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A NEW CONTINUOUS DEPENDENCE RESULT FOR IMPULSIVE RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We consider a large class of impulsive retarded functional differential equations (IRFDEs) and prove a result concerning uniqueness of solutions of impulsive FDEs. Also, we present a new result on continuous dependence of solutions on parameters for this class of equations. More precisely, we consider a sequence of initial value problems for impulsive RFDEs in the above setting, with convergent right-hand sides, convergent impulse operators and uniformly convergent initial data. We assume that the limiting equation is an impulsive RFDE whose initial condition is the uniform limit of the sequence of the initial data and whose solution exists and is unique. Then, for sufficient large indexes, the elements of the sequence of impulsive retarded initial value problem admit a unique solution and such a sequence of solutions converges to the solution of the limiting Cauchy problem.

Keywords: retarded functional differential equation; impulse local existence; impulse local existence uniqueness; continuous dependence on parameters

MSC 2010: 34K45, 34K05

1. Introduction

Our motivation for revisiting the theory of continuous dependence on parameters for RFDEs, which is described in [5] for the case of non-impulsive systems with continuous right-hand sides, is to consider more general systems (subject to impulse action). As a matter of fact, one important application of theorems about continuous dependence of a solution on a parameter is to obtain averaging methods. Indeed, we are concerned with averaging principles for RFDEs with impulses using the results of the present paper. See [2], [1]. We consider a setting of Cauchy problems for retarded

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functional differential equations (we write RFDE, for short) subject to impulse effects at preassigned moments, having a discontinuous initial function and a Lebesgue integrable right-hand side.

Let r, σ and t_0 be real numbers with $r > 0$ and $\sigma > 0$. Given $t \in [t_0, t_0 + \sigma]$ and a function $y\colon [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$, consider $y_t\colon [-r, 0] \to \mathbb{R}^n$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

We consider the initial value problem

$$\begin{cases} \dot{y}(t) = f(y_t, t), & t \neq t_k, \\ \Delta y(t_k) = I_k(y(t_k)), & k = 1, \ldots, m, \\ y_{t_0} = \varphi, \end{cases} \tag{1.1}$$

where t_k, $k = 1, \ldots, m$, are the moments of impulse action, with $t_0 < t_1 < \ldots < t_k < \ldots < t_m \leqslant t_0 + \sigma$. We assume that $y \mapsto I_k(y)$, $k = 1, \ldots, m$ maps $\mathbb{R}^n$ into itself and

$$\Delta y(t_k) := y(t_k+) - y(t_k-) = y(t_k+) - y(t_k), \quad k = 1, \ldots, m,$$

that is, y is left continuous at $t = t_k$ and the lateral limit $y(t_k+)$ exists for $k = 1, \ldots, m$. This means that y is a regulated function. We also require the initial function to be regulated and left-continuous, that is, $\varphi\colon [-r, 0] \to \mathbb{R}^n$ admits the lateral limits

$$\lim_{s \to t-} \varphi(s) = \varphi(t), \quad t \in (-r, 0], \quad \text{and} \quad \lim_{s \to t+} \varphi(s) = \varphi(t+), \quad t \in [-r, 0).$$

In addition, we assume that the mapping $t \mapsto f(y_t, t)$ is Lebesgue integrable with indefinite integral satisfying Carathéodory and Lipschitz-type conditions. Thus, the mapping $t \mapsto f(y_t, t)$ need not be piecewise continuous and the usual requirement that $f(\psi, t)$ is continuous in the first variable (which is an element of the space of regulated functions from $[-r, 0]$ to $\mathbb{R}^n$) need not be fulfilled. Under these conditions, we prove a local existence and uniqueness theorem for problem (1.1) as well as a new result on continuous dependence of the solutions on parameters.

It should be noticed that, in the above setting, it was proved in [3] that system (1.1) is equivalent to a system of generalized differential equations taking values in a Banach space and, as a consequence, local existence and uniqueness of a solution were guaranteed. In the present paper, we prove the same result without employing the theory of generalized differential equations.

With respect to continuous dependence of the solutions on parameters, we have to mention that, in the above setting, the following result is well-known. Consider a sequence of initial value problems whose right-hand sides converge to the right-hand side of an impulsive RFDE and whose initial data also converge. Let the sequence of impulse operators be convergent as well. Suppose each element of the sequence of impulsive retarded Cauchy problems admits a unique solution and that this sequence of unique solutions is uniformly convergent. Consider the limit initial value problem with limiting right-hand side, limiting impulse operators and limiting initial condition. Then, the limit of the sequence of solutions is a solution of the limiting initial value problem. See [3], Theorem 4.1, and [7].

In the present paper, we prove a certain reciprocal of the above result. We consider a sequence of initial value problems for impulsive RFDEs in the above setting, with convergent right-hand sides, convergent impulse operators and uniformly convergent initial data. We assume that the limiting equation is an impulsive RFDE whose initial condition is the uniform limit of the sequence of initial data and whose solution exists and is unique. Then, for sufficient large indexes, the elements of the sequence of impulsive retarded initial value problem admit a unique solution and such a sequence of solutions converges to the solution of the limiting Cauchy problem.

2. A class of impulsive RFDEs

Let X be a Banach space with norm $\|\cdot\|$ and $[a,b]$ a conpact interval of $\mathbb{R}$. A function $f\colon [a,b]\to X$ is called *regulated*, if the lateral limits

$$\lim_{s\to t-} f(s) = f(t-)\in X, \quad t\in(a,b], \quad \text{and} \quad \lim_{s\to t+} f(s) = f(t+)\in X, \quad t\in[a,b),$$

exist. In this case, we write $f\in G([a,b],X)$ and we endow $G([a,b],X)$ with the usual supremum norm $\|f\| = \sup_{a\leqslant t\leqslant b} |f(t)|$. Then $(G([a,b],X),\|\cdot\|)$ is a Banach space. Moreover, any function in $G([a,b],X)$ is the uniform limit of step functions. For more details about regulated functions, the reader may want to consult [4], [6].

Define

$$G^{-}([a,b],X) = \{u\in G([a,b],X)\colon\ u \text{ is left continuous at every } t\in(a,b]\}.$$

In $G^{-}([a,b],X)$, we consider the norm induced by $G([a,b],X)$. Then it is clear that given a function $y\in G^{-}([t_0-r,t_0+\sigma],\mathbb{R}^n)$ and $t\in[t_0,t_0+\sigma]$, we have $y_t\in G^{-}([-r,0],\mathbb{R}^n)$.

We consider the initial value problem for a RFDE with impulses

$$\begin{cases} \dot{y}(t) = f(y_t,t), \quad t\neq t_k, \\ \Delta y(t_k) = I_k(y(t_k)), \quad k=1,\dots,m, \\ y_{t_0} = \varphi, \end{cases} \tag{2.1}$$

where $\varphi \in G^-([-r,0],\mathbb{R}^n)$ and the moments of the impulse form a finite sequence, $\{t_k\}_{k=1,\dots,m}$, which is increasing. We assume that $y \mapsto I_k(y)$, $k = 1,\dots,m$ maps $\mathbb{R}^n$ into itself and

$$\Delta y(t_k) := y(t_k+) - y(t_k-) = y(t_k+) - y(t_k),$$

that is, y is left continuous at $t = t_k$ and the lateral limit $y(t_k+)$ exists for $k = 1,\dots,m$.

It is known that the impulsive system (2.1) is equivalent to the integral equation

$$\begin{cases} y(t) = y(t_0) + \displaystyle\int_{t_0}^{t} f(y_s,s)\,\mathrm{d}s + \sum_{t_0<t_k<t} I_k(y(t_k)), \\ y_{t_0} = \varphi, \end{cases} \tag{2.2}$$

whenever the integral on the right hand side of the equation (2.2) is defined. See [8] for more details. Throughout the paper, the integrals are understood in the Lebesgue sense.

For $T \in (t_0,\infty)$, we define the left continuous Heaviside function concentrated at T as follows:

$$H_T(t) = \begin{cases} 0 & \text{for } t_0 \leqslant t \leqslant T, \\ 1 & \text{for } T < t. \end{cases}$$

Then

$$\sum_{t_0<t_k<t} I_k(y(t_k)) = \sum_{k=1}^{m} I_k(y(t_k))H_{t_k}(t)$$

and system (2.1) can be rewritten as

$$\begin{cases} y(t) = y(t_0) + \displaystyle\int_{t_0}^{t} f(y_s,s)\,\mathrm{d}s + \sum_{k=1}^{m} I_k(y(t_k))H_{t_k}(t), \quad t \in [t_0,t_0+\sigma], \\ y_{t_0} = \varphi. \end{cases} \tag{2.3}$$

We assume that $f\colon\ G^-([-r,0],\mathbb{R}^n) \times [t_0,t_0+\sigma] \to \mathbb{R}^n$ is a function such that the mapping $t \mapsto f(y_t,t)$ is Lebesgue integrable on $[t_0,t_0+\sigma]$.

Denote by $|\cdot|$ an arbitrary norm in $\mathbb{R}^n$. For $y_s \in G^-([-r,0],\mathbb{R}^n)$, we denote $\|y_s\| = \sup\limits_{s\in[t_0-r,t_0+\sigma]} |y(s)|$. We also assume the following conditions hold:

(A) There is a Lebesgue integrable function $M\colon\ [t_0,t_0+\sigma] \to \mathbb{R}$ such that for all $y \in G^-([t_0-r,t_0+\sigma],\mathbb{R}^n)$ and all $u_1,u_2 \in [t_0,t_0+\sigma]$,

$$\left|\int_{u_1}^{u_2} f(y_s,s)\,\mathrm{d}s\right| \leqslant \int_{u_1}^{u_2} M(s)\,\mathrm{d}s.$$

(B) There is a Lebesgue integrable function $L\colon [t_0, t_0+\sigma] \to \mathbb{R}$ such that for all $x, y \in G^-([t_0-r, t_0+\sigma], \mathbb{R}^n)$ and all $u_1, u_2 \in [t_0, t_0+\sigma]$,

$$\left|\int_{u_1}^{u_2} [f(x_s, s) - f(y_s, s)]\,\mathrm{d}s\right| \leqslant \int_{u_1}^{u_2} L(s)\|x_s - y_s\|_\infty\,\mathrm{d}s.$$

Consider the following conditions concerning the impulse operators $I_k\colon \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, \ldots, m$:

(A′) There is a constant $K_1 > 0$ such that for all $k = 1, \ldots, m$ and all $x \in \mathbb{R}^n$,

$$|I_k(x)| \leqslant K_1.$$

(B′) There is a constant $K_2 > 0$ such that for all $k = 1, \ldots, m$ and all $x, y \in \mathbb{R}^n$,

$$|I_k(x) - I_k(y)| \leqslant K_2|x - y|.$$

Note that the Carathéodory and Lipschitz-type conditions (A) and (B) are required for the indefinite integral of f only and not for the function f itself. Thus, the standard requirement that $f(\psi, t)$ is continuous in ψ need not be fulfilled. Also, the mapping $t \mapsto f(y_t, t)$, $t \in [t_0, t_0+\sigma]$ need not be piecewise continuous, it is enough for the mapping to be Lebesgue integrable.

In [3], it was proved that under the conditions (A), (B), (A′) and (B′), a solution of the system (2.1) can be identified, in a one-to-one correspondence, with a solution of a system of generalized ordinary differential equations taking values in a Banach space. Local existence and uniqueness of the solution were guaranteed by [3], Theorems 2.15, 3.4 and 3.5.

In what follows, we give a direct proof of a local existence and uniqueness theorem for the impulsive RFDE (2.1) without employing the theory of generalized ODEs.

Theorem 2.1. *Consider problem* (2.1) *and suppose conditions* (A), (B), (A′) *and* (B′) *are fulfilled. Then there is a* $\Delta > 0$, *which depends only on* L, M, K_1, K_2 *from conditions* (A), (B), (A′) *and* (B′), *such that on the interval* $[t_0, t_0+\Delta]$ *there exists a unique solution* $y\colon [t_0-r, t_0+\Delta] \to \mathbb{R}^n$ *of problem* (2.1).

Proof. Our proof is inspired by the proof of [3], Theorem 2.15. For $t \in [t_0, t_0+\sigma]$, define the functions:

$$h_1(t) := \int_{t_0}^{t} [M(s) + L(s)]\,\mathrm{d}s \quad \text{and} \quad h_2(t) := \max(K_1, K_2)\sum_{k=1}^{m} H_{t_k}(t),$$

where H_{t_k} denotes the left continuous Heaviside function concentrated at t_k, that is,

$$H_{t_k}(t) = \begin{cases} 0 & \text{for } t \leqslant t_k, \\ 1 & \text{for } t > t_k, \end{cases}$$

for every $k = 1, 2, \dots, m$. Let $h = h_1 + h_2$. Then, clearly, the function h is nondecreasing and left continuous.

Let us prove the local existence and uniqueness of the solution of (2.1). Since t_0 is not a moment of impulse, h is continuous at t_0. Therefore, there exists a $\Delta > 0$ such that $[t_0, t_0 + \Delta] \subset [t_0, t_0 + \sigma)$ and $h(t_0 + \Delta) - h(t_0) < 1/2$.

Let Q be the set of all functions $z\colon [t_0 - r, t_0 + \Delta] \to \mathbb{R}^n$ such that $z \in G^-([t_0 - r, t_0 + \Delta], \mathbb{R}^n)$ and $|z(t) - \varphi(0)| \leqslant h(t) - h(t_0)$ for $t \in [t_0, t_0 + \Delta]$.

It is easy to show that the set $Q \subset G^-([t_0 - r, t_0 + \Delta], \mathbb{R}^n)$ is closed.

For $s \in [t_0 - r, t_0 + \Delta]$ and $z \in Q$, define

$$Tz(s) = \begin{cases} \varphi(s - t_0), & s \in [t_0 - r, t_0], \\ \varphi(0) + \displaystyle\int_{t_0}^{s} f(z_t, t)\,\mathrm{d}t + \sum_{t_0 < t_j < s} I_j(z(t_j)), & s \in [t_0, t_0 + \Delta]. \end{cases}$$

Then, by conditions (A) and (A′), we have

$$|Tz(s) - \varphi(0)| = \left| \int_{t_0}^{s} f(z_t, t)\,\mathrm{d}t + \sum_{t_0 < t_j \leqslant s} I_j(z(t_j)) \right| \leqslant h(s) - h(t_0), \quad s \in [t_0, t_0 + \Delta].$$

Also, the fact that Tz belongs to $G^-([t_0 - r, t_0 + \Delta], \mathbb{R}^n)$ is not difficult to prove. Thus, T maps Q into itself.

Let $z_1, z_2 \in Q$; if $t_0 - r \leqslant s_1 < s_2 \leqslant t_0$, then

$$\begin{aligned} &|Tz_2(s_2) - Tz_1(s_2) - [Tz_2(s_1) - Tz_1(s_1)]| \\ &\quad = |\varphi(s_2 - t_0) - \varphi(s_2 - t_0) - \varphi(s_1 - t_0) + \varphi(s_1 - t_0)| = 0 \\ &\quad \leqslant \|z_2 - z_1\| [h(t_0 + \Delta) - h(t_0)]. \end{aligned}$$

If $t_0 - r \leqslant s_1 \leqslant t_0$ and $t_0 \leqslant s_2 \leqslant t_0 + \Delta$, then conditions (B) and (B′) imply

$$\begin{aligned} &|Tz_2(s_2) - Tz_1(s_2) - [Tz_2(s_1) - Tz_1(s_1)]| \\ &\quad = \left| \int_{t_0}^{s_2} [f((z_2)_t, t) - f((z_1)_t, t)]\,\mathrm{d}t + \sum_{t_0 < t_j \leqslant s_2} [I_j(z_2(t_j)) - I_j(z_1(t_j))] \right| \\ &\quad \leqslant \int_{t_0}^{s_2} L(t) \|(z_2)_t - (z_1)_t\|\,\mathrm{d}t + K_2 \sum_{j=1}^{m} |z_2(t_j) - z_1(t_j)| H_{t_j}(s_2) \\ &\quad \leqslant \sup_{s \in [t_0 - r, t_0 + \Delta]} |z_2(s) - z_1(s)|\, [h(s_2) - h(t_0)] \\ &\quad \leqslant \|z_2 - z_1\| [h(t_0 + \Delta) - h(t_0)]. \end{aligned}$$

If $t_0 \leqslant s_1 < s_2 \leqslant t_0 + \Delta$, then conditions (B) and (B′) imply

$$\begin{aligned}
&|Tz_2(s_2) - Tz_1(s_2) - [Tz_2(s_1) - Tz_1(s_1)]| \\
&\quad = \left| \int_{s_1}^{s_2} [f((z_2)_t, t) - f((z_1)_t, t)]\,\mathrm{d}t + \sum_{s_1 \leqslant t_j < s_2} [I_j(z_2(t_j)) - I_j(z_1(t_j))] \right| \\
&\quad \leqslant \int_{s_1}^{s_2} L(t)\|(z_2)_t - (z_1)_t\|\,\mathrm{d}t + K_2 \sum_{j=1}^{m} |z_2(t_j) - z_1(t_j)|[H_{t_j}(s_2) - H_{t_j}(s_1)] \\
&\quad \leqslant \sup_{s \in [t_0 - r, t_0 + \Delta]} |z_2(s) - z_1(s)|\,[h(s_2) - h(t_0)] \\
&\quad \leqslant \|z_2 - z_1\|[h(t_0 + \Delta) - h(t_0)].
\end{aligned}$$

Therefore, using the above inequalities, we obtain

$$\|Tz_2 - Tz_1\| \leqslant \|z_2 - z_1\|[h(t_0 + \Delta) - h(t_0)] < \frac{1}{2}\|z_2 - z_1\|.$$

Thus, T is a contraction and by the Banach fixed-point theorem, T possesses a unique fixed point. Remember that by the definition of the operator T, x is a unique solution of (2.1) ifand only if it is a unique fixed point of T. Therefore, the result follows as well. □

Remark 2.1. We point out that Δ in the previous theorem depends only on M, L, K_1 and K_2 and also, note that by the definition of function h, $h(t_0) = 0$. Therefore, the proof of the previous theorem can be rewritten replacing $h(t_0)$ by 0.

3. Continuous dependence for impulsive RFDEs

First, regarding continuous dependence results for impulsive RFDEs, we have to mention [9] for an elucidatory discussion of the continuous dependence of solutions on parameters of an impulsive delay differential equations whose impulse operators also involve delays.

In this section, we mention some important results and definitions that are essential for proving our main result, namely, Theorem 3.3.

Definition 3.1. A set $\mathcal{A} \subset G([a,b], X)$ is called *equiregulated*, if it has the following property: for every $\varepsilon > 0$ and $t_0 \in [a,b]$, there is $\delta > 0$ such that

(1) if $y \in \mathcal{A}$, $t' \in [a,b]$ and $t_0 - \delta < t' < t_0$, then $|y(t_0{}^-) - y(t')| < \varepsilon$;
(2) if $y \in \mathcal{A}$, $t'' \in [a,b]$ and $t_0 < t'' < t_0 + \delta$, then $|y(t'') - y(t_0{}^+)| < \varepsilon$.

The next proposition can be found in [4], Theorem 2.18. It is an Arzelà-Ascoli-type theorem for regulated functions.

Theorem 3.1. *The following conditions are equivalent:*

(i) *A set $\mathcal{A} \subset G([a,b],\mathbb{R}^n)$ is relatively compact.*

(ii) *The set $\{y(a)\colon\ y \in \mathcal{A}\}$ is bounded and there is an increasing continuous function $\eta\colon\ [0,\infty) \to [0,\infty)$, $\eta(0) = 0$ and an increasing function $K\colon\ [a,b] \to \mathbb{R}$ such that*

$$\|y(t_2) - y(t_1)\| \leqslant \eta(K(t_2) - K(t_1))$$

for every $y \in \mathcal{A}$, $a \leqslant t_1 \leqslant t_2 \leqslant b$.

(iii) *$\mathcal{A}$ is equiregulated and for every $t \in [a,b]$, the set $\{y(t); y \in \mathcal{A}\}$ is bounded.*

For $p = 0, 1, 2, \ldots$, we consider the following Cauchy problem:

$$\begin{cases} \dot{y}(t) = f_p(y_t, t), & t \neq t_k, \\ \Delta y(t_k) = I_k^p(y(t_k)), & k = 1,\ldots,m, \\ y_{t_0} = \varphi_p, \end{cases} \tag{3.1}$$

where $t_0 < t_1 < \ldots < t_k < \ldots < t_m \leqslant t_0 + \sigma$, and for each $p = 0, 1, 2, \ldots$, $x \mapsto I_k^p(x)$ maps $\mathbb{R}^n$ into itself and $\Delta y(t_k) := y(t_k+) - y(t_k-) = y(t_k+) - y(t_k)$, $k = 1, 2, \ldots, m$.

The next theorem is a continuous dependence result which, together with Theorem 3.1, is essential for proving our main result. A proof of the next theorem can be found in [3], Theorem 4.1.

Theorem 3.2. *Assume that for each $p = 0, 1, \ldots$, we have $\varphi_p \in G^-([-r,0],\mathbb{R}^n)$, and moreover, $f_p\colon\ G^-([-r,0],\mathbb{R}^n) \times [t_0, t_0+\sigma] \to \mathbb{R}^n$ and $I_k^p\colon\ \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \ldots, m$, satisfy conditions* (A), (B), (A′) *and* (B′) *for the same functions M, L and the same constants K_1, K_2. Suppose*

$$\lim_{p\to\infty} \sup_{\vartheta \in [t_0, t_0+\sigma]} \left| \int_{t_0}^{\vartheta} [f_p(y_s, s) - f_0(y_s, s)]\,\mathrm{d}s \right| = 0 \tag{3.2}$$

for every $y \in G^-([t_0 - r, t_0 + \sigma],\mathbb{R}^n)$ and

$$\lim_{p\to\infty} I_k^p(x) = I_k^0(x) \tag{3.3}$$

for every $x \in \mathbb{R}^n$, $k = 1, \ldots, m$. Assume further that, for each $p = 1, 2, \ldots$, $y_p\colon\ [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ is a solution on $[t_0 - r, t_0 + \sigma]$ of the problem

$$\begin{cases} \dot{y}(t) = f_p(y_t, t), & t \neq t_k, \\ \Delta y(t_k) = I_k^p(y(t_k)), & k = 1,\ldots,m, \\ y_{t_0} = \varphi_p, \end{cases} \tag{3.4}$$

and

$$\lim_{p\to\infty} y_p = y \quad \textit{uniformly on} \quad [t_0 - r, t_0 + \sigma]. \tag{3.5}$$

Then $y\colon [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ *is a solution on* $[t_0 - r, t_0 + \sigma]$ *of the problem*

$$\begin{cases} \dot{y}(t) = f_0(y_t, t), & t \neq t_k, \\ \Delta y(t_k) = I_k^0(y(t_k)), & k = 1, \ldots, m, \\ y_{t_0} = \varphi_0. \end{cases} \tag{3.6}$$

The assumptions (3.2) and (3.3) in Theorem 3.2 ensure that, if the sequence $\{y_p\}_{p\geqslant 1}$, $y_p\colon [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$, $p = 1, 2, \ldots$, of solutions of (3.1) converges uniformly to a function $y\colon [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$, then the limit is a solution of (3.6).

The next result says that adding a uniqueness condition to the "limit" equation, then, for sufficiently large $p \in \mathbb{N}$, $y_p\colon [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ is a solution of (3.1) provided the sequence of the initial data $\{\varphi_p\}_{p\geqslant 1}$ converges uniformly on $[-r, 0]$.

Theorem 3.3. *Assume that there exist* M, L, B, K_1 *and* K_2 *such that the conditions* (A), (B), (A′) *and* (B′) *are satisfied for each* p *when* f *is replaced by* f_p *and* I_k *are replaced by* I_k^p. *Also, suppose that*

$$\lim_{p\to\infty} \int_{t_0}^{t} [f_p(y_s, s) - f_0(y_s, s)] \, \mathrm{d}s = 0, \quad t \in [t_0, t_0 + \sigma] \tag{3.7}$$

for every $y \in G^-([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, *and*

$$\lim_{p\to\infty} I_k^p(x) = I_k^0(x) \tag{3.8}$$

for every $x \in \mathbb{R}^n$ *and* $k = 1, \ldots, m$. *Let* $y\colon [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ *be a unique solution of*

$$\begin{cases} \dot{y}(t) = f_0(y_t, t), & t \neq t_k, \\ \Delta y(t_k) = I_k^0(y(t_k)), & k = 1, \ldots, m, \\ y_{t_0} = \varphi_0, \end{cases} \tag{3.9}$$

on $[t_0 - r, t_0 + \sigma]$, *where* $\varphi_0 \in G^-([-r, 0], \mathbb{R}^n)$. *Let* $\{\varphi_p\}_{p\geqslant 1}$ *be a sequence of regulated and left continuous functions from* $[-r, 0]$ *to* $\mathbb{R}^n$. *Assume further that* $\varphi_p \to \varphi_0$ *uniformly on* $[-r, 0]$ *as* $p \to \infty$. *Then, for sufficiently large* $p \in \mathbb{N}$, *there exists a solution* y_p *of*

$$\begin{cases} \dot{y}(t) = f_p(y_t, t), & t \neq t_k, \\ \Delta y(t_k) = I_k^p(y(t_k)), & k = 1, \ldots, m, \\ y_{t_0} = \varphi_p, \end{cases} \tag{3.10}$$

on $[t_0-r,t_0+\sigma]$ *and the sequence* $\{y_p\}_{p\geqslant 1}$ *possesses a subsequence which converges uniformly, that is,*

$$\lim_{l\to\infty} y_{p_l} = y \quad \textit{uniformly on } [t_0-r,t_0+\sigma]. \tag{3.11}$$

Proof. The present proof is inspired by the proof of [10], Theorem 8.6, for generalized ODEs. We strongly use the fact that the functions f_p, $p=0,1,2,\ldots$, take values in a finite dimensional space so that we can apply Theorem 3.1.

Since $\varphi_p\to\varphi_0$ uniformly on $[-r,0]$ as $p\to\infty$, it follows that $\varphi_0\in G^-([-r,0],\mathbb{R}^n)$.

For each $p\in\mathbb{N}$, all the hypotheses of Theorem 2.1 are satisfied, hence there is a $\Delta>0$ such that on the interval $[t_0,t_0+\Delta]$, there exists a local unique solution y_p of problem (3.10). Notice that, according to the proof of Theorem 2.1, $\Delta>0$ is uniform and independent of p.

Therefore, for each p, y_p is a solution of (3.10) on $[t_0,t_0+\Delta]$ and by conditions (A), (B), (A′) and (B′), for every $s_1,s_2\in[t_0,t_0+\Delta]$ such that $s_2>s_1$ we have

$$\begin{aligned}\|y_p(s_2)-y_p(s_1)\| &\leqslant \int_{s_1}^{s_2}[L(s)+M(s)]\,\mathrm{d}s+\max(K_1,K_2)\sum_{k=1}^{m}[H_{t_k}(s_2)-H_{t_k}(s_1)]\\ &< \int_{s_1}^{s_2}[L(s)+M(s)]\,\mathrm{d}s+\max(K_1,K_2)\sum_{k=1}^{m}[H_{t_k}(s_2)-H_{t_k}(s_1)]\\ &\quad+(s_2-s_1),\end{aligned}$$

where H_{t_k} denotes the left continuous Heaviside function concentrated at t_k. Then, defining the functions $\eta\colon[0,\infty)\to[0,\infty)$ and $K\colon[t_0,t_0+\Delta]\to\mathbb{R}$ by

$$K(t)=\int_{t_0}^{t}[L(s)+M(s)]\,\mathrm{d}s+\max(K_1,K_2)\sum_{k=1}^{m}[H_{t_k}(t)-H_{t_k}(t_0)]+(t-t_0)$$

and

$$\eta(t)=t,$$

we have $\|y_p(s_2)-y_p(s_1)\|\leqslant\eta(K(s_2)-K(s_1))$, where K is clearly increasing and η is a continuous function satisfying $\eta(0)=0$. Then, since φ_p is bounded, for $p=0,1,2,\ldots$, Theorem 3.1 implies that $\{y_p\}_{p\geqslant 1}$ contains a subsequence which is uniformly convergent on $[t_0,t_0+\Delta]$.

Without loss of generality, we can denote this subsequence again by $\{y_p\}_{p=1}^{\infty}$. Since $(y_p)_{t_0}=\varphi_p$, we see that $\{y_p\}_{p=1}^{\infty}$ is in fact uniformly convergent on $[t_0-r,t_0+\Delta]$. Thus,

$$\lim_{p\to\infty} y_p = y$$

uniformly on $[t_0 - r, t_0 + \Delta]$. By Theorem 3.2 and by the uniqueness of solutions, it follows that y is a solution of (3.9) on $[t_0, t_0 + \Delta]$.

Therefore, the theorem holds on $[t_0, t_0 + \Delta]$. It also holds on $[t_0 - r, t_0]$, since $\varphi_p \to \varphi_0$ uniformly.

Now, let us assume that the convergence result does not hold on the whole interval $[t_0 - r, t_0 + \sigma]$. Thus there exist a Δ', $0 < \Delta' < \sigma$ and $N \in \mathbb{N}$ sufficiently large such that for every $\Delta < \Delta'$ and for $p > N$, there is a solution y_p of (3.10) on $[t_0 - r, t_0 + \Delta]$, with $(y_p)_{t_0} = \varphi_p$, and $\lim_{p\to\infty} y_p(t) = y(t)$ for $t \in [t_0 - r, t_0 + \Delta]$, but this does not hold on $[t_0 - r, t_0 + \Delta]$ whenever $\Delta > \Delta'$.

Since

$$\|y_p(s_2) - y_p(s_1)\| \leqslant \int_{s_1}^{s_2} [L(s) + M(s)]\,\mathrm{d}s + \max(K_1, K_2) \sum_{k=1}^{m} [H_{t_k}(s_2) - H_{t_k}(s_1)],$$

for every $s_2, s_1 \in [t_0 - r, t_0 + \Delta')$ and every $p > N$, we have that the limit

$$y_p((t_0 + \Delta')-) = \lim_{\varepsilon\to 0^-} y_p(t_0 + \Delta' + \varepsilon), \quad p > N,$$

exists and $y_p((t_0 + \Delta')-) = y_p(t_0 + \Delta')$, for $p > N$, since y is left continuous.

Defining $y_p(t_0+\Delta') = y_p((t_0+\Delta')-)$ for $p > N$, then $\lim_{p\to\infty} y_p(t_0+\Delta') = y(t_0+\Delta')$. Therefore, the theorem holds on $[t_0 - r, t_0 + \Delta']$ as well. Then, using $t_0 + \Delta' < t_0 + \sigma$ as the starting point, it can be proved, analogously, that the theorem holds on the interval $[t_0 + \Delta', t_0 + \Delta' + \eta]$ for some $\eta > 0$, and this contradicts our assumption. Thus, the theorem holds on the whole interval $[t_0 - r, t_0 + \sigma]$. □

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