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AUTOMORPHISMS AND GENERALIZED SKEW DERIVATIONS WHICH ARE STRONG COMMUTATIVITY PRESERVING ON POLYNOMIALS IN PRIME AND SEMIPRIME RINGS

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Abstract. Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring and C its extended centroid. Suppose that F, G are generalized skew derivations of R with the same associated automorphism α, and $p(x_1,\dots,x_n)$ is a non-central polynomial over C such that

$$[F(x),\alpha(y)]=G([x,y])$$

for all $x,y\in\{p(r_1,\dots,r_n)\colon r_1,\dots,r_n\in R\}$. Then there exists $\lambda\in C$ such that $F(x)=G(x)=\lambda\alpha(x)$ for all $x\in R$.

Keywords: generalized skew derivation; prime ring

MSC 2010: 16W25, 16N60

1. Introduction

Let R be a prime ring of characteristic different from 2. Throughout this paper $Z(R)$ always denotes the center of R, Q_r the right Martindale quotient ring of R and $C=Z(Q_r)$ the center of Q_r (C is usually called the extended centroid of R). An additive map $G\colon R\to R$ is called the generalized derivation of R if there exists a derivation d of R such that $G(xy)=G(x)y+xd(y)$ for all $x,y\in R$.

Let α be an automorphism of R. An additive mapping $d\colon R\to R$ is called a *skew derivation* of R if

$$d(xy)=d(x)y+\alpha(x)d(y)$$

for all $x,y\in R$ and α is called the *associated automorphism of* d. An additive mapping $G\colon R\to R$ is said to be a *generalized skew derivation of* R if there exists a skew derivation d of R with an associated automorphism α such that

$$G(xy) = G(x)y + \alpha(x)d(y)$$

for all $x, y \in R$; d is said to be an *associated skew derivation* of G and α is called an *associated automorphism* of G. Any mapping of R in the form $G(x) = ax + \alpha(x)b$ for some $a, b \in R$ and $\alpha \in \mathrm{Aut}(R)$ is called an *inner generalized skew derivation.* In particular, if $a = -b$, then G is called *inner skew derivation.* If a generalized skew derivation (or a skew derivation) is not inner, then it is usually called *outer.*

In light of the above definitions, one can see that the concept of the generalized skew derivation unifies the notions of the skew derivation and the generalized derivation.

In this paper we study the structure of the prime ring R and the form of generalized skew derivations satisfying the strong commutativity preserving conditions. Specifically, if $S \subseteq R$, the map $F\colon R \to R$ is called commutativity preserving on S if $[x, y] = 0$ implies $[F(x), F(y)] = 0$; it is called strong commutativity preserving (SCP) on S if $[F(x), F(y)] = [x, y]$ for all $x, y \in S$.

Additive mapping preserving commutativity was studied by Bresar and Miers [2]. They showed that any additive mapping F which is SCP on a semiprime ring R is of the form $F(x) = \lambda x + \mu(x)$, where $\lambda \in C$, $\lambda^2 = 1$ and $\mu\colon R \to C$ is an additive map of R into C.

Recently in [18] Lin and Liu extended this result to Lie ideals, in case the ring R is prime. More precisely, they proved that if L is a non-central Lie ideal of R and F is an additive mapping satisfying $[F(x), F(y)] - [x, y] \in C$ for all $x, y \in L$, then $F(x) = \lambda x + \mu(x)$, where $\lambda \in C$, $\lambda^2 = 1$ and $\mu\colon R \to C$, unless when $\mathrm{char}(R) = 2$ and R satisfies the standard identity s_4 of degree 4.

In some recent papers many authors study generalized derivations which are SCP on some subsets of a prime and semiprime ring. In [21] Ma, Xiu and Niu described the structure of a generalized derivation which is SCP on one-sided ideals of a prime ring. More precisely, in case of a prime ring R with a right ideal I, in [21] it is proved that any generalized derivation F which is SCP on I is of the form $F(x) = ax + xb$ for all $x \in R$, with $aI = (0)$.

This last cited result is extended in [19] by Liu to prime rings. He studied the case when I is a right ideal of R, $F\colon I \to R$ is a map and G is a generalized derivation of R such that $[F(x), G(y)] = [x, y]$ for all $x, y \in I$ and obtained the complete description of F and G, and also the description of the action of F, G on I. Moreover, in Theorem 1.3 of [19] the case when both F and G are generalized derivations of R is analysed.

Finally, in [20] Liu and Liau studied the case when L is a non-central Lie ideal of R, $F\colon L \to R$ is a map and G is a generalized derivation of R such that $[F(x), G(y)] =$

$[x, y]$ for all $x, y \in L$. They proved that either $R \subseteq M_2(K)$ for a field K, or there exist $0 \neq \alpha \in C$ and a map $\mu\colon L \to C$ such that $G(x) = \alpha x$ for all $x \in R$ and $F(x) = \alpha^{-1}x + \mu(x)$ for all $x \in L$. In particular, if F is also a generalized derivation of R, then $F(x) = \alpha^{-1}x$ for all $x \in R$.

In light of all the previous cited papers, one natural question could be whether the results obtained for two SCP additive maps can be extended to the case when there exist three additive maps $f, g, h\colon R \to R$ such that $[f(x), g(y)] = h([x, y])$ for all $x, y \in S$, where S is a suitable subset of R.

Here we consider the case that S is the set of all the evaluations of a non-central polynomial, f and h are generalized skew derivations of R and g is their associated automorphism. We prove that $f = h = \lambda g$ for a fixed element $\lambda \in C$.

It is well known that automorphisms, derivations and skew derivations of R can be extended to Q_r. In [3] Chang extends the definition of the generalized skew derivation to the right Martindale quotient ring Q_r of R as follows: by a (right) generalized skew derivation we mean an additive mapping $G\colon Q_r \to Q_r$ such that $G(xy) = G(x)y + \alpha(x)d(y)$ for all $x, y \in Q$, where d is a skew derivation of R and α is an automorphism of R. Moreover, there exists $G(1) = a \in Q_r$ such that $G(x) = ax + d(x)$ for all $x \in R$.

The main result of this article is:

Theorem 1.1. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring and C its extended centroid. Suppose that F, G are generalized skew derivations of R, with the same associated automorphism α, and $p(x_1, \ldots, x_n)$ is a non-central polynomial over C such that*

$$[F(x), \alpha(y)] = G([x, y])$$

for all $x, y \in \{p(r_1, \ldots, r_n)\colon r_1, \ldots, r_n \in R\}$. Then there exists $\lambda \in C$ such that $F(x) = G(x) = \lambda\alpha(x)$ for all $x \in R$.

In the last section of the paper, we apply Theorem 1.1 and obtain some results for skew derivations preserving commutativity in semiprime rings.

We now fix some notation and collect some existing results which will be used in the sequel.

Let us denote by $\mathrm{SDer}(Q_r)$ the set of all skew-derivations of Q_r. By a *skew-derivation word* we mean an additive mapping Δ of the form $\Delta = d_1 d_1 \ldots d_m$, where $d_i \in \mathrm{SDer}(Q_r)$. A *skew-differential polynomial* is a generalized polynomial with coefficients in Q of the form $\Phi(\Delta_j(x_i))$ involving noncommutative indeterminates x_i on which the derivation words Δ_j act as unary operations. The skew-differential polynomial $\Phi(\Delta_j(x_i))$ is said to be a *skew-differential identity* on a subset T of Q_r if it vanishes on any assignment of values from T to its indeterminates x_i.

Let R be a prime ring, $\mathrm{SD}_{\mathrm{int}}$ the C-subspace of $\mathrm{SDer}(Q_r)$ consisting of all inner skew-derivations of Q_r, and let d and δ be two nonzero skew-derivations of Q_r.

We will make frequent and important use of the following facts which follow from results in [4]–[7].

Fact 1.2. If d and δ are C-linearly dependent modulo $\mathrm{SD}_{\mathrm{int}}$, then there exist $\lambda, \mu \in C$, $a \in Q_r$ and $\alpha \in \mathrm{Aut}(Q_r)$ such that $\lambda d(x) + \mu\delta(x) = ax - \alpha(x)a$ for all $x \in R$.

Fact 1.3. Let d and δ be skew derivations of R associated with the same automorphism α. If d and δ are C-linearly independent modulo $\mathrm{SD}_{\mathrm{int}}$ and $\Phi(\Delta_j(x_i))$ is a skew-differential identity on R, where $\Delta_j \in \{\delta, d\}$, then $\Phi(y_{ji})$ is a generalized polynomial identity of R, where y_{ji} are distinct indeterminates.

In particular, we have

Fact 1.4. In [9] Chuang and Lee investigate polynomial identities with skew derivations. They prove that if $\Phi(x_i, D(x_i))$ is a generalized polynomial identity for R, where R is a prime ring and D is an outer skew derivation of R, then R also satisfies the generalized polynomial identity $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates.

Fact 1.5. Let R be a prime ring and I a two-sided ideal of R. Then I, R, and Q_r satisfy the same generalized polynomial identities with coefficients in Q_r (see [7]). Furthermore, I, R, and Q_r satisfy the same generalized polynomial identities with automorphisms (see [5], Theorem 1).

2. The case of inner generalized skew derivations

In this section we will prove the following:

Proposition 2.1. *Let R be a non-commutative prime ring of characteristic different from 2, Q_r its right Martindale quotient ring and C its extended centroid. Suppose that F, G are inner generalized skew derivations of R defined, respectively, as follows:*

$$F(x) = ax + qxq^{-1}b, \quad G(x) = cx + qxq^{-1}u$$

for all $x \in R$ and suitable fixed $a, b, c, q, u \in Q_r$, with an invertible element q of Q_r. If

$$[F(x), \alpha(y)] = G([x, y])$$

for all $x, y \in [R, R]$, then one of the following holds:

(1) $a, b, c, u, q \in C$, $(a+b) = (c+u)$;

(2) $q^{-1}a, q^{-1}c \in C$ *and there exists* $\lambda \in C$ *such that* $b = \lambda - a$, $u = \lambda - c$.

Proposition 2.2. *Let R be a non-commutative prime ring of characteristic different from 2, Q_r its right Martindale quotient ring and C its extended centroid. Suppose that F, G are inner generalized skew derivations of R, with an associated automorphism α, defined, respectively, as follows:*

$$F(x) = ax + \alpha(x)b, \quad G(x) = cx + \alpha(x)u$$

for all $x \in R$ and suitable fixed $a, b, c, u \in Q_r$. If

$$[F(x), \alpha(y)] = G([x, y])$$

for all $x, y \in [R, R]$, then there exists $\lambda \in C$ such that $F(x) = \lambda\alpha(x)$ for all $x \in R$.

We always assume that R satisfies the generalized polynomial identity

$$\begin{aligned}\Psi(x_1, x_2, y_1, y_2) = {} & [a[x_1, x_2] + q[x_1, x_2]q^{-1}b, q[y_1, y_2]q^{-1}] \\ & - c[[x_1, x_2], [y_1, y_2]] - q[[x_1, x_2], [y_1, y_2]]q^{-1}u.\end{aligned} \tag{2.1}$$

We begin with the following facts:

Fact 2.3 (Lemma 1.5 in [11]). Let H be an infinite field and $n \geqslant 2$. If $A_1, \ldots, A_k$ are non scalar matrices in $M_m(H)$ then there exists an invertible matrix $P \in M_m(H)$ such that each matrix $PA_1P^{-1}, \ldots, PA_kP^{-1}$ has all nonzero entries.

Fact 2.4. Let F be a generalized derivation of R and let $\gamma \in Z(R)$ be such that $F(x) - \gamma x \in Z(R)$ for all $x \in [R, R]$. If R is not commutative, then $F(x) = \gamma x$ for all $x \in R$.

Fact 2.5. Let $a, b \in R$ and let be $F(x) = ax + xb$ for all $x \in R$. If there exists $\gamma \in Z(R)$ such that $F(x) = \gamma x$ for all $x \in R$, then $a, b \in Z(R)$ and $a + b = \gamma$.

Fact 2.6. Let $a, b \in R$ and $F(x) = ax + xb$ for all $x \in R$. If $F(x) = 0$ for all $x \in R$, then $a, b \in Z(R)$ and $a + b = 0$.

Lemma 2.7. *Let $R = M_m(C)$ be the algebra of $m \times m$ matrices over C, $Z(R)$ the center of R, a, b, c elements of R. If R satisfies*

$$[a[x_1, x_2] + [x_1, x_2]b, [y_1, y_2]] - c[[x_1, x_2], [y_1, y_2]] - [[x_1, x_2], [y_1, y_2]]u \tag{2.2}$$

then $a, b, c, u \in C$ and $(a+b) = (c+u)$.

Proof. Let e_{ij} be the usual matrix unit, with 1 as the (i,j)-entry and zero elsewhere. For any $i \neq j$ and $[x_1,x_2] = [e_{ii}, e_{ij}] = e_{ij}$, $[y_1,y_2] = [e_{ij}, e_{ji}] = e_{ii} - e_{jj}$ in (2.2), we have

$$[ae_{ij} + e_{ij}b, e_{ii} - e_{jj}] - c[e_{ij}, e_{ii} - e_{jj}] - [e_{ij}, e_{ii} - e_{jj}]u = 0. \tag{2.3}$$

Left multiplying (2.3) by e_{jj}, we get $-2e_{jj}ce_{ij} = 0$. Analogously, right multiplying (2.3) by e_{ii}, we have $-2e_{ij}ue_{ii} = 0$ for all $i \neq j$. Therefore both c and u are diagonal matrices in R. In this case, the standard argument shows that both $c \in Z(R)$ and $u \in Z(R)$.

Hence R satisfies

$$[a[x_1,x_2] + [x_1,x_2]b - \lambda[x_1,x_2], [y_1,y_2]] \tag{2.4}$$

where $\lambda = c + u$. Therefore, for any $x_1, x_2 \in R$, $a[x_1,x_2] + [x_1,x_2]b - \lambda[x_1,x_2]$ centralizes $[R,R]$, that is $a[x_1,x_2] + [x_1,x_2]b - \lambda[x_1,x_2] \in Z(R)$ for all $x_1, x_2 \in R$. Thus the conclusion follows from Facts 2.4 and 2.5. $\square$

Lemma 2.8. *Let $R = M_m(C)$ be the algebra of $m \times m$ matrices over C, $Z(R)$ the center of R, a,b,c,q elements of R. If R satisfies $\Psi(x_1,x_2,y_1,y_2)$ and if $q^{-1}a \in Z(R)$, then $q^{-1}c \in Z(R)$ and there exists $\lambda \in Z(R)$ such that $b = \lambda - a$, $u = \lambda - c$.*

Proof. Let $i \neq j$ and $[x_1,x_2] = [y_1,y_2] = [e_{ii}, e_{ij}] = e_{ij}$ in (2.1), then

$$[ae_{ij} + qe_{ij}q^{-1}b, qe_{ij}q^{-1}] = 0; \tag{2.5}$$

moreover, by $q^{-1}a = \nu \in Z(R)$, it is easy to see that $aq^{-1} \in Z(R)$.

Multiplying (2.5) by q^{-1} and using $q^{-1}a \in Z(R)$, it follows that $e_{ij}ae_{ij}q^{-1} + e_{ij}q^{-1}bqe_{ij}q^{-1} = 0$, that is $e_{ij}ae_{ij} + e_{ij}q^{-1}bqe_{ij} = 0$ for all $i \neq j$. Therefore $a + q^{-1}bq$ is a diagonal matrix in R and, as above, $a + q^{-1}bq = \lambda \in Z(R)$. Since $a = \nu q$, it follows that $\nu q + q^{-1}bq = \lambda \in Z(R)$. Both right multiplying by q^{-1} and left multiplying by q, we get $\nu q + b = \lambda$, which means $a + b = \lambda$.

Since $\Psi(r_1,r_2,s_1,s_2) = 0$ for all $r_1,r_2,s_1,s_2 \in R$, we have $q^{-1}\Psi(r_1,r_2,s_1,s_2)q = 0$ for all $r_1,r_2,s_1,s_2 \in R$, that R satisfies $q^{-1}\Psi(x_1,x_2,y_1,y_2)q$. By computation and using $q^{-1}bq = \lambda - a$, it follows that

$$\lambda[[x_1,x_2],[y_1,y_2]] - q^{-1}c[[x_1,x_2],[y_1,y_2]]q - [[x_1,x_2],[y_1,y_2]]q^{-1}uq \tag{2.6}$$

is satisfied by R. In particular, for $[x_1,x_2] = e_{ii} - e_{jj}$ and $[y_1,y_2] = e_{ij}$ in (2.6), left multiplying by e_{jj}, we get $e_{jj}q^{-1}ce_{ij}q = 0$, that is $e_{jj}q^{-1}ce_{ij} = 0$ for all $i \neq j$. Thus

$q^{-1}c$ is a diagonal matrix in R, and as above $q^{-1}c \in Z(R)$. This implies easily that $cq^{-1} \in Z(R)$. In other words, there exists $\mu \in Z(R)$ such that $c = \mu q$. Moreover, (2.6) reduces to

$$\lambda[[x_1, x_2], [y_1, y_2]] - [[x_1, x_2], [y_1, y_2]]c - [[x_1, x_2], [y_1, y_2]]q^{-1}uq \tag{2.7}$$

and by Fact 2.6 we have $c+q^{-1}uq = \lambda$, that is $\mu q+q^{-1}uq = \lambda$. Both right multiplying by q^{-1} and left multiplying by q we get $\mu q+u = \lambda$, which means $c+u = \lambda$. Therefore, for all $x \in R$, $F(x) = G(x) = \lambda qxq^{-1}$, as required. $\square$

Lemma 2.9. *Let $R = M_m(C)$, $m \geqslant 2$ and let C be infinite, $Z(R)$ the center of R, a, b, c, q elements of R and q invertible. If R satisfies $\Psi(x_1, x_2, y_1, y_2)$ then one of the following holds:*

(1) $a, b, c, u, q \in C$, $(a + b) = (c + u)$;
(2) $q^{-1}a, q^{-1}c \in C$ *and there exists* $\lambda \in C$ *such that* $b = \lambda - a$, $u = \lambda - c$.

P r o o f. If $q \in Z(R)$, then the conclusion follows from Lemma 2.7. Analogously, in the case $q^{-1}a \in Z(R)$, we are done by Lemma 2.8.

We assume that $q^{-1}a \notin Z(R)$ and $q \notin Z(R)$, that is both $q^{-1}a$ and q are nonscalar matrices, and prove that a contradiction follows. By Fact 2.3, there exists an invertible matrix $P \in M_m(C)$ such that each of the matrices $P(q^{-1}a)P^{-1}$, PqP^{-1} has all nonzero entries. Denote by $\varphi(x) = PxP^{-1}$ the inner automorphism induced by P. Without loss of generality we may replace q and $q^{-1}a$ with $\varphi(q)$ and $\varphi(q^{-1}a)$, respectively. Let $i \neq j$ and $[x_1, x_2] = [y_1, y_2] = [e_{ii}, e_{ij}] = e_{ij}$ in (2.1), then

$$[ae_{ij} + qe_{ij}q^{-1}b, qe_{ij}q^{-1}] = 0. \tag{2.8}$$

Multiplying by $e_{jj}q^{-1}$ and right multiplying by q, it follows that $e_{jj}q^{-1}ae_{ij}qe_{ij} = 0$, that is either the (j, i)-entry of $(q^{-1}a)$ or the (j, i)-entry of q is zero, which is a contradiction. $\square$

Lemma 2.10. *Let $R = M_m(C)$ $(m \geqslant 2)$. Then Proposition* 2.2 *holds.*

P r o o f. If one assumes that C is infinite, the conclusion follows from Lemma 2.9.

Now, let E be an infinite field which is an extension of the field C and let $\overline{R} = M_t(E) \cong R \otimes_C E$. Consider the generalized polynomial $\Psi(x_1, x_2, y_1, y_2)$, which is a multilinear generalized polynomial identity for R. Clearly, $\Psi(x_1, x_2, y_1, y_2)$ is a generalized polynomial identity for $\overline{R}$ as well, and the conclusion follows from Lemma 2.9. $\square$

We also need the following result:

Lemma 2.11. *Let R be a prime ring, $a, q \in R$ and q invertible. If R satisfies*

$$\Phi(x) = (q^{-1}a)xqx - x(q^{-1}aq)x - x(q^{-1}a)xq + x^2(q^{-1}aq)$$

then $q^{-1}a \in C$.

Proof. Assume that $q^{-1}a \notin C$ and prove that a contradiction follows. Notice that, if $q \in C$, then $\Phi(x)$ reduces to $[a, x]_2$, that is $[a, r]_2 = 0$ for all $r \in R$. Thus, by [23] we get $a \in C$, which is a contradiction.

Therefore we may also assume that $q \notin C$, then the generalized polynomial $\Phi(x)$ is a nontrivial generalized polynomial identity for R. By [7] it follows that $\Phi(x)$ is a nontrivial generalized polynomial identity for Q_r. By the well-known Martindale's theorem of [22], Q_r is a primitive ring having a nonzero socle with the field C as its associated division ring. By [16], page 75, Q_r is isomorphic to a dense subring of the ring of linear transformations of a vector space V over C, containing nonzero linear transformations of finite rank. Assume first that $\dim_C V = k \geqslant 2$ is a finite positive integer, then $Q_r \cong M_k(C)$.

First, suppose that C is infinite. Since neither $q^{-1}a$ nor q are scalar matrices, by Fact 2.3 there exists an invertible matrix $P \in M_m(C)$ such that each of the matrices $P(q^{-1}a)P^{-1}$, PqP^{-1} has all nonzero entries. Denote by $\varphi(x) = PxP^{-1}$ the inner automorphism induced by P. Without loss of generality we may replace q and $q^{-1}a$ by $\varphi(q)$ and $\varphi(q^{-1}a)$, respectively. Setting $x = e_{ij}$ in $\Phi(x)$ and left multiplying by e_{ij}, we obtain $e_{ij}q^{-1}ae_{ij}qe_{ij} = 0$, that is either the (j, i)-entry of $(q^{-1}a)$ or the (j, i)-entry of q is zero, which is a contradiction.

Now, let E be an infinite field which is an extension of the field C and let $\overline{R} = M_t(E) \cong R \otimes_C E$. Consider the generalized polynomial $\Phi(x)$ which is a generalized polynomial identity for R. Moreover, it is multi-homogeneous of multi-degree 2 in the indeterminate x. Hence the complete linearization of $\Phi(x)$ is a multilinear generalized polynomial $\Theta(x, y)$. Moreover,

$$\Theta(x, x) = 2^2\Phi(x).$$

Clearly, the multilinear polynomial $\Theta(x, x)$ is a generalized polynomial identity for R and $\overline{R}$ as well. Since $\mathrm{char}(C) \neq 2$, we obtain $\Phi(r_1, r_2) = 0$ for all $r_1, r_2 \in \overline{R}$, and the conclusion follows from the above argument.

Let now $\dim_C V = \infty$. Notice that eRe satisfies $\Phi(x)$ for all $e^2 = e \in \mathrm{Soc}(R) = H$. Since $q^{-1}a \notin C$ and $q \notin C$, there exist $h_1, h_2 \in H$ such that $[q^{-1}a, h_1] \neq 0$, $[q, h_2] \neq 0$. By Litoff's theorem in [13], there exists $e^2 = e \in H$ such that for all $i = 1, 2$

$$q^{-1}ah_i, h_iq^{-1}a, qh_i, h_iq, ah_i, h_ia, q^{-1}aqh_i, h_iq^{-1}aq \in eRe;$$

moreover, eRe is a central simple algebra finite dimensional over its center. Then $eRe \cong M_t(C)$ for $t \geqslant 2$. We know that

$$(eq^{-1}ae)xeqex - x(eq^{-1}aqe)x - x(eq^{-1}ae)xeqe + x^2(eq^{-1}aqe) \tag{2.9}$$

is a generalized polynomial identity for eRe, hence by the above matrix case we have that either $e(q^{-1}a)e \in eCe$ or $eqe \in eCe$. Thus one of the following equalities is a contradiction:

$$(q^{-1}a)h_1 = e(q^{-1}a)h_1 = e(q^{-1}a)eh_1 = h_1e(q^{-1}a)e = h_1(q^{-1}a)e = h_1(q^{-1}a)$$
$$qh_2 = eqh_2 = eqeh_2 = h_2eqe = h_2qe = h_2q.$$

□

Lemma 2.12. *Either $\Psi(x_1, x_2, y_1, y_2)$ is a nontrivial generalized polynomial identity for R or $q^{-1}a, q^{-1}c \in C$ and there exists $\gamma \in C$ such that $b = \gamma - a$, $u = \gamma - c$.*

P r o o f. Consider the generalized polynomial

$$\begin{aligned}\Psi(x_1, x_2, y_1, y_2) = {} & [a[x_1, x_2] + q[x_1, x_2]q^{-1}b, q[y_1, y_2]q^{-1}] \\ & - c[[x_1, x_2], [y_1, y_2]] - q[[x_1, x_2], [y_1, y_2]]q^{-1}u.\end{aligned} \tag{2.10}$$

By our hypothesis, R satisfies this generalized polynomial identity. Replacing $[x_1, x_2]$ by $q^{-1}[x_1, x_2]q$ and $[y_1, y_2]$ by $q^{-1}[y_1, y_2]q$ in (2.10), we have that R satisfies the generalized polynomial identity

$$\begin{aligned}& [aq^{-1}[x_1, x_2]q + [x_1, x_2]b, [y_1, y_2]] \\ & - c[q^{-1}[x_1, x_2]q, q^{-1}[y_1, y_2]q] - [[x_1, x_2], [y_1, y_2]]u.\end{aligned} \tag{2.11}$$

If $\{aq^{-1}, cq^{-1}, 1\}$ are linearly independent over C then (2.11) is a nontrivial generalized polynomial identity for R. Therefore, we may assume in what follows that $\{aq^{-1}, cq^{-1}, 1\}$ are linearly dependent over C and there exist $\lambda_1, \lambda_2, \lambda_3 \in C$ such that $\lambda_1 aq^{-1} + \lambda_2 cq^{-1} + \lambda_3 = 0$.

If $\lambda_2 = 0$ then $aq^{-1} \in C$ and (2.11) reduces to

$$\begin{aligned}& [x_1, x_2](a + b)[y_1, y_2] - [y_1, y_2][x_1, x_2](a + b) \\ & - cq^{-1}[[x_1, x_2], [y_1, y_2]]q - [[x_1, x_2], [y_1, y_2]]u.\end{aligned} \tag{2.12}$$

If $cq^{-1} \notin C$ then (2.12) is a nontrivial generalized polynomial identity for R, a contradiction. Thus we get $cq^{-1} \in C$.

On the other hand, in case $\lambda_2 \neq 0$ we have $cq^{-1} = \lambda aq^{-1} + \mu$ for suitable $\lambda, \mu \in C$, and (2.11) reduces to

$$aq^{-1}[x_1, x_2]q[y_1, y_2] + [x_1, x_2]b[y_1, y_2] - [y_1, y_2]aq^{-1}[x_1, x_2]q - [y_1, y_2][x_1, x_2]b - \lambda aq^{-1}[[x_1, x_2], [y_1, y_2]]q - \mu[[x_1, x_2], [y_1, y_2]]q - [[x_1, x_2], [y_1, y_2]]u. \tag{2.13}$$

If $aq^{-1} \notin C$ then (2.13) is a nontrivial generalized polynomial identity for R, a contradiction again. Thus we assume $aq^{-1} \in C$, so that $cq^{-1} = \lambda aq^{-1} + \mu \in C$.

The previous argument shows that, in any case, we may assume that both $aq^{-1} = \eta \in C$ and $cq^{-1} = \omega \in C$. Therefore, by (2.11), it follows that

$$[[x_1, x_2](a + b), [y_1, y_2]] - [[x_1, x_2], [y_1, y_2]](c + u) \tag{2.14}$$

is a nontrivial generalized polynomial identity for R, unless $a + b = c + u \in C$. Here we denote $a + b = c + u = \gamma \in C$, moreover it is easy to see that $q^{-1}a = \eta \in C$, $q^{-1}c = \omega \in C$, $q^{-1}u = \gamma q^{-1} - \omega$ and $q^{-1}b = \gamma q^{-1} - \eta$. $\square$

P r o o f of Proposition 2.1. The generalized polynomial $\Psi(x_1, x_2, y_1, y_2)$ is a generalized polynomial identity for R. By Lemma 2.12 we may assume that $\Psi(x_1, x_2, y_1, y_2)$ is a nontrivial generalized polynomial identity for R and, by [7], it follows that $\Psi(x_1, x_2, y_1, y_2)$ is a nontrivial generalized polynomial identity for Q_r. By the well-known Martindale's theorem of [22], Q_r is a primitive ring having a nonzero socle with the field C as its associated division ring. By [16], page 75, Q_r is isomorphic to a dense subring of the ring of linear transformations of a vector space V over C, containing nonzero linear transformations of finite rank. Assume first that $\dim_C V = k \geqslant 2$ is a finite positive integer, then $Q_r \cong M_k(C)$ and the conclusion follows from Lemma 2.10.

Let now $\dim_C V = \infty$. Since the set $[R, R]$ is dense on R, as in Lemma 2 in [24] and by the fact that $\Psi(x_1, x_2, y_1, y_2)$ is a generalized polynomial identity of Q_r, we know that Q_r satisfies

$$[ax + qxq^{-1}b, qyq^{-1}] - c[x, y] - q[x, y]q^{-1}u. \tag{2.15}$$

Replacing x by $x + 1$ in (2.15) we have that Q_r satisfies

$$[a + b, qyq^{-1}] \tag{2.16}$$

that is $a + b = \lambda \in C$. Therefore, for $x = y$ and $q^{-1}b = \lambda q^{-1} - q^{-1}a$ in (2.16) it follows that

$$[ax - qxq^{-1}a, qxq^{-1}] \tag{2.17}$$

is satisfied by Q_r. Left multiplying (2.17) by q^{-1}, one has that

$$(q^{-1}a)xqx - x(q^{-1}aq)x - x(q^{-1}a)xq + x^2(q^{-1}aq) \tag{2.18}$$

is a generalized polynomial identity for Q_r. By Lemma 2.11 we get $q^{-1}a \in C$ and (2.15) reduces to

$$\lambda[qxq^{-1}, qyq^{-1}] - c[x,y] - q[x,y]q^{-1}u. \tag{2.19}$$

Here we introduce the element $p = \lambda - u$, so that, by (2.19), we have $q[x,y]q^{-1}p - c[x,y] = 0$ for all $x, y \in Q_r$. Assume that $q^{-1}p \notin C$, then there exists $v \in V$ such that v and $q^{-1}pv$ are linearly C-independent and, by the density of Q_r, there exist $r_1, r_2 \in Q_r$ such that

$$r_1 v = r_2 v = 0, \quad r_1(q^{-1}pv) = v, \quad r_2(q^{-1}pv) = q^{-1}pv.$$

Hence the following contradiction occurs:

$$0 = (q[r_1, r_2]q^{-1}p - c[r_1, r_2])v = qv \neq 0.$$

Therefore $q^{-1}p \in C$ and $(p-c)[x,y] = 0$ for all $x, y \in Q_r$. Thus $p - c = 0$, that is $q^{-1}c \in C$ and $u + c = \lambda \in C$. $\square$

Here we recall some useful known results:

Remark 2.13. Let R be a prime ring of characteristic different from 2.

If $[[x_1, x_2], [y_1, y_2]] \in Z(R)$ for all $x_1, x_2, y_1, y_2 \in R$, then R is commutative.

Proof. Since R is a prime ring satisfying the polynomial identity

$$[[[x_1, x_2], [y_1, y_2]], x_3]$$

there exists a field K such that R and $M_t(K)$, the ring of all $t \times t$ matrices over K, satisfy the same polynomial identities (see [15]).

Suppose $t \geqslant 2$. Let $x_1 = e_{11}$, $x_2 = e_{22}$, $y_1 = e_{22}$ and $y_2 = e_{21}$. By calculation we obtain $[[x_1, x_2], [y_1, y_2]] = e_{11} - e_{22} \notin Z(R)$, a contradiction. So $t = 1$ and R is commutative. $\square$

Remark 2.14. Let R be a prime ring of characteristic different from 2 and $a \in R$. If $a[[x_1, x_2], [y_1, y_2]] = 0$ (or $[[x_1, x_2], [y_1, y_2]]a = 0$) for all $x_1, x_2, y_1, y_2 \in R$, then either $a = 0$ or R is commutative.

P r o o f. By Remark 2.13 we may assume that the polynomial $[[x_1, x_2], [y_1, y_2]]$ is not central in R. Therefore $a = 0$ follows from [10]. □

Remark 2.15. Let R be a prime ring of characteristic different from 2 and $a \in R$. If $[a[x_1, x_2], [x_1, x_2]] = 0$ ($[[x_1, x_2]a, [x_1, x_2]] = 0$, respectively) for all $x_1, x_2 \in R$, then $a \in Z(R) = 0$.

P r o o f. It is an easy consequence of [1]. □

P r o o f of Proposition 2.2. If there exists an invertible element $q \in Q_r$ such that $\alpha(x) = qxq^{-1}$ for all $x \in R$, then the conclusion follows from Proposition 2.1. Hence we may assume that α is not an inner automorphism of R. Thus, since R satisfies

$$\begin{aligned}(2.20)\qquad & [a[x_1, x_2] + \alpha([x_1, x_2])b, \alpha([y_1, y_2])] \\ & \quad - c[[x_1, x_2], [y_1, y_2]] - \alpha([[x_1, x_2], [y_1, y_2]])u\end{aligned}$$

it follows that

$$(2.21)\qquad [a[x_1, x_2] + [z_1, z_2]b, [t_1, t_2]] - c[[x_1, x_2], [y_1, y_2]] - [[z_1, z_2], [t_1, t_2]]u$$

is a generalized identity for R. In particular R satisfies $c[[x_1, x_2], [y_1, y_2]]$, which implies that $c = 0$ (see Remark 2.14), moreover R satisfies $[a[x_1, x_2], [t_1, t_2]]$, which implies $a = 0$ (again by Remark 2.14). Thus (2.21) reduces to

$$(2.22)\qquad [[z_1, z_2]b, [t_1, t_2]] - [[z_1, z_2], [t_1, t_2]]u.$$

For $[z_1, z_2] = [t_1, t_2]$ in (2.22), it follows that $[[t_1, t_2]b, [t_1, t_2]]$ is a generalized identity for R. Thus $b \in C$ (see Remark 2.15) and by (2.22), we have that $[[z_1, z_2], [t_1, t_2]] \times (b - u)$ is satisfied by R. Hence by Remark 2.14 we have $b = u = \lambda \in C$, so that $F(x) = G(x) = \lambda\alpha(x)$ for all $x \in R$, as required. □

3. The proof of Theorem 1.1

We will make frequent use of the following:

Lemma 3.1. *Let R be a prime ring, α an automorphism of R. If R satisfies*

$$(3.1)\qquad (z_1x_2 - \alpha(x_2)z_1)[y_1, y_2] - \alpha([y_1, y_2])(z_1x_2 - \alpha(x_2)z_1)$$

then R is commutative.

P r o o f. Notice that, in case α is the identity map on R, then (3.1) reduces to

$$[z_1, x_2][y_1, y_2] - [y_1, y_2][z_1, x_2], \tag{3.2}$$

which implies that R is commutative (see Remark 2.13).

Thus we assume that α is not the identity map and there exists an invertible element $q \in Q_r$ such that $q \notin C$ and $\alpha(x) = qxq^{-1}$ for all $x \in R$. Replacing any z_i by qz_i in (3.1) and left multiplying by q^{-1}, one has that $[z_1, x_2][y_1, y_2] - [y_1, y_2][z_1, x_2]$ is a polynomial identity for R. As above it follows that R is commutative.

On the other hand, in case α is not inner, then by (3.1) it follows that R satisfies the generalized identity

$$(z_1x_2 - t_2z_1)[y_1, y_2] - [w_1, w_2](z_1x_2 - t_2z_1) \tag{3.3}$$

and for $w_1 = w_2 = t_2 = 0$, R satisfies the polynomial identity $z_1x_2[y_1, y_2] = 0$, which implies again that R is commutative. $\square$

By using the same argument, one can prove the following result (even if the proof is similar to the previous one, here we prefer to insert it for sake of completeness):

Lemma 3.2. *Let R be a prime ring, α an automorphism of R. If R satisfies*

$$(t_1x_2 - \alpha(x_2)t_1)[y_1, y_2] - [y_1, y_2](t_1x_2 - \alpha(x_2)t_1) \tag{3.4}$$

then R is commutative.

P r o o f. By our assumption

$$[t_1x_2 - \alpha(x_2)t_1, [y_1, y_2]] \tag{3.5}$$

is a generalized identity for R.

Notice that, in case α is the identity map on R, (3.5) reduces to

$$[[t_1, x_2], [y_1, y_2]], \tag{3.6}$$

which implies that R is commutative (by Remark 2.13).

Thus we assume that α is not the identity map and there exists an invertible element $q \in Q_r$ such that $q \notin C$ and $\alpha(x) = qxq^{-1}$ for all $x \in R$. Replacing any t_1 by qt_1 in (3.5) one has that

$$[q[t_1, x_2], [y_1, y_2]] \tag{3.7}$$

is a generalized polynomial identity for R. In this case, Remark 2.15 implies $q \in C$, which is a contradiction.

On the other hand, in case α is not inner, then by (3.5) it follows that R satisfies the generalized identity

$$[t_1x_2 - t_2t_1, [y_1, y_2]] \tag{3.8}$$

and as above we have that R is commutative. □

Remark 3.3. As mentioned in Introduction, we can write $F(x) = ax + f(x)$, $G(x) = bx + g(x)$ for all $x \in R$, where $a, b \in Q_r$ and f, g are skew derivations of R. Let α be the automorphism associated with f and g. That is, $f(xy) = f(x)y + \alpha(x)f(y)$ and $g(xy) = g(x)y + \alpha(x)g(y)$ for all $x, y \in R$.

Remark 3.4. Let S be the additive subgroup generated by the set

$$p(R) = \{p(r_1, \dots, r_n)\colon\ r_1, \dots, r_n \in R\} \neq 0.$$

Of course $[F(x), \alpha(y)] = G([x, y])$ for all $x, y \in S$. Since $p(x_1, \dots, x_n)$ is not central in R, by [8] and $\mathrm{char}(R) \neq 2$ it follows that there exists a non-central Lie ideal L of R such that $L \subseteq S$. Moreover, it is well known that there exists a nonzero ideal I of R such that $[I, R] \subseteq L$ (see [14], pages 4–5, [12], Lemma 2, Proposition 1, [17], Theorem 4).

By Remark 3.4 we assume there exists a non-central ideal I of R such that

$$\begin{aligned}[][a[x_1, x_2] + f([x_1, x_2]), \alpha([y_1, y_2])]\\ - b[[x_1, x_2], [y_1, y_2]] - g([[x_1, x_2], [y_1, y_2]])\end{aligned} \tag{3.9}$$

is satisfied by I. Since I and R satisfy the same generalized identities with derivations and automorphisms, (3.9) is a generalized differential identity for R, that is R satisfies

$$\begin{aligned}&[a[x_1, x_2] + f(x_1)x_2 + \alpha(x_1)f(x_2) - f(x_2)x_1 - \alpha(x_2)f(x_1), \alpha([y_1, y_2])]\\ &\quad - b[[x_1, x_2], [y_1, y_2]] - (g(x_1)x_2 + \alpha(x_1)g(x_2) - g(x_2)x_1 - \alpha(x_2)g(x_1))\\ &\quad \times [y_1, y_2] - \alpha([x_1, x_2])(g(y_1)y_2 + \alpha(y_1)g(y_2) - g(y_2)y_1 - \alpha(y_2)g(y_1))\\ &\quad + (g(y_1)y_2 + \alpha(y_1)g(y_2) - g(y_2)y_1 - \alpha(y_2)g(y_1))[x_1, x_2]\\ &\quad + \alpha([y_1, y_2])(g(x_1)x_2 + \alpha(x_1)g(x_2) - g(x_2)x_1 - \alpha(x_2)g(x_1)).\end{aligned} \tag{3.10}$$

Moreover, in all what follows we assume R is not commutative.

Remark 3.5. First we suppose $g = 0$, so that R satisfies

$$\begin{aligned}(3.11)\qquad &[a[x_1, x_2] + f(x_1)x_2 + \alpha(x_1)f(x_2) - f(x_2)x_1 \\ &\quad - \alpha(x_2)f(x_1), \alpha([y_1, y_2])] - b[[x_1, x_2], [y_1, y_2]].\end{aligned}$$

We may assume that $0 \neq f$ is not inner, otherwise we are done by Proposition 2.2.

Then R satisfies

$$(3.12)\quad [a[x_1, x_2] + t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1, \alpha([y_1, y_2])] - b[[x_1, x_2], [y_1, y_2]]$$

and in particular

$$(3.13)\qquad [t_1x_2 - \alpha(x_2)t_1, \alpha([y_1, y_2])]$$

is satisfied by R. By the arbitrariness of y_1, y_2 in (3.13), it follows that

$$(3.14)\qquad [t_1x_2 - \alpha(x_2)t_1, [y_1, y_2]]$$

is a generalized identity for R and the conclusion follows from Lemma 3.2.

In light of the previous remark, in all what follows we assume that g is not zero.

3.1. Let f and g be C-linearly independent modulo $\mathrm{SD}_{\mathrm{int}}$**.** Assume that both f, g are a nonzero skew derivations of R. In this case, by (3.10), R satisfies

$$\begin{aligned}(3.15)\qquad &[a[x_1, x_2] + t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1, \alpha([y_1, y_2])] \\ &\quad - b[[x_1, x_2], [y_1, y_2]] - (z_1x_2 + \alpha(x_1)z_2 - z_2x_1 - \alpha(x_2)z_1)[y_1, y_2] \\ &\quad - \alpha([x_1, x_2])(v_1y_2 + \alpha(y_1)v_2 - v_2y_1 - \alpha(y_2)v_1) \\ &\quad + (v_1y_2 + \alpha(y_1)v_2 - v_2y_1 - \alpha(y_2)v_1)[x_1, x_2] \\ &\quad + \alpha([y_1, y_2])(z_1x_2 + \alpha(x_1)z_2 - z_2x_1 - \alpha(x_2)z_1)\end{aligned}$$

and in particular R satisfies

$$(3.16)\qquad (z_1x_2 - \alpha(x_2)z_1)[y_1, y_2] - \alpha([y_1, y_2])(z_1x_2 - \alpha(x_2)z_1).$$

By Lemma 3.1 we get the contradiction that R is commutative.

Assume now that $f = 0$ and $g \neq 0$. In this case, by (3.10), R satisfies

$$\begin{aligned}(3.17)\qquad &-b[[x_1, x_2], [y_1, y_2]] - (z_1x_2 + \alpha(x_1)z_2 - z_2x_1 - \alpha(x_2)z_1)[y_1, y_2] \\ &\quad - \alpha([x_1, x_2])(v_1y_2 + \alpha(y_1)v_2 - v_2y_1 - \alpha(y_2)v_1) \\ &\quad + (v_1y_2 + \alpha(y_1)v_2 - v_2y_1 - \alpha(y_2)v_1)[x_1, x_2] \\ &\quad + \alpha([y_1, y_2])(z_1x_2 + \alpha(x_1)z_2 - z_2x_1 - \alpha(x_2)z_1).\end{aligned}$$

In particular R satisfies

$$(z_1x_2 - \alpha(x_2)z_1)[y_1, y_2] - \alpha([y_1, y_2])(z_1x_2 - \alpha(x_2)z_1) \tag{3.18}$$

and we argue as above by Lemma 3.1.

3.2. Let f and g be C-linearly dependent modulo $\mathrm{SD}_{\mathrm{int}}$. Assume now there exist $\lambda, \mu \in C$, $c \in Q$ and $\beta \in \mathrm{Aut}(R)$ such that

$$\lambda f(x) + \mu g(x) = cx - \beta(x)c, \quad x \in R.$$

We notice that if $f = 0$ then $F(x) = ax$ and $G(x) = (b + \mu^{-1}c)x - \beta(x)(\mu^{-1}c)$ for all $x \in R$, and the conclusion follows from Proposition 2.2. Thus, in all what follows both $f \neq 0$ and $g \neq 0$.

Assume first $\lambda = 0$ and $\mu \neq 0$.

Hence $g(x) = vx - \beta(x)v$, where $v = \mu^{-1}c$. We recall that any inner skew derivation has a unique associated automorphism, hence $\alpha = \beta$. Moreover f is not an inner skew derivation of R, if it is we are done by Proposition 2.2.

By (3.10), R satisfies

$$\begin{aligned}[a[x_1, x_2] + t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1, \alpha([y_1, y_2])]\\ - (b + v)[[x_1, x_2], [y_1, y_2]] + \alpha([[x_1, x_2], [y_1, y_2]])v\end{aligned} \tag{3.19}$$

and in particular R satisfies

$$[t_1x_2 - \alpha(x_2)t_1, \alpha([y_1, y_2])]. \tag{3.20}$$

By the arbitrariness of y_1, y_2 in (3.20), it follows that

$$[t_1x_2 - \alpha(x_2)t_1, [y_1, y_2]] \tag{3.21}$$

is a generalized identity for R and we proceed by Lemma 3.2.

Let now $\lambda \neq 0$ and $\mu = 0$.

Hence $f(x) = ux - \beta(x)u$, where $u = \lambda^{-1}c$, moreover, as above, $\alpha = \beta$. By Proposition 2.2 we may also assume that g is not an inner skew derivation of R. Due to (3.10), R satisfies

$$\begin{aligned}&[(a + u)[x_1, x_2] - \alpha([x_1, x_2]), \alpha([y_1, y_2])]\\ &\quad - b[[x_1, x_2], [y_1, y_2]] - (t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1)[y_1, y_2]\\ &\quad - \alpha([x_1, x_2])(z_1y_2 + \alpha(y_1)z_2 - z_2y_1 - \alpha(y_2)z_1)\\ &\quad + (z_1y_2 + \alpha(y_1)z_2 - z_2y_1 - \alpha(y_2)z_1)[x_1, x_2]\\ &\quad + \alpha([y_1, y_2])(t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1).\end{aligned} \tag{3.22}$$

In particular R satisfies

$$(t_1x_2 - \alpha(x_2)t_1)[y_1, y_2] - \alpha([y_1, y_2])(t_1x_2 - \alpha(x_2)t_1) \tag{3.23}$$

and by Lemma 3.1 we are done.

Now we study the case when both $\lambda \neq 0$ and $\mu \neq 0$. Thus $g(x) = vx - \beta(x)v - \theta f(x)$, where $v = \mu^{-1}c$ and $\theta = \lambda\mu^{-1} \neq 0$. By (3.10) it follows that

$$\begin{aligned}
&[a[x_1, x_2] + f(x_1)x_2 + \alpha(x_1)f(x_2) - f(x_2)x_1 - \alpha(x_2)f(x_1), \alpha([y_1, y_2])] \\
&\quad - (b + v)[[x_1, x_2], [y_1, y_2]] + \beta([[x_1, x_2], [y_1, y_2]])v \\
&\quad + \theta(f(x_1)x_2 + \alpha(x_1)f(x_2) - f(x_2)x_1 - \alpha(x_2)f(x_1))[y_1, y_2] \\
&\quad + \theta\alpha([x_1, x_2])(f(y_1)y_2 + \alpha(y_1)f(y_2) - f(y_2)y_1 - \alpha(y_2)f(y_1)) \\
&\quad - \theta(f(y_1)y_2 + \alpha(y_1)f(y_2) - f(y_2)y_1 - \alpha(y_2)f(y_1))[x_1, x_2] \\
&\quad - \theta\alpha([y_1, y_2])(f(x_1)x_2 + \alpha(x_1)f(x_2) - f(x_2)x_1 - \alpha(x_2)f(x_1))
\end{aligned} \tag{3.24}$$

is satisfied by R. If f is not inner, then R satisfies

$$\begin{aligned}
&[a[x_1, x_2] + t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1, \alpha([y_1, y_2])] \\
&\quad - (b + v)[[x_1, x_2], [y_1, y_2]] + \beta([[x_1, x_2], [y_1, y_2]])v \\
&\quad + \theta(t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1)[y_1, y_2] \\
&\quad + \theta\alpha([x_1, x_2])(z_1y_2 + \alpha(y_1)z_2 - z_2y_1 - \alpha(y_2)z_1) \\
&\quad - \theta(z_1y_2 + \alpha(y_1)z_2 - z_2y_1 - \alpha(y_2)z_1)[x_1, x_2] \\
&\quad - \theta\alpha([y_1, y_2])(t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1).
\end{aligned} \tag{3.25}$$

Thus R satisfies the blended component

$$\theta\alpha([x_1, x_2])(z_1y_2 - \alpha(y_2)z_1) - \theta(z_1y_2 - \alpha(y_2)z_1)[x_1, x_2]. \tag{3.26}$$

By Lemma 3.1 and since R is not commutative, we get a contradiction.

We finally assume that f is an inner skew derivation of R, that is there exists $u \in Q_r$ such that $f(x) = ux - \alpha(x)u$ for all $x \in R$. Hence $g(x) = (v - \theta u)x - \beta(x)v + \alpha(x)(\theta u)$, moreover, in light of Proposition 2.1 we may suppose that g is not an inner skew derivation of R.

We remark that for all $x, y \in R$

$$g(xy) = (v - \theta u)xy - \beta(x)\beta(y)v + \alpha(x)\alpha(y)(\theta u). \tag{3.27}$$

On the other hand

$$\begin{aligned}
g(xy) &= g(x)y + \alpha(x)g(y) \\
&= (v - \theta u)xy - \beta(x)vy + \alpha(x)(\theta u)y \\
&\quad + \alpha(x)(v - \theta u)y - \alpha(x)\beta(y)v + \alpha(x)\alpha(y)(\theta u).
\end{aligned} \tag{3.28}$$

Comparision of (3.27) and (3.28) implies $(\alpha(x)-\beta(x))(\beta(y)v-vy)=0$ for all $x, y \in R$. Since $\alpha-\beta$ is an automorphism of R and by the arbitrariness of $x \in R$ we get that either $\alpha=\beta$ or $\beta(y)v-vy=0$ for all $y \in R$. In the latter case, either β is not inner and $v=0$, or there exists an invertible element $p \in Q_r$ such that $\beta(x)=pxp^{-1}$ and $p^{-1}v \in C$.

In any case, by computation we get $g(x)=-(\theta u)x+\alpha(x)(\theta u)$, which is a contradiction, since g is not inner. □

As a consequence of Theorem 1.1 we get the following:

Theorem 3.6. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring and C its extended centroid. Suppose that F, G are skew derivations of R, with the same associated automorphism α, and $p(x_1,\ldots,x_n)$ a polynomial over C such that*

$$[F(x),\alpha(y)]=G([x,y])$$

for all $x, y \in \{p(r_1,\ldots,r_n)\colon\ r_1,\ldots,r_n \in R\}$. Then either $p(x_1,\ldots,x_n)$ is central valued on R or $F=G=0$.

In particular:

Theorem 3.7. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring and C its extended centroid, I a two-sided ideal of R. Suppose that F, G are skew derivations of R, with the same associated automorphism α, such that*

$$[F(x),\alpha(y)]=G([x,y])$$

for all $x, y \in I$. Then either R is commutative or $F=G=0$.

4. An application for skew derivations in semiprime rings

In this final section we extend Theorem 3.7 to semiprime rings. We premit the following easy results, which will be useful in the sequel:

Remark 4.1. Let R be a prime ring, G a nonzero skew derivation of R, I a two-sided ideal of R such that $G([x,y])=0$ for all $x, y \in I$. Then R is commutative.

P r o o f. It is a reduced case of Theorem 3.7. □

Remark 4.2. Let R be a prime ring, α an automorphism of R, I a two-sided ideal of R such that $\alpha(x)y - yx = 0$ for all $x \in R$ and $y \in I$. Then R is commutative (and α is the identity map on R).

Proof. Notice that, in case α is the identity map on R, $[x, y] = 0$ for all $x \in R$ and $y \in I$ and we are done.

Thus we assume that α is not the identity map and there exists an invertible element $q \in Q_r$ such that $q \notin C$ and $\alpha(x) = qxq^{-1}$ for all $x \in R$. Hence $qxq^{-1}y - yx = 0$ for all $x \in R$ and $y \in I$. Since I and R satisfy the same generalized polynomial identities, it follows that $qxq^{-1}y - yx = 0$ for all $x, y \in R$. Replacing any y by qy in the previous relation and left multiplying by q^{-1}, we get $[x, y] = 0$ for all $x, y \in R$. In particular, for $x = q$ we get the contradiction $q \in C$.

On the other hand, in case α is not inner, then $ty - yx = 0$ for all $x, y, t \in R$. For $t = 0$ one has the contradiction $R^2 = 0$. $\square$

We are ready to prove the following:

Theorem 4.3. *Let R be a semiprime ring of characteristic different from 2, Q_r its right Martindale quotient ring and C its extended centroid. Suppose that F, G are skew derivations of R, with the same associated automorphism α, such that*

$$[F(x), \alpha(y)] = G([x, y])$$

for all $x, y \in I$. Then either $F = G = 0$ or R contains a nonzero central ideal.

Proof. Let P be a prime ideal of R. Set $\overline{R} = R/P$ and write $\overline{x} = x + P \in \overline{R}$ for all $x \in R$. We start from

$$[\overline{F(x)}, \overline{\alpha(y)}] = \overline{G([x, y])} \quad \text{for all } \overline{x}, \overline{y} \in \overline{R}. \tag{4.1}$$

Case 1: $\alpha(P) \not\subseteq P$. In this case both $\overline{\alpha(P)}$ and $\overline{\alpha^{-1}(P)}$ are nonzero ideals of $\overline{R}$. Moreover, for any $x \in R$, $p \in P$, replacing y by $\alpha^{-1}(p)$ in (4.1), we have that

$$\overline{G([x, \alpha^{-1}(p)])} = \overline{0} \quad \text{for all } \overline{x} \in \overline{R},\ p \in P. \tag{4.2}$$

Thus, by Remark 4.1, either $\overline{G(x)} = \overline{0}$, that is $G(R) \subseteq P$, or $\overline{R}$ is commutative, that is $[R, R] \subseteq P$. In either case $\overline{[G(R), R]} = \overline{0}$, moreover by (4.1) it follows that

$$\overline{[F(x), \alpha(y)]} = \overline{0} \quad \text{for all } x, y \in R.$$

Therefore, for all $x \in R$, $\overline{F(x)}$ centralizes the nonzero ideal $\overline{\alpha(P)}$ of $\overline{R}$, that is $\overline{[F(R), R]} = \overline{0}$.

Case 2: $F(P) \subseteq P$, $\alpha(P) \subseteq P$. In this case $\overline{F}$ is a skew derivation of $\overline{R}$. If $G(P) \subseteq P$, then also $\overline{G}$ is a skew derivation of $\overline{R}$, and by the primeness of $\overline{R}$ and Theorem 3.7 we have that either $\overline{R}$ is commutative, that is $[R, R] \subseteq P$, or both $F(R) \subseteq P$ and $G(R) \subseteq P$.

Let now $G(P) \nsubseteq P$, then $\overline{G(P)}$ is a nonzero ideal of $\overline{R}$. For any $x, y \in R$ and $p \in P$, we get

$$[F(x), \alpha(yp)] = G([x, yp]) \quad \text{for all } x, y \in R.$$

By computation and since $\alpha(P) \subseteq P$, it follows that

$$\overline{\alpha(x)(\alpha(y)G(p)) - (\alpha(y)G(p))x} = \overline{0} \quad \text{for all } \overline{x}, \overline{y} \in \overline{R}. \tag{4.3}$$

By the primeness of $\overline{R}$ and since $\overline{\alpha(R)G(P)}$ is a nonzero ideal of $\overline{R}$, we may apply the result in Remark 4.2. More precisely, we obtain that $\overline{R}$ is commutative, that is $[R, R] \subseteq P$.

Case 3: $G(P) \subseteq P$, $\alpha(P) \subseteq P$, $F(P) \nsubseteq P$. In this case $\overline{G}$ is a skew derivation of $\overline{R}$, moreover $\overline{\alpha(R)F(P)}$ is a nonzero ideal of $\overline{R}$. For any $x, y \in R$ and $p \in P$ we get $G([xp, y]) = G(xpy - ypx) = G(x)py + \alpha(x)G(p)y + \alpha(x)\alpha(p)G(y) \in P$, so that

$$\overline{[F(xp), \alpha(y)]} = \overline{0} \quad \text{for all } x, y \in R$$

that is

$$\overline{[\alpha(x)F(p), \alpha(y)]} = \overline{0} \quad \text{for all } x, y \in R$$

implying that

$$\overline{[\alpha(R)F(P), \alpha(R)]} = \overline{0}.$$

Therefore the nonzero ideal $\overline{\alpha(R)F(P)}$ of $\overline{R}$ is central, that is $\overline{R}$ is commutative, i.e. $[R, R] \subseteq P$.

Case 4: $F(P) \nsubseteq P$, $G(P) \nsubseteq P$, $\alpha(P) \subseteq P$. In this case $\overline{\alpha(R)G(P)}$ is a nonzero ideal of $\overline{R}$.

For any $x, y \in R$ and $p \in P$, by (4.1) and since $[F(x), \alpha(yp)] \subseteq P$, we get

$$\overline{G([x, yp])} = \overline{0} \quad \text{for all } \overline{x}, \overline{y} \in \overline{R} \tag{4.4}$$

that is

$$\overline{\alpha(x)(\alpha(y)G(p)) - (\alpha(y)G(p))x} = \overline{0} \quad \text{for all } \overline{x}, \overline{y} \in \overline{R}. \tag{4.5}$$

Since $\overline{\alpha(R)G(P)}$ is a nonzero ideal of $\overline{R}$ and by Remark 4.2, we have that $\overline{R}$ is commutative, i.e. $[R, R] \subseteq P$.

Therefore in either case $[F(R),R] \subseteq P$ and $[G(R),R] \subseteq P$ for any prime ideal P of R. Then $[F(R),R] \subseteq \bigcap_i P_i = (0)$ and $[G(R),R] \subseteq \bigcap_i P_i = (0)$ (where P_i are all prime ideals of R), that is $F(R) \subseteq Z(R)$ and $G(R) \subseteq Z(R)$. Assume that F and G are not simultaneously zero. For instance, let $x_0 \in R$ be such that $0 \neq F(x_0) \in Z(R)$. Hence R contains the nonzero central ideal generated by the element $F(x_0)$, and we are done. $\square$

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