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Takao Ohno; Tetsu Shimomura

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MUSIELAK-ORLICZ-SOBOLEV SPACES WITH ZERO BOUNDARY VALUES ON METRIC MEASURE SPACES

Takao Ohno, Oita, Tetsu Shimomura, Hiroshima

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Abstract. We define and study Musielak-Orlicz-Sobolev spaces with zero boundary values on any metric space endowed with a Borel regular measure. We extend many classical results, including completeness, lattice properties and removable sets, to Musielak-Orlicz-Sobolev spaces on metric measure spaces. We give sufficient conditions which guarantee that a Sobolev function can be approximated by Lipschitz continuous functions vanishing outside an open set. These conditions are based on Hardy type inequalities.

Keywords: Sobolev space; metric measure space; Hajłasz-Sobolev space; Musielak-Orlicz space; capacity; variable exponent; zero boundary values

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1. Introduction

Sobolev spaces on metric measure spaces have been studied during the last two decades, see [6], [12], [13], [22], [33], etc. The theory was generalized to Orlicz-Sobolev spaces on metric measure spaces in [4], [3], [34]. We refer to [1], [2], [9], [35] for Sobolev spaces on $\mathbb{R}^N$, [7], [8] for variable exponent Sobolev spaces and [31] for Musielak-Orlicz spaces. Variable exponent Sobolev spaces on metric measure spaces have been developed during the past decades (see e.g. [10], [11], [21], [20], [30]).

We recall the definition due to Hajłasz [12] of the first order Sobolev spaces on metric measure spaces. He showed that a p-integrable function u, $1 < p < \infty$, belongs to $W^{1,p}(\mathbb{R}^N)$ if and only if there exists a nonnegative p-integrable function g such that

$$|u(x) - u(y)| \leqslant |x - y|(g(x) + g(y)) \tag{1.1}$$

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for almost every $x, y \in \mathbb{R}^N$. If we replace $|x - y|$ by the distance of the points x and y, (1.1) can be stated in metric measure spaces. Spaces defined by using (1.1) are called Hajłasz-Sobolev spaces. See also [13], [22]. The theory was generalized to Orlicz-Sobolev spaces by Aïssaoui ([4], [3]). Kilpeläinen, Kinnunen and Martio [24] generalized the definition of the first order Sobolev spaces with zero boundary values to an arbitrary metric space endowed with a Borel regular measure. In order to define the first order Sobolev spaces with zero boundary values, the notion of the Sobolev capacity was needed in the metric setting, and the rudiments were established in [28]. In [24], the authors extended many classical results, including completeness, lattice properties and removable sets, to the metric setting. For Newtonian spaces, see e.g. [6], [25], [33], [34].

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth conditions (see [7], [8]). For the Sobolev capacity on variable exponent Sobolev spaces, see [16], [17], [19], etc. Harjulehto, Hästö, Koskenoja and Varonen [18] studied variable exponent Sobolev spaces with zero boundary values in the Euclidean setting. See also [15], [25].

In [20], basic properties of the variable exponent Hajłasz-Sobolev space were studied. Recently, we defined Musielak-Orlicz-Sobolev spaces on metric measure spaces and proved the basic properties of such spaces (see [32]). For example, we showed that Lipschitz continuous functions are dense, as well as other basic properties and studied a related Sobolev type capacity on Musielak-Orlicz-Hajłasz-Sobolev spaces. We also dealt with the boundedness of the Hardy-Littlewood maximal operator on Musielak-Orlicz spaces on metric measure spaces.

In this paper, to develop the theory of Musielak-Orlicz-Sobolev spaces, we study Musielak-Orlicz-Sobolev spaces with zero boundary values on metric measure spaces, as an extension of [18], [24].

The present paper is organized as follows. In Section 2, we define Musielak-Orlicz spaces on metric measure spaces.

In Section 3, we study Sobolev capacity on Musielak-Orlicz-Hajłasz-Sobolev spaces. We give a characterization of the capacity in terms of quasicontinuous functions (see Theorem 3.6), as an extension of [24], Theorem 3.4.

In Section 4, we define Musielak-Orlicz-Hajłasz-Sobolev spaces with zero boundary values on metric measure spaces. We show that the sets of capacity zero are removable in Musielak-Orlicz-Hajłasz-Sobolev spaces with zero boundary values (see Theorem 4.5), as an extension of [24], Theorem 4.6.

In Section 5, we give sufficient conditions which guarantee that a Sobolev function can be approximated by Lipschitz continuous functions vanishing outside an open set (see Theorem 5.1), as an extension of [24], Theorem 5.1. These conditions are based on Hardy type inequalities (see Theorem 5.2).

In Section 6, we discuss Musielak-Orlicz-Sobolev spaces with zero boundary values in the Euclidean setting, as an extension of [18].

2. Musielak-Orlicz spaces

Throughout this paper, let C denote various positive constants independent of the variables in question.

We denote by (X, d, μ) a metric measure space, where X is a set, d is a metric on X and μ is a nonnegative complete Borel regular outer measure on X which is finite in every bounded set. For simplicity, we often write X instead of (X, d, μ). For $x \in X$, $r > 0$ and a set $E \subset X$, we denote by $B(x, r)$ the open ball centered at x with radius r, $d_E = \sup\{d(x, y)\colon\ x,\ y \in E\}$ and $\operatorname{dist}(x, E) = \inf\{d(x, y)\colon\ y \in E\}$.

We say that the measure μ is a doubling measure, if there exists a constant $c_1 > 0$ such that

$$\mu(B(x, 2r)) \leqslant c_1 \mu(B(x, r))$$

for every $x \in X$ and $0 < r < d_X$. A nonempty set $E \subset X$ is uniformly μ-thick if there exist constants $0 < c_2 \leqslant 1$ and $0 < r_0 \leqslant 1$ such that

$$\mu(B(x, r) \cap E) \geqslant c_2 \mu(B(x, r))$$

for every $x \in E$ and $0 < r < r_0$. This condition is often called the measure or regularity condition (see [14]).

We consider a function

$$\Phi(x, t) = t\varphi(x, t)\colon\ X \times [0, \infty) \to [0, \infty)$$

satisfying the following conditions ($\Phi1$)–($\Phi3$):

($\Phi1$) $\varphi(\cdot, t)$ is measurable on X for each $t \geqslant 0$ and $\varphi(x, \cdot)$ is continuous on $[0, \infty)$ for each $x \in X$;

($\Phi2$) there exists a constant $A_1 \geqslant 1$ such that

$$A_1^{-1} \leqslant \varphi(x, 1) \leqslant A_1 \quad \text{for all } x \in X;$$

($\Phi3$) $\varphi(x, \cdot)$ is uniformly almost increasing, namely there exists a constant $A_2 \geqslant 1$ such that

$$\varphi(x, t) \leqslant A_2 \varphi(x, s) \quad \text{for all } x \in X \text{ whenever } 0 \leqslant t < s.$$

Let $\overline{\varphi}(x,t) = \sup_{0\leqslant s\leqslant t} \varphi(x,s)$ and

$$\overline{\Phi}(x,t) = \int_0^t \overline{\varphi}(x,r)\,\mathrm{d}r$$

for $x \in X$ and $t \geqslant 0$. Then $\overline{\Phi}(x,\cdot)$ is convex and

$$\Phi\Big(x, \frac{t}{2}\Big) \leqslant \overline{\Phi}(x,t) \leqslant A_2\Phi(x,t)$$

for all $x \in X$ and $t \geqslant 0$.

We shall also consider the following conditions:

(Φ4) there exists a constant $A_3 > 1$ such that

$$\varphi(x,2t) \leqslant A_3\varphi(x,t) \quad \text{for all } x \in X \text{ and } t > 0;$$

(Φ5) for every $\gamma_1, \gamma_2 > 0$, there exists a constant $B_{\gamma_1,\gamma_2} \geqslant 1$ such that

$$\varphi(x,t) \leqslant B_{\gamma_1,\gamma_2}\varphi(y,t)$$

whenever $d(x,y) \leqslant \gamma_1 t^{-1/\gamma_2}$ and $t \geqslant 1$;

(Φ6) there exist $x_0 \in X$, a function $g \in L^1(X)$ and a constant $B_\infty \geqslant 1$ such that $0 \leqslant g(x) < 1$ for all $x \in X$ and

$$B_\infty^{-1}\Phi(x,t) \leqslant \Phi(x',t) \leqslant B_\infty\Phi(x,t)$$

whenever $d(x',x_0) \geqslant d(x,x_0)$ and $g(x) \leqslant t \leqslant 1$.

Note from (Φ4) that

$$\overline{\Phi}(x,at) \leqslant a^{\log_2 A_3+1}A_3\overline{\Phi}(x,t) \tag{2.1}$$

for all $a \geqslant 1$, $x \in X$ and $t \geqslant 0$. In fact, if we choose a positive integer k such that $2^{k-1} \leqslant a \leqslant 2^k$, then we have by ($\Phi$4)

$$\begin{aligned}\overline{\Phi}(x,at) &= \int_0^{at} \overline{\varphi}(x,r)\,\mathrm{d}r = a\int_0^t \overline{\varphi}(x,ar)\,\mathrm{d}r \\ &\leqslant aA_3^k \int_0^t \overline{\varphi}(x,r)\,\mathrm{d}r = a^{\log_2 A_3+1}A_3\overline{\Phi}(x,t).\end{aligned}$$

Example 2.1. Let $p(\cdot)$ and $q_j(\cdot)$, $j = 1,\ldots,k$, be measurable functions on X such that

(P1) $1 \leqslant p^- := \inf_{x\in X} p(x) \leqslant \sup_{x\in X} p(x) =: p^+ < \infty$

and

(Q1) $-\infty < q_j^- := \inf_{x\in X} q_j(x) \leqslant \sup_{x\in X} q_j(x) =: q_j^+ < \infty$

for all $j = 1, \ldots, k$.

Set $L_c(t) = \log(c+t)$ for $c \geqslant \mathrm{e}$ and $t \geqslant 0$, $L_c^{(1)}(t) = L_c(t)$, $L_c^{(j+1)}(t) = L_c(L_c^{(j)}(t))$ and

$$\Phi(x,t) = t^{p(x)} \prod_{j=1}^{k} (L_c^{(j)}(t))^{q_j(x)}.$$

Then $\Phi(x,t)$ satisfies (Φ1), (Φ2), (Φ3) and (Φ4) if $p(\cdot)$ and $q_j(\cdot)$ satisfy the following condition: for every $x \in X$, there exists a nonnegative integer $0 \leqslant j_0(x) \leqslant k$ such that $q_{j_0(x)}(x) > 0$, $q_j(x) = 0$ whenever $0 \leqslant j < j_0(x)$ and

$$\sup_{x\in X} \max_{j_0(x)<j\leqslant k} \frac{-q_j(x)}{q_{j_0(x)}(x)} < \infty, \tag{2.2}$$

where $q_0(x) = p(x) - 1$. In fact, it is trivial that $\Phi(x,t)$ satisfies (Φ1), (Φ2) and (Φ4). For (Φ3), it is sufficient to prove that

$$\widetilde{\varphi}(x,t) = t^{q_0(x)} \prod_{j=1}^{k} (L_0^{(j)}(t))^{q_j(x)}$$

is uniformly almost increasing on $[\widetilde{c}, \infty)$ for some $\widetilde{c} > 0$. Note that

$$\frac{\mathrm{d}\widetilde{\varphi}(x,t)}{\mathrm{d}t} = t^{q_0(x)-1} \prod_{j=1}^{k} (L_0^{(j)}(t))^{q_j(x)-1} h(x,t),$$

where

$$h(x,t) = \sum_{n=0}^{k-1} \left(q_n(x) \prod_{l=n+1}^{k} L_0^{(l)}(t) \right) + q_k(x).$$

Then (2.2) implies that there exists a constant $\widetilde{c} > 0$ such that $h(x,t) > 0$ for all $t \geqslant \widetilde{c}$, so that $\widetilde{\varphi}(x,t)$ is uniformly almost increasing on $[\widetilde{c}, \infty)$.

Moreover, we see that $\Phi(x,t)$ satisfies (Φ5) if

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \leqslant \frac{C_p}{L_{\mathrm{e}}(1/d(x,y))}$$

with a constant $C_p \geqslant 0$ and

(Q2) $q_j(\cdot)$ is $(j+1)$-log-Hölder continuous, namely

$$|q_j(x) - q_j(y)| \leqslant \frac{C_{q_j}}{L_{\mathrm{e}}^{(j+1)}(1/d(x,y))}$$

with constants $C_{q_j} \geqslant 0$, $j = 1, \ldots, k$.

Fix $x_0 \in X$. Let κ and c be positive constants. If μ satisfies $\mu(B(x_0, r)) \leqslant cr^{\kappa}$ for all $r \geqslant 1$ and

(P3) $p(\cdot)$ is log-Hölder continuous at ∞, namely

$$|p(x') - p(x)| \leqslant \frac{C_\infty}{L_{\mathrm{e}}(d(x,x_0))} \quad \text{for} \quad d(x', x_0) \geqslant d(x, x_0)$$

with a constant $C_\infty \geqslant 0$, then $\Phi(x,t)$ satisfies (Φ6) with $g(x) = 1/(1 + d(x, x_0))^{\kappa+1}$. Here note that if μ is a doubling measure, then $\mu(B(x_0, r)) \leqslant cr^{\kappa}$ for all $r \geqslant 1$ and some κ, $c > 0$.

Example 2.2. Let $p_1(\cdot)$, $p_2(\cdot)$, $q_1(\cdot)$ and $q_2(\cdot)$ be measurable functions on X satisfying (P1) and (Q1). Then,

$$\Phi(x,t) = (1+t)^{p_1(x)} \Big(1 + \frac{1}{t}\Big)^{-p_2(x)} L_c(t)^{q_1(x)} L_c\Big(\frac{1}{t}\Big)^{-q_2(x)}$$

satisfies (Φ1), (Φ2) and (Φ4). It satisfies (Φ3) if $p_j^- > 1$, $j = 1, 2$ or $q_j^- \geqslant 0$, $j = 1, 2$. As a matter of fact, it satisfies (Φ3) if and only if $p_j(\cdot)$ and $q_j(\cdot)$ satisfy the following conditions:

(1) $q_j(x) \geqslant 0$ at points x where $p_j(x) = 1$, $j = 1, 2$;
(2) $\sup_{x\,:\, p_j(x) > 1} \{\min(q_j(x), 0) \log(p_j(x) - 1)\} < \infty$.

Moreover, we see that $\Phi(x,t)$ satisfies (Φ5) if $p_1(\cdot)$ is log-Hölder continuous and $q_1(\cdot)$ is 2-log-Hölder continuous.

Fix $x_0 \in X$. Let κ and c be positive constants. If μ satisfies $\mu(B(x_0, r)) \leqslant cr^{\kappa}$ for all $r \geqslant 1$, $p_2(\cdot)$ satisfies (P3) and

(Q3) $q_2(\cdot)$ is 2-log-Hölder continuous at ∞, namely

$$|q_2(x) - q_2(x')| \leqslant \frac{C_{q_2,\infty}}{L_c^{(2)}(d(x,x_0))} \quad \text{for } d(x', x_0) \geqslant d(x, x_0)$$

with a constant $C_{q_2,\infty} \geqslant 0$,

then $\Phi(\cdot, \cdot)$ satisfies (Φ6) with $g(x) = 1/(1 + d(x, x_0))^{\kappa+1}$.

We say that u is a locally integrable function on X if u is an integrable function on all balls B in X. From now on, we assume that $\Phi(x,t)$ satisfies ($\Phi1$), ($\Phi2$) and ($\Phi3$). The associated Musielak-Orlicz space

$$L^{\Phi}(X) = \Big\{ f \in L^1_{\rm loc}(X)\colon \int_X \Phi\Big(y, \frac{|f(y)|}{\lambda}\Big)\,\mathrm{d}\mu(y) < \infty \text{ for some } \lambda > 0 \Big\}$$

is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}(X)} = \inf \Big\{ \lambda > 0\colon \int_X \overline{\Phi}\Big(y, \frac{|f(y)|}{\lambda}\Big)\,\mathrm{d}\mu(y) \leqslant 1 \Big\}$$

(cf. [31]). Note that if $\Phi(x,t)$ satisfies ($\Phi1$) and ($\Phi2$), then $\|\cdot\|_{L^{\Phi}(X)}$ is a norm and $L^{\Phi}(X)$ is complete. If $\Phi(x,t)$ satisfies ($\Phi1$), ($\Phi2$) and ($\Phi3$), then the Luxemburg norm with $\Phi(x,t)$ instead of $\overline{\Phi}(x,t)$ gives a quasinorm, these (quasi)norms are equivalent, and $L^{\Phi}(X)$ contains simple functions. We also note that if $\Phi(x,t)$ satisfies ($\Phi1$), then $L^{\Phi}(X)$ is a lattice.

For a measurable function f on X, we define the modular $\varrho_{\Phi}(f)$ by

$$\varrho_{\Phi}(f) = \int_X \overline{\Phi}(y, |f(y)|)\,\mathrm{d}\mu(y).$$

Remark 2.3. Let f, f_n and g be measurable functions in X. Then note that the following statements hold.

(1) If $\Phi(x,t)$ satisfies ($\Phi1$) and ($\Phi4$), then $\|f\|_{L^{\Phi}(X)} < \infty$ if and only if $\varrho_{\Phi}(f) < \infty$, and $\|\cdot\|_{L^{\Phi}(X)}$ is absolutely continuous.

(2) If $\Phi(x,t)$ satisfies ($\Phi1$), ($\Phi2$), ($\Phi3$) and ($\Phi4$) and the conjugate function of $\Phi(x,\cdot)$ satisfies ($\Phi4$), then $L^{\Phi}(X)$ is reflexive ([5], Corollary 4.4, and [8], Corollary 2.7.18).

(3) Let $0 \leqslant f_n \nearrow f$ μ-a.e. in X. If $\Phi(x,t)$ satisfies ($\Phi1$), then $\|f_n\|_{L^{\Phi}(X)} \nearrow \|f\|_{L^{\Phi}(X)}$. It follows from this fact that if $0 \leqslant f \leqslant g$ μ-a.e. in X, then $\|f\|_{L^{\Phi}(X)} \leqslant \|g\|_{L^{\Phi}(X)}$.

(4) Let E be a measurable set in X with $\mu(E) < \infty$. If $\Phi(x,t)$ satisfies ($\Phi1$), then there exists a constant $C_E > 0$ such that

$$\int_E f(x)\,\mathrm{d}\mu(x) \leqslant C_E \|f\|_{L^{\Phi}(X)}.$$

(5) If $\Phi(x,t)$ satisfies ($\Phi1$), ($\Phi2$) and ($\Phi3$), then $L^{\Phi}(X)$ is a Banach function space (see [5], Definition 1.3).

Lemma 2.4 ([31], Theorem 1.6)**.** *Let $\{f_i\}$ be a sequence in $L^{\Phi}(X)$. Then $\varrho_{\Phi}(\lambda f_i)$ converges to* 0 *for any $\lambda > 0$ if and only if $\|f_i\|_{L^{\Phi}(X)}$ converges to* 0.

3. Sobolev capacity on Musielak-Orlicz-Hajłasz-Sobolev spaces $M^{1,\Phi}(X)$

We say that a function $u \in L^{\Phi}(X)$ belongs to Musielak-Orlicz-Hajłasz-Sobolev space $M^{1,\Phi}(X)$ if there exists a nonnegative function $g \in L^{\Phi}(X)$ such that

$$|u(x) - u(y)| \leqslant d(x,y)(g(x) + g(y))$$

for μ-almost every $x, y \in X$. Here, we call the function g a Hajłasz gradient of u. We define the norm

$$\|u\|_{M^{1,\Phi}(X)} = \|u\|_{L^{\Phi}(X)} + \inf \|g\|_{L^{\Phi}(X)},$$

where the infimum is taken over all Hajłasz gradients of u. For the case when $\Phi(x,t) = t^p$, the spaces $M^{1,p}(X)$ were first introduced by Hajłasz [12] as a generalization of the classical Sobolev spaces $W^{1,p}(\mathbb{R}^N)$ to the general setting of the quasi-metric measure spaces. For variable exponent spaces $M^{1,p(\cdot)}(X)$, see [20].

For $u \in M^{1,\Phi}(X)$, we define

$$\widetilde{\varrho}_{\Phi}(u) = \varrho_{\Phi}(u) + \inf \varrho_{\Phi}(g),$$

where the infimum is taken over all Hajłasz gradients of u.

Remark 3.1. For all $0 < \varepsilon < 1$, $\|u\|_{M^{1,\Phi}(X)} < \varepsilon$ implies $\widetilde{\varrho}_{\Phi}(u) < \varepsilon$ due to the convexity of $\overline{\Phi}$.

For $E \subset X$, we denote

$$S_{\Phi}(E) = \{u \in M^{1,\Phi}(X)\colon\ u \geqslant 1 \text{ in an open set containing } E\}.$$

The Sobolev capacity in Musielak-Orlicz-Hajłasz-Sobolev spaces is defined by

$$C_{\Phi}(E) = \inf_{u \in S_{\Phi}(E)} \widetilde{\varrho}_{\Phi}(u).$$

In case $S_{\Phi}(E) = \emptyset$, we set $C_{\Phi}(E) = \infty$.

Remark 3.2. Suppose $\Phi(x,t)$ satisfies (Φ4). Then since $u \in M^{1,\Phi}(X)$ if and only if $\widetilde{\varrho}_{\Phi}(u) < \infty$, note that

$$C_{\Phi}(E) = \inf_{u \in \widehat{S}_{\Phi}(E)} \widetilde{\varrho}_{\Phi}(u),$$

where

$$\widehat{S}_\Phi(E) = \{u \text{ measurable on } X\colon\ u \geqslant 1 \text{ in an open set containing } E\}.$$

Remark 3.3. We can redefine the Sobolev capacity in Musielak-Orlicz-Hajłasz-Sobolev spaces by

$$C_\Phi(E) = \inf_{u \in S'_\Phi(E)} \widetilde{\varrho}_\Phi(u)$$

since $M^{1,\Phi}(X)$ is a lattice (see [28], Lemma 2.4 and Remark 3.1), where

$$S'_\Phi(E) = \{u \in S_\Phi(X)\colon\ 0 \leqslant u \leqslant 1\}.$$

For the Sobolev capacity in Musielak-Orlicz-Hajłasz-Sobolev spaces the following results hold.

Lemma 3.4 ([20], Theorem 3.11, [28], Theorem 3.2, Remark 3.3 and Lemma 3.4, and [27], Theorem 4.1)**.** *The set function $C_\Phi(\cdot)$ satisfies the following conditions:*

(1) *$C_\Phi(\cdot)$ is an outer measure.*
(2) *$C_\Phi(E) = \inf\limits_{E \subset U,\, U\colon \text{open}} C_\Phi(U)$ for $E \subset X$ ($C_\Phi(\cdot)$ is an outer capacity).*
(3) *If $K_1 \supset K_2 \supset \ldots$ are compact sets on X, then*

$$\lim_{i\to\infty} C_\Phi(K_i) = C_\Phi\Bigl(\bigcap_{i=1}^{\infty} K_i\Bigr).$$

(4) *If $L^\Phi(X)$ is reflexive and $E_1 \subset E_2 \subset \ldots$ are subsets of X, then*

$$\lim_{i\to\infty} C_\Phi(E_i) = C_\Phi\Bigl(\bigcup_{i=1}^{\infty} E_i\Bigr).$$

We say that a property holds C_Φ-q.e. (quasi everywhere) in X, if it holds except of a set $F \subset X$ with $C_\Phi(F) = 0$. A function u is C_Φ-quasicontinuous on X if, for any $\varepsilon > 0$, there is an open set E such that $C_\Phi(E) < \varepsilon$ and $u|_{X\setminus E}$ is continuous.

Proposition 3.5 ([24], Theorem 3.2 and Remark 3.3)**.** *Let u and v be C_Φ-quasicontinuous on an open set $O \subset X$.*

(1) *If $u = v$ μ-a.e. in O, then $u = v$ C_Φ-q.e. in O.*
(2) *If $u \leqslant v$ μ-a.e. in O, then $u \leqslant v$ C_Φ-q.e. in O.*

In fact, we can prove (1) by [23], since $C_\Phi(\cdot)$ is an outer capacity and satisfies the compatibility condition: if $O \subset X$ is an open set and $E \subset X$ is a set such that $\mu(E) = 0$, then $C_\Phi(O) = C_\Phi(O \setminus E)$. As in [24], Remark 3.3, we obtain (2) by (1).

Next, we consider a Sobolev capacity in Musielak-Orlicz-Hajłasz-Sobolev spaces in terms of C_Φ-quasicontinuous functions. For $E \subset X$, we denote

$$\widetilde{S}_\Phi(E) = \{u \in M^{1,\Phi}(X)\colon\ u \text{ is } C_\Phi\text{-quasicontinuous and } u \geqslant 1\ C_\Phi\text{-q.e. in } E\}.$$

We define

$$\widetilde{C}_\Phi(E) = \inf_{u \in \widetilde{S}_\Phi(E)} \widetilde{\varrho}_\Phi(u).$$

In case $\widetilde{S}_\Phi(E) = \emptyset$, we set $\widetilde{C}_\Phi(E) = \infty$.

Theorem 3.6 ([24], Theorem 3.4)**.** *Let $E \subset X$.*

(1) $C_\Phi(E) \leqslant \widetilde{C}_\Phi(E)$.

(2) *If continuous functions are dense in $M^{1,\Phi}(X)$, then $\widetilde{C}_\Phi(E) = C_\Phi(E)$.*

P r o o f. First, we prove (1). Let $v \in \widetilde{S}_\Phi(E)$. Then we may assume that $0 \leqslant v \leqslant 1$ as in Remark 3.3. For $0 < \varepsilon < 1$, there exists an open set $V \subset X$ such that $C_\Phi(V) < \varepsilon$, $v = 1$ on $E \setminus V$ and $v|_{X\setminus V}$ is continuous by Lemma 3.4 (2). Since $v|_{X\setminus V}$ is continuous, there is an open set $U \subset X$ such that

$$U \setminus V = \{x \in X \setminus V\colon\ v(x) > 1 - \varepsilon\} = \{x \in X\colon\ v(x) > 1 - \varepsilon\} \setminus V.$$

Note that $E \setminus V \subset U \setminus V$. Since $C_\Phi(V) < \varepsilon$, we can take $u \in S_\Phi(V)$ such that $\widetilde{\varrho}_\Phi(u) < \varepsilon$, $u = 1$ on V and $0 \leqslant u \leqslant 1$ by Remark 3.3. We define $w = \max\{v/(1-\varepsilon), u\}$. Then $w \geqslant 1$ on $(U \setminus V) \cup V = U \cup V$, which is an open neighbourhood of E and hence $w \in S_\Phi(E)$. We have

$$\varrho_\Phi(w) \leqslant \varrho_\Phi\Big(\frac{v}{1-\varepsilon}\Big) + \varrho_\Phi(u) \leqslant \varrho_\Phi\Big(\frac{v}{1-\varepsilon}\Big) + \varepsilon \to \varrho_\Phi(v)$$

as $\varepsilon \to 0$. Similarly, we see that $\max\{g/(1-\varepsilon), h\}$ is a Hajłasz gradients of w and

$$\varrho_\Phi\Big(\max\Big\{\frac{g}{1-\varepsilon}, h\Big\}\Big) \leqslant \varrho_\Phi\Big(\frac{g}{1-\varepsilon}\Big) + \varrho_\Phi(h) \leqslant \varrho_\Phi\Big(\frac{g}{1-\varepsilon}\Big) + \varepsilon \to \varrho_\Phi(g)$$

as $\varepsilon \to 0$, where h, g are Hajłasz gradients of u, v with $\varrho_\Phi(h) < \varepsilon$, respectively. Hence we obtain $C_\Phi(E) \leqslant \widetilde{C}_\Phi(E)$.

Next, to prove (2), we show the inequality $\widetilde{C}_\Phi(E) \leqslant C_\Phi(E)$. Take $u \in S_\Phi(E)$. Then there exists an open set $E \subset O$ such that $u \geqslant 1$ on O. Since [32], Proposition 3.10, holds by our assumption, there exists a C_Φ-quasicontinuous function $v \in M^{1,\Phi}(X)$ such that $v = u$ μ-a.e. in X, so that $v \geqslant 1$ μ-a.e. in O. Then we see from Proposition 3.5 (2) that $v \geqslant 1$ C_Φ-q.e. in O. Hence $v \in \widetilde{S}_\Phi(E)$, so that $\widetilde{C}_\Phi(E) \leqslant C_\Phi(E)$. □

Remark 3.7. If $\Phi(x,t)$ satisfies (Φ4), then continuous functions are dense in $M^{1,\Phi}(X)$ (see [32], Proposition 3.4).

Lemma 3.8 ([24], Lemma 3.5). *Suppose $\{u_i\} \in M^{1,\Phi}(X)$ is a sequence of C_Φ-quasicontinuous functions on X such that u_i converges to u in $M^{1,\Phi}(X)$. Then there exist $\widetilde{u} \in M^{1,\Phi}(X)$ and a subsequence of $\{u_i\}$ such that $\widetilde{u}$ is a C_Φ-quasicontinuous function on X, $\widetilde{u} = u$ μ-a.e. in X and a subsequence of $\{u_i\}$ converges pointwise to $\widetilde{u}$ C_Φ-q.e. in X.*

Proof. We can take a subsequence of $\{u_i\}$, which we denote again by $\{u_i\}$, such that $\|u_i - u_{i+1}\|_{M^{1,\Phi}(X)} \leqslant 4^{-i}$ for each positive integer i. Then note that $\widetilde{\varrho}_\Phi(2^i|u_i - u_{i+1}|) \leqslant 2^{-i}$. Consider the sets

$$E_i = \{x \in X\colon\ |u_i(x) - u_{i+1}(x)| > 2^{-i}\}$$

and $F_j = \bigcup\limits_{i=j}^{\infty} E_i$. Since $2^i|u_i - u_{i+1}| \in \widetilde{S}(E_i)$ by C_Φ-quasicontinuity of u_i, we have by Theorem 3.6 and (2.1) that

$$C_\Phi(E_i) \leqslant \widetilde{\varrho}_\Phi(2^i|u_i - u_{i+1}|) \leqslant 2^{-i}.$$

Then it follows from Lemma 3.4 (1) that

$$C_\Phi(F_j) \leqslant \sum_{i=j}^{\infty} C_\Phi(E_i) \leqslant 2^{-j+1},$$

so that

$$C_\Phi\biggl(\bigcap_{j=1}^{\infty} F_j\biggr) \leqslant \lim_{j\to\infty} C_\Phi(F_j) = 0.$$

Hence there exists $\widetilde{u} \in M^{1,\Phi}(X)$ such that $\widetilde{u} = u$ μ-a.e. in X and $\{u_i\}$ converges pointwise to $\widetilde{u}$ C_Φ-q.e. in X.

Next, we show that $\widetilde{u}$ is a C_Φ-quasicontinuous function on X. For $\varepsilon > 0$, there is a set F_j such that $C_\Phi(F_j) < \varepsilon/2$ and $\{u_i\}$ uniformly converges to $\widetilde{u}$ in $X \setminus F_j$. Since $\{u_i\}$ is a sequence of C_Φ-quasicontinuous functions on X, there exists an open set $G_i \subset X$ such that $C_\Phi(G_i) < \varepsilon/2^{i+1}$ and $u_i|_{X\setminus G_i}$ is continuous. Set $G = \bigcup\limits_{i=1}^{\infty} G_i$. Then we see from Lemma 3.4 (1) that

$$C_\Phi(G) \leqslant \sum_{i=1}^{\infty} C_\Phi(G_i) < \frac{\varepsilon}{2}$$

and

$$C_\Phi(F_j \cup G) \leqslant C_\Phi(F_j) + C_\Phi(G) < \varepsilon.$$

Since $\{u_i\}$ uniformly converges to $\widetilde{u}$ in $X \setminus (F_j \cup G)$, we conclude that $\widetilde{u}$ is a C_Φ-quasicontinuous function on X. $\square$

4. Musielak-Orlicz-Hajłasz-Sobolev spaces with zero boundary values

Let E be a subset of X. We say that u belongs to the Musielak-Orlicz-Hajłasz-Sobolev space with zero boundary values and write $u \in M_0^{1,\Phi}(E)$ if there is a C_Φ-quasicontinuous function $\widetilde{u} \in M^{1,\Phi}(X)$ such that $\widetilde{u} = u$ μ-a.e. in E and $\widetilde{u} = 0$ C_Φ-q.e. in $X \setminus E$. The space $M_0^{1,\Phi}(E)$ is endowed with the norm

$$\|u\|_{M_0^{1,\Phi}(E)} = \|\widetilde{u}\|_{M^{1,\Phi}(X)}.$$

By [32], Lemma 3.11, it follows that the norm does not depend on the choice of the C_Φ-quasicontinuous representative. Since $M^{1,\Phi}(X)$ is a linear space, so is $M_0^{1,\Phi}(E)$.

Theorem 4.1 ([24], Theorem 4.1)**.** *Let $E \subset X$. Then $M_0^{1,\Phi}(E)$ is a Banach space.*

P r o o f. Let $\{u_i\}$ be a Cauchy sequence in $M_0^{1,\Phi}(E)$. Then for every u_i there exists a C_Φ-quasicontinuous function $\widetilde{u}_i \in M^{1,\Phi}(X)$ such that $\widetilde{u}_i = u_i$ μ-a.e. in E and $\widetilde{u}_i = 0$ C_Φ-q.e. in $X \setminus E$. By [32], Proposition 3.3, there exists $u \in M^{1,\Phi}(X)$ such that $\widetilde{u}_i$ converge to u in $M^{1,\Phi}(X)$. Lemma 3.8 yields that there exist $\widetilde{u} \in M^{1,\Phi}(X)$ and a subsequence of $\{\widetilde{u}_i\}$ such that $\widetilde{u}$ is a C_Φ-quasicontinuous function on X, $\widetilde{u} = u$ μ-a.e. in X and a subsequence of $\{\widetilde{u}_i\}$ converges pointwise to $\widetilde{u}$ C_Φ-q.e. in X. This shows that $\widetilde{u} = 0$ C_Φ-q.e. in $X \setminus E$, so that $u \in M_0^{1,\Phi}(E)$. Thus the theorem is proved. $\square$

By straightforward arguments, we obtain the following lattice properties.

Lemma 4.2 ([24], Theorem 4.3)**.** *Let $E \subset X$ and let u, $v \in M_0^{1,\Phi}(E)$.*

(1) *If $\lambda \geqslant 0$, then $\min(u,\lambda) \in M_0^{1,\Phi}(E)$ and $\|\min(u,\lambda)\|_{M_0^{1,\Phi}(E)} \leqslant \|u\|_{M_0^{1,\Phi}(E)}$.*
(2) *If $\lambda \leqslant 0$, then $\max(u,\lambda) \in M_0^{1,\Phi}(E)$ and $\|\max(u,\lambda)\|_{M_0^{1,\Phi}(E)} \leqslant \|u\|_{M_0^{1,\Phi}(E)}$.*
(3) *$|u| \in M_0^{1,\Phi}(E)$ and $\||u|\|_{M_0^{1,\Phi}(E)} \leqslant \|u\|_{M_0^{1,\Phi}(E)}$.*
(4) *$\min(u,v) \in M_0^{1,\Phi}(E)$.*
(5) *$\max(u,v) \in M_0^{1,\Phi}(E)$.*

Lemma 4.3 ([24], Theorem 4.5)**.** *Let $E \subset X$. Suppose $u \in M_0^{1,\Phi}(E)$ and $v \in M^{1,\Phi}(X)$ are bounded functions. If v is C_Φ-quasicontinuous in X, then $uv \in M_0^{1,\Phi}(E)$.*

P r o o f. Since $u \in M_0^{1,\Phi}(E)$, there exists a C_Φ-quasicontinuous function $\widetilde{u} \in M^{1,\Phi}(X)$ such that $\widetilde{u} = u$ μ-a.e. in E and $\widetilde{u} = 0$ C_Φ-q.e. in $X \setminus E$. Here note that a Hajłasz gradient of $\widetilde{u}v$ is included in $L^\Phi(X)$ since u and v are bounded functions. Therefore, $\widetilde{u}v \in M^{1,\Phi}(X)$ is C_Φ-quasicontinuous in X and may be nonzero outside E in a set $A \cup B$, where

$$A = \{x \in X \setminus E\colon\ \widetilde{u}(x) \neq 0\} \quad \text{and} \quad B = \{x \in X \setminus E\colon\ v(x) = \infty\}.$$

Noting that $C_\Phi(A) = C_\Phi(B) = 0$, we have $C_\Phi(A \cup B) = 0$ in view of Lemma 3.4 (1). Hence $\widetilde{u}v = 0$ C_Φ-q.e. in $X \setminus E$. Since $\widetilde{u}v = uv$ μ-a.e. in E, we see that $uv \in M_0^{1,\Phi}(E)$. $\square$

As in the proof of [24], Theorem 4.4, we have the following result.

Proposition 4.4 ([24], Theorem 4.4)**.** *Let E be a μ-measurable set in X. Assume that continuous functions are dense in $M^{1,\Phi}(X)$. Suppose $u \in M_0^{1,\Phi}(E)$ and $v \in M^{1,\Phi}(E)$. If $|v| \leqslant u$ μ-a.e. in E, then $v \in M_0^{1,\Phi}(E)$.*

We will show that the sets of capacity zero are removable in Musielak-Orlicz-Hajłasz-Sobolev spaces with zero boundary values.

Theorem 4.5 ([24], Theorems 4.6 and 4.8)**.** *Let $E \subset X$ be open and let $N \subset X$.*

(1) *If $C_\Phi(N \cap E) = 0$, then $M_0^{1,\Phi}(E) = M_0^{1,\Phi}(E \setminus N)$.*
(2) *If $\mu(N) = 0$ and $M_0^{1,\Phi}(E) = M_0^{1,\Phi}(E \setminus N)$, then $C_\Phi(N \cap E) = 0$.*

P r o o f. First we show the case (1). Since $C_\Phi(N \cap E) = 0$, we have $\mu(N \cap E) = 0$ by [32], Lemma 3.11. Hence $M_0^{1,\Phi}(E \setminus N) \subset M_0^{1,\Phi}(E)$. If $u \in M_0^{1,\Phi}(E)$, then there exists a C_Φ-quasicontinuous function $\widetilde{u} \in M^{1,\Phi}(X)$ such that $\widetilde{u} = u$ μ-a.e. in E and $\widetilde{u} = 0$ C_Φ-q.e. in $X \setminus E$. Since $C_\Phi(N \cap E) = 0$, we see that $\widetilde{u} = 0$ C_Φ-q.e. in $X \setminus (E \setminus N)$. This implies $M_0^{1,\Phi}(E) \subset M_0^{1,\Phi}(E \setminus N)$.

Next we show the case (2). We may assume that $N \subset E$. Let $x_0 \in E$ and set

$$E_i = B(x_0, i) \cap \Big\{x \in E\colon\ \operatorname{dist}(x, X \setminus E) > \frac{1}{i}\Big\}$$

for all positive integers i. Define $u_i(x) = \max(0, 1 - \operatorname{dist}(x, N \cap E_i))$ for $x \in X$. Then we see that $u_i \in M^{1,\Phi}(X)$ is C_Φ-quasicontinuous on X, $0 \leqslant u_i \leqslant 1$ and $u_i = 1$ on $N \cap E_i$. Define $v_i(x) = \operatorname{dist}(x, X \setminus E_i)$ for $x \in X$. Then we find that $w_i = u_i v_i \in M_0^{1,\Phi}(E) = M_0^{1,\Phi}(E \setminus N)$ by Lemma 4.3 since $v_i \in M_0^{1,\Phi}(E_i) \subset M_0^{1,\Phi}(E)$. Since $w_i \in M_0^{1,\Phi}(E \setminus N)$, there exists a C_Φ-quasicontinuous function $\widetilde{w}_i \in M^{1,\Phi}(X)$ such that $\widetilde{w}_i = w_i$ μ-a.e. in $E \setminus N$ and $\widetilde{w}_i = 0$ C_Φ-q.e. in $X \setminus (E \setminus N)$. By $\mu(N) = 0$, we have $\widetilde{w}_i = w_i$ μ-a.e. in E, so that Proposition 3.5 implies $\widetilde{w}_i = w_i$ C_Φ-q.e. in E. In particular, $\widetilde{w}_i = w_i > 0$ C_Φ-q.e. in $N \cap E_i$. On the other hand, since $\widetilde{w}_i = 0$ C_Φ-q.e. in $X \setminus (E \setminus N)$, we have $\widetilde{w}_i = 0$ C_Φ-q.e. in $N \cap E_i$. This is possible only if $C_\Phi(N \cap E_i) = 0$, so that we have by Lemma 3.4 (1)

$$C_\Phi(N) \leqslant \sum_{i=1}^{\infty} C_\Phi(N \cap E_i) = 0,$$

as required. □

5. Equivalence of function spaces

Our aim in this section is to describe $M_0^{1,\Phi}(E)$ as the completion of

$$\mathrm{Lip}_0^{1,\Phi}(E) = \{u \in M^{1,\Phi}(X)\colon\ u \text{ is Lipschitz in } X \text{ and } u = 0 \text{ in } X \setminus E\}$$

in the norm $\|\cdot\|_{M^{1,\Phi}(X)}$. By [32], Proposition 3.3, this completion is the closure of $\mathrm{Lip}_0^{1,\Phi}(E)$ in $M^{1,\Phi}(X)$. We denote it by $H_0^{1,\Phi}(E)$.

Theorem 5.1 ([24], Theorem 5.1)**.** *Assume that $\Phi(x,t)$ satisfies $(\Phi 4)$. Let $E \subset X$ be open and let $u \in M^{1,\Phi}(E)$. If*

$$\frac{u(x)}{\mathrm{dist}(x, X \setminus E)} \in L^\Phi(E),$$

then $u \in H_0^{1,\Phi}(E)$.

Proof. Let $g \in L^\Phi(E)$ be a Hajłasz gradient of u and define

$$\overline{g}(x) = \begin{cases} \max\Big(g(x), \dfrac{|u(x)|}{\mathrm{dist}(x, X \setminus E)}\Big) & x \in E, \\ 0 & x \in X \setminus E. \end{cases}$$

Let $\overline{u}$ be the zero extension of u to $X \setminus E$. As in the proof of [24], Theorem 5.1, there exists a set $N \subset E$ such that $\mu(N) = 0$ and

$$|\overline{u}(x) - \overline{u}(y)| \leqslant d(x,y)(\overline{g}(x) + \overline{g}(y))$$

for all $x, y \in X \setminus N$. Hence $\overline{g} \in L^\Phi(X)$ is a Hajłasz gradient of $\overline{u}$. Thus $\overline{u} \in M^{1,\Phi}(X)$.

Set

$$F_\lambda = \{x \in E \setminus N\colon\ |\overline{u}(x)| \leqslant \lambda, \overline{g}(x) \leqslant \lambda\} \cup (X \setminus E)$$

for $\lambda \geqslant 1$. Since $\overline{u}|_{F_\lambda}$ is 2λ-Lipschitz continuous, we extend it to a 2λ-Lipschitz continuous function on X using the McShane extension

$$\overline{u}_\lambda(x) = \inf\{\overline{u}(y) + 2\lambda d(x, y)\colon\ y \in F_\lambda\}.$$

Further, set

$$u_\lambda(x) = \min(\max(\overline{u}_\lambda(x), -\lambda), \lambda).$$

By the same arguments as in [32] Proposition 3.4, we obtain that $u_\lambda \in \mathrm{Lip}_0^{1,\Phi}(E)$ and u_λ converges to $\overline{u}$ in $M^{1,\Phi}(X)$. Thus $\overline{u} \in H_0^{1,\Phi}(E)$. $\square$

We shall give a condition for the open set E such that the assumptions of Theorem 5.1 hold for every $u \in M_0^{1,\Phi}(E)$. Recall that a nonempty set $E \subset X$ is uniformly μ-thick if there exist constants $0 < c_2 \leqslant 1$ and $0 < r_0 \leqslant 1$ such that

$$\mu(B(x, r) \cap E) \geqslant c_2 \mu(B(x, r))$$

for every $x \in E$ and $0 < r < r_0$.

For a locally integrable function u on X, the Hardy-Littlewood maximal function Mu is defined by

$$Mu(x) = \sup_{r>0} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} |u(y)| \,\mathrm{d}\mu(y).$$

Theorem 5.2 ([24], Theorem 5.6). *Suppose μ is a doubling measure and the Hardy-Littlewood maximal operator is bounded on $L^\Phi(X)$. Let $E \subset X$ be an open set such that $X \setminus E$ is uniformly μ-thick. Then there exists a constant $C^* > 0$ such that*

$$\int_E \overline{\Phi}\Big(x, \frac{|u(x)|}{C^* \operatorname{dist}(x, X \setminus E)}\Big) \,\mathrm{d}\mu(x) \leqslant 1$$

for all $u \in M_0^{1,\Phi}(E)$ with $\|u\|_{M_0^{1,\Phi}(E)} \leqslant 1$.

Proof. Since $u \in M_0^{1,\Phi}(E)$, there exist a function $\widetilde{u} \in M^{1,\Phi}(X)$ and a set $N \subset X$ such that $\mu(N) = 0$, $\widetilde{u}$ is a C_Φ-quasicontinuous function on X, $u = \widetilde{u}$ in $E \setminus N$, $\widetilde{u} = 0$, C_Φ-q.e. in $X \setminus E$ and $\|\widetilde{u}\|_{M^{1,\Phi}(X)} \leqslant 1$. Let $g \in L^\Phi(X)$ be a Hajłasz gradient of $\widetilde{u}$ with $\|g\|_{L^\Phi(X)} \leqslant 1$. Set

$$E_0 = \{x \in E\colon\ \operatorname{dist}(x, X \setminus E) < r_0\}.$$

Fix $x \in E_0$ and let n be a positive integer. Choose $x_n \in X \setminus E$ such that $r_n = d(x, x_n) < r_0$ and $r_n \leqslant (1 + 1/n) \operatorname{dist}(x, X \setminus E)$. Then, for all positive integers n, we have by the uniform μ-thickness and the doubling condition

$$\begin{aligned}
\frac{1}{\mu(B(x_n, r_n) \setminus E)} \int_{B(x_n, r_n)\setminus E} g(y) \,\mathrm{d}\mu(y) &\leqslant \frac{1}{c_2 \mu(B(x_n, r_n))} \int_{B(x_n, r_n)} g(y) \,\mathrm{d}\mu(y) \\
&\leqslant \frac{c_1^2}{c_2 \mu(B(x, 2r_n))} \int_{B(x, 2r_n)} g(y) \,\mathrm{d}\mu(y) \\
&\leqslant \frac{c_1^2}{c_2} Mg(x)
\end{aligned}$$

for $x \in E_0$. Here note that for μ-a.e. $\xi \in B(x_n, r_n) \setminus E$,

$$g(\xi) > \frac{1}{\mu(B(x_n, r_n) \setminus E)} \int_{B(x_n, r_n)\setminus E} g(y) \,\mathrm{d}\mu(y)$$

does not hold, so that there exists $z_n \in B(x_n, r_n) \setminus E$ such that $\widetilde{u}(z_n) = 0$,

$$|u(x)| = |\widetilde{u}(x) - \widetilde{u}(z_n)| \leqslant d(x, z_n)(g(x) + g(z_n)) \leqslant 2r_n(g(x) + g(z_n))$$

for all $x \in E_0 \setminus N$ and

$$g(z_n) \leqslant \frac{1}{\mu(B(x_n, r_n) \setminus E)} \int_{B(x_n, r_n)\setminus E} g(y) \,\mathrm{d}\mu(y).$$

Therefore, we obtain for all $x \in E_0 \setminus N$

$$|u(x)| \leqslant \frac{2c_1^2}{c_2} r_n(g(x) + Mg(x)) \leqslant \frac{4c_1^2}{c_2}\Big(1 + \frac{1}{n}\Big) \operatorname{dist}(x, X \setminus E) Mg(x).$$

Letting $n \to \infty$, we have for all $x \in E_0 \setminus N$

$$|u(x)| \leqslant \frac{4c_1^2}{c_2} \operatorname{dist}(x, X \setminus E) Mg(x).$$

Since the Hardy-Littlewood maximal operator is bounded on $L^\Phi(X)$, that is, there exists a constant $c_M \geqslant 1$ such that $\|Mg\|_{L^\Phi(X)} \leqslant c_M \|g\|_{L^\Phi(X)}$, we have

$$\int_{E_0} \overline{\Phi}\Big(x, \frac{c_2 |u(x)|}{8c_1^2 c_M \operatorname{dist}(x, X \setminus E)}\Big) \,\mathrm{d}\mu(x) \leqslant \frac{1}{2} \int_{E_0} \overline{\Phi}\Big(x, \frac{Mg(x)}{c_M}\Big) \,\mathrm{d}\mu(x) \leqslant \frac{1}{2}.$$

On the other hand, we have

$$\int_{E\setminus E_0} \overline{\Phi}\Big(x, \frac{r_0 |u(x)|}{2 \operatorname{dist}(x, X \setminus E)}\Big) \,\mathrm{d}\mu(x) \leqslant \frac{1}{2} \int_{E\setminus E_0} \overline{\Phi}(x, |u(x)|) \,\mathrm{d}\mu(x) \leqslant \frac{1}{2}.$$

Consequently, for $C^* = \max\{8c_1^2 c_M/c_2, 2/r_0\}$, we have

$$\int_E \overline{\Phi}\Big(x, \frac{|u(x)|}{C^* \operatorname{dist}(x, X \setminus E)}\Big) \, \mathrm{d}\mu(x) \leqslant 1,$$

as required. $\square$

By Theorems 5.1 and 5.2, we obtain the following corollary.

Corollary 5.3. *Assume that $\Phi(x,t)$ satisfies ($\Phi4$). Suppose μ is a doubling measure and the Hardy-Littlewood maximal operator is bounded on $L^{\Phi}(X)$. Let $E \subset X$ be an open set such that $X \setminus E$ is uniformly μ-thick. Then $M_0^{1,\Phi}(E) = H_0^{1,\Phi}(E)$.*

As in the proof of [26], Theorem 4.5, we can show the following result by [32], Lemma 3.13.

Lemma 5.4. *Assume that $\Phi(x,t)$ satisfies ($\Phi4$). Let μ be a doubling measure and let $u \in M^{1,\Phi}(X)$. Suppose the Hardy-Littlewood maximal operator is bounded on $L^{\Phi}(X)$. Then*

$$\widetilde{u}(x) = \lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u(y) \, \mathrm{d}\mu(y)$$

for C_Φ-q.e. in X, where $\widetilde{u}$ is the C_Φ-quasicontinuous representative of u.

By Lemma 5.4, we can show the following characterization of Musielak-Orlicz-Hajłasz-Sobolev spaces with zero boundary values (see [25], Theorem 2.8).

Proposition 5.5. *Assume that $\Phi(x,t)$ satisfies ($\Phi4$). Let μ be a doubling measure and let $u \in M^{1,\Phi}(X)$. Let $E \subset X$. Suppose the Hardy-Littlewood maximal operator is bounded on $L^{\Phi}(X)$ and*

$$\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u(y) \, \mathrm{d}\mu(y) = 0$$

for C_Φ-q.e. in $X \setminus E$. Then $u \in M_0^{1,\Phi}(E)$.

P r o o f. Let $u \in M^{1,\Phi}(X)$ and let $\widetilde{u}$ be a C_Φ-quasicontinuous representative of u. By Lemma 5.4, we have

$$\widetilde{u}(x) = \lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u(y) \, \mathrm{d}\mu(y) = 0$$

for C_Φ-q.e. in $X \setminus E$. Hence $u \in M_0^{1,\Phi}(E)$. $\square$

A.1. Sobolev capacity on Musielak-Orlicz-Sobolev spaces $W^{1,\Phi}(\mathbb{R}^N)$. Let $\mathbb{R}^N$ be an N-dimensional Euclidean space. In the case $X = \mathbb{R}^N$, let μ be the Lebesgue measure on $\mathbb{R}^N$ and let d be the Euclidean metric. We define the Musielak-Orlicz-Sobolev space $W^{1,\Phi}(\mathbb{R}^N)$ by

$$W^{1,\Phi}(\mathbb{R}^N) = \{u \in L^{\Phi}(\mathbb{R}^N) \colon\ |\nabla u| \in L^{\Phi}(\mathbb{R}^N)\}.$$

The norm

$$\|u\|_{W^{1,\Phi}(\mathbb{R}^N)} = \|u\|_{L^{\Phi}(\mathbb{R}^N)} + \||\nabla u|\|_{L^{\Phi}(\mathbb{R}^N)}$$

makes $W^{1,\Phi}(\mathbb{R}^N)$ a Banach space. We know the following result.

Lemma A.1 ([32], Proposition 5.1)**.** $M^{1,\Phi}(\mathbb{R}^N) \subset W^{1,\Phi}(\mathbb{R}^N)$. *Moreover, if the Hardy-Littlewood maximal operator is bounded on* $L^{\Phi}(\mathbb{R}^N)$, *then* $M^{1,\Phi}(\mathbb{R}^N) = W^{1,\Phi}(\mathbb{R}^N)$.

Remark A.2. Let $N = 2$. Then there exists a function $\Phi(x,t)$ on $B(0,1) \times [0,\infty)$ such that $\Phi(x,t)$ satisfies ($\Phi1$), ($\Phi2$), ($\Phi3$) and ($\Phi4$) and $M^{1,\Phi}(B(0,1)) \neq W^{1,\Phi}(B(0,1))$. In fact, there exists a function $\Phi(x,t)$ on $B(0,1) \times [0,\infty)$ such that $\Phi(x,t)$ satisfies ($\Phi1$), ($\Phi2$), ($\Phi3$) and ($\Phi4$) and smooth functions are not dense in $W^{1,\Phi}(B(0,1))$ (see [8], Example 9.2.6). By [32], Proposition 3.4, and the fact that $M^{1,\Phi}(B(0,1)) \hookrightarrow W^{1,\Phi}(B(0,1))$, $M^{1,\Phi}(B(0,1))$ is included in the closure of Lipschitz functions in $W^{1,\Phi}(B(0,1))$. However, since smooth functions are not dense in $W^{1,\Phi}(B(0,1))$, we obtain $M^{1,\Phi}(B(0,1)) \neq W^{1,\Phi}(B(0,1))$.

For $u \in W^{1,\Phi}(\mathbb{R}^N)$, we define

$$\breve{\varrho}_{\Phi}(u) = \varrho_{\Phi}(u) + \varrho_{\Phi}(\nabla u).$$

For $E \subset \mathbb{R}^N$, we denote

$$s_{\Phi}(E) = \{u \in W^{1,\Phi}(\mathbb{R}^N) \colon\ u \geqslant 1 \text{ in an open set containing } E\}.$$

The Musielak-Orlicz-Sobolev c_{Φ}-capacity is defined by

$$c_{\Phi}(E) = \inf_{u \in s_{\Phi}(E)} \breve{\varrho}_{\Phi}(u).$$

In case $s_{\Phi}(E) = \emptyset$, we set $c_{\Phi}(E) = \infty$. For the Sobolev capacity in Musielak-Orlicz-Sobolev spaces the following results hold.

Lemma A.3 ([19], Theorems 3.1 and 3.2)**.** *The set function $c_\Phi(\cdot)$ satisfies the following conditions:*

(1) $c_\Phi(\emptyset) = 0$;

(2) *if* $E_1 \subset E_2 \subset \mathbb{R}^N$, *then* $c_\Phi(E_1) \leqslant c_\Phi(E_2)$;

(3) $c_\Phi(\cdot)$ *is an outer capacity;*

(4) *for* E_1, $E_2 \subset \mathbb{R}^N$,

$$c_\Phi(E_1 \cup E_2) + c_\Phi(E_1 \cap E_2) \leqslant c_\Phi(E_1) + c_\Phi(E_2);$$

(5) *if* $K_1 \supset K_2 \supset \ldots$ *are compact sets on* $\mathbb{R}^N$, *then*

$$\lim_{i\to\infty} c_\Phi(K_i) = c_\Phi\left(\bigcap_{i=1}^{\infty} K_i\right);$$

(6) *if* $L^\Phi(\mathbb{R}^N)$ *is reflexive and* $E_1 \subset E_2 \subset \ldots$ *are subsets of* $\mathbb{R}^N$, *then*

$$\lim_{i\to\infty} c_\Phi(E_i) = c_\Phi\left(\bigcup_{i=1}^{\infty} E_i\right);$$

(7) *if* $E_i \subset \mathbb{R}^N$ *for* $i = 1, 2, \ldots$, *then*

$$c_\Phi\left(\bigcup_{i=1}^{\infty} E_i\right) \leqslant \sum_{i=1}^{\infty} c_\Phi(E_i).$$

P r o o f. We prove only (7). We may assume that $\sum\limits_{i=1}^{\infty} c_\Phi(E_i) < \infty$. Then, for $\varepsilon > 0$, we can take $u_i \in s_\Phi(E_i)$ such that

$$\breve{\varrho}_\Phi(u_i) \leqslant c_\Phi(E_i) + 2^{-i}\varepsilon.$$

Set $v = \sup\limits_{1\leqslant i<\infty} u_i$ and $h = \sup\limits_{1\leqslant i<\infty} |\nabla u_i|$. Then note that

$$\varrho_\Phi(v) \leqslant \int_{\mathbb{R}^N} \sum_{i=1}^{\infty} \overline{\Phi}(x, |u_i(x)|)\,\mathrm{d}x \leqslant \sum_{i=1}^{\infty}(c_\Phi(E_i) + 2^{-i}\varepsilon) = \sum_{i=1}^{\infty} c_\Phi(E_i) + \varepsilon < \infty$$

and

$$\varrho_\Phi(h) \leqslant \sum_{i=1}^{\infty} \varrho_\Phi(|\nabla u_i(x)|) \leqslant \sum_{i=1}^{\infty} c_\Phi(E_i) + \varepsilon < \infty,$$

so that v, $h \in L^{\Phi}(\mathbb{R}^N)$. By [9], Lemma 2 (iii), Section 4.7, we have $|\nabla v| \leqslant h$ a.e. in $\mathbb{R}^N$. Hence, since $v \in s_\Phi\Bigl(\bigcup\limits_{i=1}^{\infty} E_i\Bigr)$, we conclude

$$c_\Phi\biggl(\bigcup_{i=1}^{\infty} E_i\biggr) \leqslant \breve{\varrho}_\Phi(v) \leqslant \sum_{i=1}^{\infty} \breve{\varrho}_\Phi(u_i) \leqslant \sum_{i=1}^{\infty} c_\Phi(E_i) + \varepsilon,$$

as required. □

We say that a property holds c_Φ-q.e. in $\mathbb{R}^N$ if it holds except of a set $F \subset \mathbb{R}^N$ with $c_\Phi(F) = 0$, and a function $u\colon\ \mathbb{R}^N \to \mathbb{R}$ is c_Φ-quasicontinuous if for every $\varepsilon > 0$ there exists an open set E with $c_\Phi(E) < \varepsilon$ such that u restricted to $\mathbb{R}^N \setminus E$ is continuous.

Lemma A.4 ([19], Theorem 5.2). *Suppose continuous functions are dense in $W^{1,\Phi}(\mathbb{R}^N)$. Then $u \in W^{1,\Phi}(\mathbb{R}^N)$ has a c_Φ-quasicontinuous representative.*

Remark A.5. By [29], Theorem 3.5, we know that C^∞-functions are dense in $W^{1,\Phi}(\mathbb{R}^N)$ if $\Phi(x,t)$ satisfies (Φ4), (Φ5) and (Φ6).

We can show the following result by [23].

Proposition A.6 ([18], Lemma 2.1). *Let u and v be c_Φ-quasicontinuous on an open set $O \subset \mathbb{R}^N$.*

(1) *If $u = v$ μ-a.e. in O, then $u = v$ c_Φ-q.e. in O.*
(2) *If $u \leqslant v$ μ-a.e. in O, then $u \leqslant v$ c_Φ-q.e. in O.*

Next, we consider a Sobolev capacity in Musielak-Orlicz-Sobolev spaces in terms of c_Φ-quasicontinuous functions. For $E \subset \mathbb{R}^N$, we denote

$$\widetilde{s}_\Phi(E) = \{u \in W^{1,\Phi}(\mathbb{R}^N)\colon\ u \text{ is } c_\Phi\text{-quasicontinuous and } u \geqslant 1 \ c_\Phi\text{-q.e. in } E\}.$$

We define

$$\widetilde{c}_\Phi(E) = \inf_{u \in \widetilde{s}_\Phi(E)} \breve{\varrho}_\Phi(u).$$

In case $\widetilde{s}_\Phi(E) = \emptyset$, we set $\widetilde{c}_\Phi(E) = \infty$.

As in the proof of Theorem 3.6, we have the following result.

Theorem A.7 ([18], Theorem 2.2). *Let $E \subset \mathbb{R}^N$.*

(1) $c_\Phi(E) \leqslant \widetilde{c}_\Phi(E)$.
(2) *If continuous functions are dense in $W^{1,\Phi}(\mathbb{R}^N)$, then $\widetilde{c}_\Phi(E) = c_\Phi(E)$.*

As in the proof of Lemma 3.8, we have the following lemma.

Lemma A.8 ([18], Lemma 2.3)**.** *Suppose $\{u_i\} \in W^{1,\Phi}(\mathbb{R}^N)$ is a sequence of c_Φ-quasicontinuous functions on $\mathbb{R}^N$ such that u_i converge to u in $W^{1,\Phi}(\mathbb{R}^N)$. Then there exist $\widetilde{u} \in W^{1,\Phi}(\mathbb{R}^N)$ and a subsequence of $\{u_i\}$ such that $\widetilde{u}$ is a c_Φ-quasicontinuous function on $\mathbb{R}^N$, $\widetilde{u} = u$ μ-a.e. in $\mathbb{R}^N$, and a subsequence of $\{u_i\}$ converges pointwise to $\widetilde{u}$ c_Φ-q.e. in $\mathbb{R}^N$.*

A.2. Musielak-Orlicz-Sobolev spaces with zero boundary values. Let E be a subset of $\mathbb{R}^N$. We say that u belongs to the Musielak-Orlicz-Sobolev space with zero boundary values and write $u \in W_0^{1,\Phi}(E)$ if there is a c_Φ-quasicontinuous function $\widetilde{u} \in W^{1,\Phi}(\mathbb{R}^N)$ such that $\widetilde{u} = u$ μ-a.e. in E and $\widetilde{u} = 0$ c_Φ-q.e. in $\mathbb{R}^N \setminus E$. The space $W_0^{1,\Phi}(E)$ is endowed with the norm

$$\|u\|_{W_0^{1,\Phi}(E)} = \|\widetilde{u}\|_{W^{1,\Phi}(\mathbb{R}^N)}.$$

It follows that the norm does not depend on the choice of the c_Φ-quasicontinuous representative. Since $W^{1,\Phi}(\mathbb{R}^N)$ is a linear space, so is $W_0^{1,\Phi}(E)$.

As in the proof of Theorem 4.1, we have the following result.

Theorem A.9 ([18], Theorem 3.1)**.** *Let $E \subset \mathbb{R}^N$. Then $W_0^{1,\Phi}(E)$ is a Banach space.*

As in the proof of Lemma 4.3, we have the following result by Lemma A.3 (4).

Lemma A.10 ([18], Lemma 3.6)**.** *Let $E \subset \mathbb{R}^N$. Suppose $u \in W_0^{1,\Phi}(E)$ and $v \in W^{1,\Phi}(\mathbb{R}^N)$ are bounded functions. If v is a c_Φ-quasicontinuous function on $\mathbb{R}^N$, then $uv \in W_0^{1,\Phi}(E)$.*

As in the proof of Theorem 4.5, we have the following result.

Theorem A.11 ([18], Theorem 3.7)**.** *Let $E \subset \mathbb{R}^N$ be open and let $N \subset \mathbb{R}^N$.*

(1) *If $c_\Phi(N \cap E) = 0$, then $W_0^{1,\Phi}(E) = W_0^{1,\Phi}(E \setminus N)$.*
(2) *If $\mu(N) = 0$, $W_0^{1,\Phi}(E) = W_0^{1,\Phi}(E \setminus N)$, then $c_\Phi(N \cap E) = 0$.*

A.3. Equivalence of function spaces. Let E be a subset of $\mathbb{R}^N$. By $D_0^{1,\Phi}(E)$ we denote the closure of $C_0^\infty(E)$ in the space $W^{1,\Phi}(E)$. By Theorem A.9, we have the following result.

Lemma A.12 ([18], Corollary 3.2)**.** *Let $E \subset \mathbb{R}^N$. Then $D_0^{1,\Phi}(E) \subset W_0^{1,\Phi}(E) \subset W^{1,\Phi}(E)$.*

Theorem A.13 ([18], Theorem 3.3). *Let $E \subset \mathbb{R}^N$. Assume that $\Phi(x,t)$ satisfies (Φ4). If continuous functions are dense in $W^{1,\Phi}(\mathbb{R}^N)$. Then $D_0^{1,\Phi}(E) = W_0^{1,\Phi}(E)$.*

Proof. By Lemma A.12, it is sufficient to show $D_0^{1,\Phi}(E) \supset W_0^{1,\Phi}(E)$. Let $u \in W_0^{1,\Phi}(E)$. Then there is a c_Φ-quasicontinuous function $\widetilde{u} \in W^{1,\Phi}(\mathbb{R}^N)$ such that $\widetilde{u} = u$ μ-a.e. in E and $\widetilde{u} = 0$ c_Φ-q.e. in $\mathbb{R}^N \setminus E$. As in the proof of [18], Theorem 3.3, we may assume that $\widetilde{u}$ is a nonnegative and bounded function with compact support.

For $\varepsilon > 0$, we set $\widetilde{u}_\varepsilon = \max\{\widetilde{u} - \varepsilon, 0\}$. Since $\widetilde{u}$ is a c_Φ-quasicontinuous function and $\widetilde{u} = 0$ c_Φ-q.e. in $\mathbb{R}^N \setminus E$, there exist an open set G and $\delta > 0$ such that $\widetilde{u}|_{\mathbb{R}^N \setminus G}$ is continuous, $\widetilde{u} = 0$ in $(\mathbb{R}^N \setminus E) \setminus G$ and $c_\Phi(G) < \delta$ by Lemma A.3 (3). Then there exists a function $w \in W^{1,\Phi}(\mathbb{R}^N)$ such that $0 \leqslant w \leqslant 1, w|_G = 1$ and $\breve{\varrho}_\Phi(w) < \delta$. Define $v = (1-w)\widetilde{u}_\varepsilon$. Here note that v vanishes in a neighborhood of $\mathbb{R}^N \setminus E$. Further, we find that

$$\|\widetilde{u} - v\|_{W^{1,\Phi}(\mathbb{R}^N)} \leqslant \|\widetilde{u} - \widetilde{u}_\varepsilon\|_{W^{1,\Phi}(\mathbb{R}^N)} + \|w\widetilde{u}_\varepsilon\|_{W^{1,\Phi}(\mathbb{R}^N)}.$$

Since

$$\|\widetilde{u} - \widetilde{u}_\varepsilon\|_{W^{1,\Phi}(\mathbb{R}^N)} \leqslant \varepsilon\|\chi_{\operatorname{spt}\widetilde{u}}\|_{L^\Phi(\mathbb{R}^N)} + \|\chi_{\{0<\widetilde{u}\leqslant\varepsilon\}}\nabla\widetilde{u}\|_{L^\Phi(\mathbb{R}^N)},$$

we have $\|\widetilde{u} - \widetilde{u}_\varepsilon\|_{W^{1,\Phi}(\mathbb{R}^N)} \to 0$ as $\varepsilon \to 0$. We also find that

$$\begin{aligned}
\breve{\varrho}_\Phi(w\widetilde{u}_\varepsilon) \leqslant & \int_{\mathbb{R}^N} \overline{\Phi}(x, w(x)\widetilde{u}(x))\,\mathrm{d}\mu(x) \\
& + 2A_3^2\Big(\int_{\mathbb{R}^N} \overline{\Phi}(x, w(x)|\nabla\widetilde{u}(x)|)\,\mathrm{d}\mu(x) + \int_{\mathbb{R}^N} \overline{\Phi}(x, |\nabla w(x)|\widetilde{u}(x))\,\mathrm{d}\mu(x)\Big) \\
\leqslant & A_3(2A_3^2+1)\delta\max\Big\{\sup_{x\in\mathbb{R}^N}\widetilde{u}(x)^{\log_2 A_3+1}, 1\Big\} \\
& + 2A_3^2\int_{\mathbb{R}^N} \overline{\Phi}(x, w(x)|\nabla\widetilde{u}(x)|)\,\mathrm{d}\mu(x)
\end{aligned}$$

by (2.1). Since w converges to 0 in $L^\Phi(\mathbb{R}^N)$ as $\delta \to 0$, we can choose a sequence w_i which tends to 0 pointwise a.e. Hence, by the dominated converge theorem, we have $\breve{\varrho}_\Phi(w\widetilde{u}_\varepsilon) \to 0$ as $\delta \to 0$ and so also $\|w\widetilde{u}_\varepsilon\|_{W^{1,\Phi}(\mathbb{R}^N)} \to 0$ as $\delta \to 0$ by Lemma 2.4. Thus we see that v converges to $\widetilde{u}$ in $W^{1,\Phi}(\mathbb{R}^N)$ as $\varepsilon, \delta \to 0$.

Let $\varphi_i \in C^\infty(\mathbb{R}^N)$ be functions in $W^{1,\Phi}(\mathbb{R}^N)$ which tend to v. Let $\psi \in C_0^\infty(E)$ be a function satisfying $\psi = 1$ on $\operatorname{spt} v$ since v vanishes in a neighborhood of $\mathbb{R}^N \setminus E$. Then, since there exists a constant $M \geqslant 1$ such that $|\psi(x)| \leqslant M$ for all $x \in \mathbb{R}^N$, we have by (2.1)

$$\varrho_\Phi(v - \psi\varphi_i) = \int_{\operatorname{spt} v} \overline{\Phi}(x, |v(x) - \varphi_i(x)|)\,\mathrm{d}\mu(x) + \int_{\mathbb{R}^N\setminus\operatorname{spt} v} \overline{\Phi}(x, |\psi(x)\varphi_i(x)|)\,\mathrm{d}\mu(x)$$

$$
\begin{aligned}
&\leqslant \int_{\operatorname{spt} v} \overline{\Phi}(x, |v(x) - \varphi_i(x)|)\, \mathrm{d}\mu(x) \\
&\quad + A_3 M^{\log_2 A_3 + 1} \int_{\mathbb{R}^N \setminus \operatorname{spt} v} \overline{\Phi}(x, |\varphi_i(x)|)\, \mathrm{d}\mu(x).
\end{aligned}
$$

Since φ_i tend to v in $W^{1,\Phi}(\mathbb{R}^N)$ and φ_i converge to $v = 0$ μ-a.e. in $\mathbb{R}^N \setminus \operatorname{spt} v$, we conclude $\varrho_\Phi(v - \psi\varphi_i) \to 0$ as $i \to \infty$. Similarly, we see that $\varrho_\Phi(\nabla(v - \psi\varphi_i)) \to 0$ as $i \to \infty$. Thus we obtain the required result by Lemma 2.4. $\square$

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Authors' addresses: Takao Ohno, Faculty of Education and Welfare Science, Oita University, 700 Dannoharu, Oita-city, 870-1192, Oita, Japan, e-mail: `t-ohno@oita-u.ac.jp`; Tetsu Shimomura, Department of Mathematics, Graduate School of Education, Hiroshima University, 1-1-1, Kagamiyama, Higashi-Hiroshima 739-8524, Hiroshima, Japan, e-mail: `tshimo@hiroshima-u.ac.jp`.