1}b$ and*

$$(ar + qrq^{-1}b)[ar + qrq^{-1}b, r]^n = 0 \tag{2.3}$$

for all $r \in [R, R]$. Then either $q \in C$ or $q^{-1}b \in C$. In any case either there exists $\lambda \in Z(R)$ such that $F(x) = \lambda x$ for all $x \in R$, or $k = 2$ and there exist $a' \in R$ and $\lambda \in Z(R)$ such that $F(x) = a'x + xa' + \lambda x$ for any $x \in R$.

Proof. In what follows we assume that both $q^{-1}b \notin C$ and $q \notin C$; if not we are done by Theorem 1.5.

Thus

$$(2.4)\qquad (a[x_1,x_2]+q[x_1,x_2]q^{-1}b)\bigl[a[x_1,x_2]+q[x_1,x_2]q^{-1}b,[x_1,x_2]\bigr]^n$$

is a nontrivial generalized polynomial identity for R. By [21] Q_r is a primitive ring, which is isomorphic to a dense subring of the ring of linear transformations of a vector space V over a division ring D, and D is finite-dimensional over its center $C=Z(D)$. If $\dim_D V=k$ is finite, then R is a simple ring which satisfies a nontrivial generalized polynomial identity. By Lemma 2 in [16] (see also Theorem 2.3.29 in [23]), $R\subseteq M_t(K)$ for a suitable field K, moreover, $M_t(K)$ satisfies the same generalized identity of R, hence $M_t(K)$ satisfies (2.4). In this case we are done by using Lemma 2.5.

Let now $\dim_D V=\infty$. As in Lemma 2 in [25], the set $[R,R]$ is dense on R. By the fact that (2.4) is a generalized polynomial identity of R, we know that R satisfies

$$(2.5)\qquad (ax+qxq^{-1}b)[ax+qxq^{-1}b,x]^n.$$

Suppose first that there exist $v\in V$ such that $\{v,q^{-1}bv\}$ are linearly D-independent. Since $\dim_D V=\infty$, there exists $w\in V$ such that $\{v,q^{-1}bv,w\}$ are linearly D-independent. By the density of R, there exists $s\in R$ such that $sv=0$, $sq^{-1}bv=q^{-1}w$ and $sw=-v$. In this case we also have $[as+qsq^{-1}b,s]^nv=v$ and (2.5) implies the contradiction

$$0=(as+qsq^{-1}b)[as+qsq^{-1}b,s]^nv=w\neq 0.$$

This means that for any choice of $v\in V$, $v,q^{-1}bv$ are linearly D-dependent. Standard arguments prove that there exists $\beta\in D$ such that $q^{-1}bv=v\beta$ for all $v\in V$ and also, by using this fact, that $q^{-1}b\in Z(R)$. Thus R satisfies

$$(2.6)\qquad (a+b)x[(a+b)x,x]^n$$

and by Theorem 1.4, we have that $a+b=\lambda\in Z(R)$ and $F(x)=\lambda x$ for all $x\in R$. □

Proposition 2.8. *Let R be a non-commutative prime ring of characteristic different from 2, $a,b\in Q_r$, $\alpha\colon R\to R$ an outer automorphism of R such that $(ax+\alpha(x)b)[ax+\alpha(x)b,x]^n=0$ for all $x\in[R,R]$. Then $a\in C$ and $b=0$.*

P r o o f. In the following, we assume that either $a\notin C$ or $b\neq 0$.

Hence, by [6] R is a GPI-ring and Q_r is also a GPI-ring by [7]. By Martindale's theorem in [21], Q_r is a primitive ring having nonzero socle and its associated division ring D is finite-dimensional over C. Hence Q_r is isomorphic to a dense subring of the ring of linear transformations of a vector space V over D, containing nonzero linear transformations of finite rank.

By [15], page 79, there exists a semi-linear automorphism $T \in \mathrm{End}(V)$ such that $\alpha(x) = TxT^{-1}$ for all $x \in Q_r$. Hence, Q_r satisfies $(ax + TxT^{-1}b)[ax + TxT^{-1}b, x]^n$.

If for any $v \in V$ there exists $\lambda_v \in D$ such that $T^{-1}cv = v\lambda_v$, then, by a standard argument, it follows that there exists a unique $\lambda \in D$ such that $T^{-1}bv = v\lambda$ for all $v \in V$. In this case

$$\begin{aligned}(ax + \alpha(x)b)v &= (ax + TxT^{-1}b)v = axv + T(xv\lambda) = axv + T((xv)\lambda) \\ &= axv + T(T^{-1}bxv) = axv + bxv = (a+b)xv.\end{aligned}$$

Hence, for all $v \in V$,

$$(ax + \alpha(x)b - (a+b)x)v = 0$$

which implies $ax + \alpha(x)b = (a+b)x$ for all $x \in Q_r$, since V is faithful. Therefore we have both $(a+b)x[(a+b)x, x]^n = 0$ and $\alpha(x)b = bx$ for all $x \in Q$. Thus $a + b \in C$ follows from Theorem 1.5. Moreover, since Q_r satisfies $\alpha(x)b = bx$ and the $\alpha(x)$-word degree is 1, Theorem 3 in [5] yields that $yb - bx$ is an identity for Q. This implies $b = 0$, which is a contradiction.

In light of the previous argument, we may suppose there exists $v \in V$ such that $\{v, T^{-1}bv\}$ is linearly D-independent.

Consider first the case $\dim_D V \geqslant 4$.

Thus there exist $w, w' \in V$ such that $\{w, w', v, T^{-1}bv\}$ are linearly D-independent. Moreover, by the density of Q_r, there exists $r, s \in Q_r$ such that

$$rv = sv = v, \quad rT^{-1}bv = 0, \quad sT^{-1}bv = w, \quad rw = T^{-1}w', \quad rw' = 0, \quad sw' = v.$$

Hence, by the main assumption, we get the contradiction

$$0 = (a[r,s] + T[r,s]T^{-1}b)\big[a[r,s] + T[r,s]T^{-1}b, [r,s]\big]^n v = w' \neq 0.$$

Therefore, we have just to consider the case when $\dim_D V \leqslant 3$.

Of course in this case Q_r satisfies

$$(a[x_1,x_2] + \alpha([x_1,x_2])b)\big[a[x_1,x_2] + \alpha([x_1,x_2])b, [x_1,x_2]\big]^3.$$

Therefore the $\alpha(x_i)$-word degree is 4. Since either $\mathrm{char}(R) = 0$ or $\mathrm{char}(R) \geqslant 5$, Theorem 3 in [5] implies that Q_r satisfies

$$(a[x_1,x_2] + [t_1,t_2]b)\big[a[x_1,x_2] + [t_1,t_2]b, [x_1,x_2]\big]^3. \tag{2.7}$$

In particular, Q_r satisfies both

$$a[x_1,x_2]\big[a[x_1,x_2], [x_1,x_2]\big]^3 \tag{2.8}$$

and

$$(2.9)\qquad (a[x_1,x_2]+[x_1,x_2]b)\big[a[x_1,x_2]+[x_1,x_2]b,[x_1,x_2]\big]^3.$$

Applying Theorems 1.4 and 1.5 respectively to (2.8) and (2.9) we have simultaneously that $a \in C$ and $a-b \in C$, that is both $a \in C$ and $b \in C$. Since if $b=0$ we are done, here we assume $b \neq 0$ and prove that a contradiction follows.

In fact, if $a,b \in C$ and $b \neq 0$ then (2.7) is a polynomial identity for Q_r with coefficients in C. By the well known Posner's theorem, there exists a field $\mathcal{K}$ such that Q_r and the matrix ring $M_m(\mathcal{K})$ satisfy the same polynomial identities, in particular $M_m(\mathcal{K})$ satisfies (2.7). Moreover, we may assume $m \geqslant 2$ since Q_r is not commutative. Therefore, for $[x_1,x_2]=e_{12}$ and $[t_1,t_2]=e_{21}$ in relation (2.7) we have the contradiction $ae_{12}+(-1)^n be_{21}=0$. □

3. The proof of main result

Here we can finally prove the main theorem of this paper. We remark that Chang, in [4] showed that any (right) generalized skew derivation of R can be uniquely extended to the right Martindale quotient ring Q_r of R as follows: a (right) generalized skew derivation is an additive mapping $F\colon Q_r \to Q_r$ such that $F(xy) = F(x)y+\alpha(x)d(y)$ for all $x,y \in Q_r$, where d is a skew derivation of R and α is an automorphism of R. Notice that there exists $F(1)=a \in Q_r$ such that $F(x)=ax+d(x)$ for all $x \in R$.

P r o o f of Theorem 1.7. It is easy to see that R is non-commutative as L is non-central. Notice that, in case α is the identity map on R, then F is a generalized derivation of R and we conclude by Theorem 1.5. Moreover, since $\operatorname{char}(R) \neq 2$, there exists an ideal I of R such that $0 \neq [I,R] \subseteq L$ (see [14], pages 4–5, [13], Lemma 2, Proposition 1, [18], Theorem 4). By the assumption, we have $F([x,y])[F([x,y]),[x,y]]^n=0$ for all $x,y \in I$ and also for all $x,y \in Q_r$ (see [8], Theorem 2). This implies that

$$(3.1)\quad (a[x,y]+d(x)y+\alpha(x)d(y)-d(y)x-\alpha(y)d(x))\big[a[x,y]+d(x)y \\ +\alpha(x)d(y)-d(y)x-\alpha(y)d(x),[x,y]\big]^n=0,\quad x,y \in Q_r,$$

that is

$$(3.2)\quad (a[x_1,x_2]+d(x_1)x_2+\alpha(x_1)d(x_2)-d(x_2)x_1-\alpha(x_2)d(x_1))\big[a[x_1,x_2] \\ +d(x_1)x_2+\alpha(x_1)d(x_2)-d(x_2)x_1-\alpha(x_2)d(x_1),[x_1,x_2]\big]^n$$

is an identity for Q_r.

In what follows we may assume that the associated automorphism α is not the identity map and also that $d \neq 0$. In fact, if either $\alpha = \mathrm{id}$ or $d = 0$, then F is a generalized derivation of R and the result follows from Theorem 1.5.

Suppose that d is X-inner. Then there exist $c \in Q_r$ and $\alpha \in \mathrm{Aut}(Q_r)$ such that $d(x) = cx - \alpha(x)c$ for all $x \in R$. In this case $F(x) = (a + c)x - \alpha(x)c$. It follows from Propositions 2.7 and 2.8 that either $F(x) = \lambda x$, where $\lambda \in C$, or $R \subseteq M_2(C)$ and $F(x) = a'x + xa' + \lambda x$, with $a' \in Q_r$ and $\lambda \in C$.

Assume that d is outer. By [8], Theorem 1, and (3.2) it follows that Q_r satisfies the generalized polynomial identity

$$(a[x_1, x_2] + t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1)\big[a[x_1, x_2] + t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1, [x_1, x_2]\big]^n$$

and in particular,

$$(a[x_1, x_2] + t_1x_2 - \alpha(x_2)t_1)\big[a[x_1, x_2] + t_1x_2 - \alpha(x_2)t_1, [x_1, x_2]\big]^n \tag{3.3}$$

is an identity for Q_r.

Moreover, for $t_1 = 0$ in (3.3) we have that Q_r satisfies $a[x_1, x_2]\big[a[x_1, x_2], [x_1, x_2]\big]^n$, and by Theorem 1.5 it follows easily that $a \in C$.

Let us first consider the case when α is an inner automorphism of R. Then there exists an invertible element $q \in Q_r$ such that $\alpha(x) = qxq^{-1}$. Since $1 \neq \alpha \in \mathrm{Aut}(R)$, we may assume $q \notin C$. Thus we may write (3.3) as

$$(a[x_1, x_2] + t_1x_2 - qx_2q^{-1}t_1)\big[a[x_1, x_2] + t_1x_2 - qx_2q^{-1}t_1, [x_1, x_2]\big]^n. \tag{3.4}$$

Replace in (3.4) t_1 by qx_1, then it follows that Q_r satisfies

$$(a + q)[x_1, x_2]\big[(a + q)[x_1, x_2], [x_1, x_2]\big]^n$$

and as above we get $a + q \in C$, that is $q \in C$, which is a contradiction.

Finally, we assume that α is outer. By [6] R is a GPI-ring and Q_r is also GPI-ring by [7]. By Martindale's theorem in [21], Q_r is a primitive ring having nonzero socle and its associated division ring D is finite-dimensional over C. Hence Q_r is isomorphic to a dense subring of the ring of linear transformations of a vector space V over D, containing nonzero linear transformations of finite rank.

Moreover, we know that there exists a semi-linear automorphism $T \in \mathrm{End}(V)$ such that $\alpha(x) = TxT^{-1}$ for all $x \in Q_r$. Hence, by (3.3), Q_r satisfies

$$(a[x_1, x_2] + t_1x_2 - Tx_2T^{-1}t_1)\big[a[x_1, x_2] + t_1x_2 - Tx_2T^{-1}t_1, [x_1, x_2]\big]^n. \tag{3.5}$$

Notice that, if for any $v \in V$ there exists $\lambda_v \in D$ such that $T^{-1}v = v\lambda_v$, then, by a standard argument, it follows that there exists a unique $\lambda \in D$ such that $T^{-1}v = v\lambda$ for all $v \in V$. In this case

$$\alpha(x)v = (TxT^{-1})v = Txv\lambda$$

and

$$(\alpha(x) - x)v = T(xv\lambda) - xv = T(T^{-1}xv) - xv = 0,$$

which implies the contradiction that α is the identity map, since V is faithful.

Therefore, there exists $v \in V$ such that $\{v, T^{-1}v\}$ is linearly D-independent.

Consider first the case $\dim_D V \geqslant 3$. Thus there exists $w \in V$ such that $\{w, v, T^{-1}v\}$ is linearly D-independent. Moreover, by the density of Q_r, there exists $r, s, t \in Q_r$ such that

$$rv = sv = tv = v, \quad sT^{-1}bv = T^{-1}w, \quad rw = 0, \quad sw = v.$$

Hence, by (3.5), we get the contradiction

$$0 = (a[r,s] + ts - TsT^{-1}t)\big[a[r,s] + ts - TsT^{-1}t, [r,s]\big]^n v = v - w \neq 0.$$

Therefore, we have just to consider the case when $\dim_D V \leqslant 2$.

In this case, by (3.3), since $a \in C$, $\alpha(x_i)$-word degree is 3 and either $\operatorname{char}(R) = 0$ or $\operatorname{char}(R) \geqslant 5$, it follows by Theorem 3 in [5] that Q_r satisfies

$$(a[x_1, x_2] + t_1x_2 - y_2t_1)\big[t_1x_2 - y_2t_1, [x_1, x_2]\big]^2. \tag{3.6}$$

For $x_1 = e_{12}$, $x_2 = e_{21}$, $t_1 = e_{22}$, $y_2 = e_{12}$ in (3.6) it follows that

$$4(ae_{11} - ae_{22} + e_{21} - e_{12}) = 0$$

and easy computations show that $a = 0$ and $4(e_{21} - e_{12}) = 0$, which is a contradiction. □

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