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Kybernetika, Vol. 52 (2016), No. 1, 131–152

Persistent URL: `http://dml.cz/dmlcz/144867`

KYBERNETIKA — VOLUME 52 (2016), NUMBER 1, PAGES 131–152

STABILIZATION OF HOMOGENEOUS POLYNOMIAL SYSTEMS IN THE PLANE

Hamadi Jerbi, Thouraya Kharrat and Khaled Sioud

In this paper, we study the problem of stabilization via homogeneous feedback of single-input homogeneous polynomial systems in the plane. We give a complete classification of systems for which there exists a homogeneous stabilizing feedback that is smooth on $\mathbb{R}^2\setminus\{(0,0)\}$ and preserve the homogeneity of the closed loop system. Our results are essentially based on Theorem of Hahn in which the author gives necessary and sufficient conditions of stability of homogeneous systems in the plane.

Keywords: polynomial system, control system, homogeneous feedback, stabilization

Classification: 93D15

1. INTRODUCTION

For affine in the control systems in the form

$$\dot{x} = f(x) + ug(x) \tag{1}$$

where the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}$, $f(0) = 0$, and f, g are smooth vector fields, the basic stabilization Lyapunov condition provided in [1, 10, 11] and [12] can be expressed as follows. There exists a positive definite real function $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ (i. e., $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ near zero) such that for any $x \neq 0$ near zero with $\nabla V.g(x) = 0$ it holds $\nabla V.f(x) < 0$. In [1] it was shown that if the above condition is fulfilled, then the system (1) is stabilizable at the origin by means of a nonlinear feedback law which is smooth for $x \neq 0$. The same result was proved independently in [10, 11] and [12], where the corresponding stabilizing feedback laws are more explicitly identified. One of the interesting topics in control theory is the stabilization of homogeneous systems affine in control. The importance of such systems is due to the fact that they model several phenomena. In addition, under the classical theorem of Massera [9], a vector field regular enough is asymptotically stable, if the first nonzero term in its Taylor expansion, which defines a homogeneous vector field, is asymptotically stable. One direct consequence of this result is: if a nonlinear control system admits a first approximation stabilizable by a homogeneous feedback, then the original system is also stabilizable by the same homogeneous feedback. In order to make use of this approximation property

DOI: 10.14736/kyb-2016-1-0131

in the design of locally stabilizing feedbacks for nonlinear systems, the main idea lies in the construction of homogeneous feedback laws that preserve the homogeneity of the resulting closed loop system. These laws can be shown to be locally stabilizing the approximated nonlinear system ([7, 10, 11] and [12]). It was shown in [8] that for general controllable homogeneous systems, the existence of a stabilizing feedback does not necessarily imply the existence of a homogeneous stabilizing feedback. It is well known that homogeneous vector fields of even degree can not be asymptotically stable at the origin [11]. Accordingly, our results are valid only if the homogeneity degree is odd.

In this paper, we give a complete characterization of single-input homogeneous systems in the plane of the form

$$\left\{ \begin{array}{ll} \dot{x}_1 = & \mathcal{P}_1(x_1,x_2) + u\mathcal{Q}_1(x_1,x_2) \\ \dot{x}_2 = & \mathcal{P}_2(x_1,x_2) + u\mathcal{Q}_2(x_1,x_2) \end{array} \right. \tag{2}$$

where $(x_1,x_2) \in \mathbb{R}^2$, $u \in \mathbb{R}$, $\mathcal{P}_1$ and $\mathcal{P}_2$ being homogeneous polynomials of degree $2k+1$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are homogeneous polynomials of degree q, with k and q are integers. The problem is to find a feedback function $(x_1,x_2) \mapsto u(x_1,x_2)$ which is homogeneous of degree $2k+1-q$ and asymptotically stabilizes the control system (2). If such a feedback exists, we say that the system (2) is globally asymptotically stabilizable (GAS) at the origin. We give some methods for the construction of a homogeneous feedback which makes the system (2) globally asymptotically stable. Obviously, asymptotic controllability at the origin is a necessary condition for asymptotic stabilizability. For this, we recall the following: we say that the system (2) is asymptotically controllable at the origin if for any given $(x_1^0,x_2^0) \in \mathbb{R}^2$, there exists a time-dependent control law u such that $\lim_{t\to+\infty}(x_1(t),x_2(t)) = 0$, $(x_1(t),x_2(t))$ denoting the solution of the control system (2), with initial condition $(x_1(0),x_2(0)) = (x_1^0,x_2^0)$. In this paper, the study of the stabilization via homogeneous feedback of the control system (2) is based essentially on a theorem given by Hahn in [3], in which the author gives necessary and sufficient conditions for stability of homogeneous systems in the plane. In [6], the authors give necessary and sufficient conditions for the existence of a stabilizing homogeneous feedback. They give an explicit construction of the stabilizing feedback in the case where the polynomial functions $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have no common linear factor and the functions $\mathcal{G}$ and $\mathcal{H}$ introduced in definition (5) have no common zeros. In [5], the authors give a classification of the globally stabilizable systems of the form (2), when $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have no common linear factor. The form of the feedback considered in this paper is complicated and hard to compute explicitly. In the present work, the goal is to simplify this form and to complete the classification of the stabilizability of the system (2) in the case where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have common linear factor and the functions $\mathcal{G}$ and $\mathcal{H}$ have common zeros. We treat two cases. In the first case the function $\mathcal{G}$ is definite and in the second one $\mathcal{G}$ has at least a linear factor in its factorization.

2. PRELIMINARY RESULTS

Let $x = (x_1,x_2)$, $y = (y_1,y_2) \in \mathbb{R}^2$. We introduce the following notations:

- $\langle x^T \mid y^T \rangle = \sum_{i=1}^{2} x_i y_i$ denotes the Euclidean inner product.
- $\|x\| = \sqrt{\langle x^T \mid x^T \rangle}$ denotes the Euclidean norm on $\mathbb{R}^2$.
- Let $M \in \mathcal{M}_{n,p}(\mathbb{R})$, M^T denotes the transpose matrix of M.

Definition. Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial function. We say that P is homogeneous of degree $d \in \mathbb{N}$, if

$$P(\lambda x) = \lambda^d P(x), \ \forall \lambda \in \mathbb{R}, \ \forall x \in \mathbb{R}^2.$$

We recall the following theorem, which gives necessary and sufficient conditions for the stability of homogeneous systems in the plane and plays an important role in our study.

Theorem 2.1. (Hahn [3]) Consider the two-dimensional system

$$\begin{cases} \dot{x_1} & = X_1(x_1, x_2) \\ \dot{x_2} & = X_2(x_1, x_2) \end{cases} \tag{3}$$

where $X_1(0,0) = 0$, $X_2(0,0) = 0$ and the vector fields X_1 and X_2 are Lipschitz, continuous and homogeneous of degree p.

Let $\Phi(x_1, x_2) = \det \begin{pmatrix} X_1(x_1, x_2) & x_1 \\ X_2(x_1, x_2) & x_2 \end{pmatrix}$.

The system (3) is asymptotically stable if and only if one of the following conditions is satisfied:

(i) the system (3) does not have any one-dimensional invariant subspace and

$$X_2(1,0) \int_0^{2\pi} \frac{\cos\theta X_1(\cos\theta, \sin\theta) + \sin\theta X_2(\cos\theta, \sin\theta)}{\cos\theta X_2(\cos\theta, \sin\theta) - \sin\theta X_1(\cos\theta, \sin\theta)} \, \mathrm{d}\theta < 0,$$

(ii) the restriction of the system (3) to each of its one-dimensional invariant subspaces is asymptotically stable, i. e. if the point $(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$ satisfies $\Phi(\xi_1, \xi_2) = 0$, then

$$\left\langle \begin{pmatrix} X_1(\xi_1, \xi_2) \\ X_2(\xi_1, \xi_2) \end{pmatrix} \mid \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \right\rangle < 0.$$

In the following, we give a result which plays an important role in the study of the stability of the system (3), when it has no one-dimensional invariant subspace.

Proposition 2.2. If $\Phi(x_1, x_2) \neq 0$ for all $(x_1, x_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$, then

$$\int_0^{2\pi} \frac{\cos\theta X_1(\cos\theta, \sin\theta) + \sin\theta X_2(\cos\theta, \sin\theta)}{\cos\theta X_2(\cos\theta, \sin\theta) - \sin\theta X_1(\cos\theta, \sin\theta)} \, \mathrm{d}\theta = 2 \lim_{a \to +\infty} \int_{-a}^{a} \frac{X_1(1,s)}{X_2(1,s) - sX_1(1,s)} \, \mathrm{d}s.$$

P r o o f. Using the homogeneity of the vector field (X_1, X_2) and the $2\pi-$periodicity of the functions *cosine* and *sine*, we can easily verify that

$$\int_0^{2\pi} \frac{\cos\theta X_1(\cos\theta,\sin\theta)+\sin\theta X_2(\cos\theta,\sin\theta)}{\cos\theta X_2(\cos\theta,\sin\theta)-\sin\theta X_1(\cos\theta,\sin\theta)}\,\mathrm{d}\theta$$

$$= 2\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta X_1(\cos\theta,\sin\theta)+\sin\theta X_2(\cos\theta,\sin\theta)}{\cos\theta X_2(\cos\theta,\sin\theta)-\sin\theta X_1(\cos\theta,\sin\theta)}\,\mathrm{d}\theta$$

$$= 2\lim_{\alpha\to\frac{\pi}{2}^-}\int_{-\alpha}^{\alpha} \frac{\cos\theta X_1(\cos\theta,\sin\theta)+\sin\theta X_2(\cos\theta,\sin\theta)}{\cos\theta X_2(\cos\theta,\sin\theta)-\sin\theta X_1(\cos\theta,\sin\theta)}\,\mathrm{d}\theta.$$

We get for $0<\alpha<\frac{\pi}{2}$

$$\int_{-\alpha}^{\alpha} \frac{\cos\theta X_1(\cos\theta,\sin\theta)+\sin\theta X_2(\cos\theta,\sin\theta)}{\cos\theta X_2(\cos\theta,\sin\theta)-\sin\theta X_1(\cos\theta,\sin\theta)}\,\mathrm{d}\theta$$

$$= \int_{-\alpha}^{\alpha} \frac{\cos^{p+1}\theta X_1(1,\frac{\sin\theta}{\cos\theta})+\sin\theta\cos^p\theta X_2(1,\frac{\sin\theta}{\cos\theta})}{\cos^{p+1}\theta X_2(1,\frac{\sin\theta}{\cos\theta})-\cos^p\theta\sin\theta X_1(1,\frac{\sin\theta}{\cos\theta})}\,\mathrm{d}\theta\cdot$$

By the change of coordinates $u=\tan\theta$, we deduce

$$\int_0^{2\pi} \frac{\cos\theta X_1(\cos\theta,\sin\theta)+\sin\theta X_2(\cos\theta,\sin\theta)}{\cos\theta X_2(\cos\theta,\sin\theta)-\sin\theta X_1(\cos\theta,\sin\theta)}\,\mathrm{d}\theta$$

$$= 2\lim_{a\to+\infty}\int_{-a}^{a}\frac{X_1(1,u)+uX_2(1,u)}{X_2(1,u)-uX_1(1,u)}\frac{\mathrm{d}u}{(1+u^2)}.$$

Let $a>0$, one has

$$\int_{-a}^{a}\frac{X_1(1,u)+uX_2(1,u)}{X_2(1,u)-uX_1(1,u)}\frac{\mathrm{d}u}{(1+u^2)}$$

$$= \int_{-a}^{a}\frac{X_1(1,u)+uX_2(1,u)+u^2X_1(1,u)-u^2X_1(1,u)}{X_2(1,u)-uX_1(1,u)}\frac{\mathrm{d}u}{(1+u^2)}$$

$$= \int_{-a}^{a}\frac{(1+u^2X_1(1,u))+u(X_2(1,u)-uX_1(1,u))}{X_2(1,u)-uX_1(1,u)}\frac{\mathrm{d}u}{(1+u^2)}$$

$$= \int_{-a}^{a}\frac{(1+u^2)X_1(1,u)\mathrm{d}u}{(1+u^2)(X_2(1,u)-uX_1(1,u))}+\int_{-a}^{a}\frac{u(X_2(1,u)-uX_1(1,u))}{X_2(1,u)-uX_1(1,u)}\frac{\mathrm{d}u}{(1+u^2)}$$

$$= \int_{-a}^{a}\frac{X_1(1,u)}{X_2(1,u)-uX_1(1,u)}\mathrm{d}u+\int_{-a}^{a}\frac{u}{(1+u^2)}\,\mathrm{d}u$$

$$= \int_{-a}^{a}\frac{X_1(1,u)}{X_2(1,u)-uX_1(1,u)}\,\mathrm{d}u.$$

We conclude by

$$\int_0^{2\pi} \frac{\cos\theta X_1(\cos\theta,\sin\theta)+\sin\theta X_2(\cos\theta,\sin\theta)}{\cos\theta X_2(\cos\theta,\sin\theta)-\sin\theta X_1(\cos\theta,\sin\theta)}\,\mathrm{d}\theta = 2\lim_{a\to+\infty}\int_{-a}^{a}\frac{X_1(1,u)}{X_2(1,u)-uX_1(1,u)}\mathrm{d}u.$$

□

Now, we consider the homogeneous polynomial system

$$\begin{cases} \dot{x}_1 = \mathcal{P}_1(x_1,x_2) + u\mathcal{Q}_1(x_1,x_2) \\ \dot{x}_2 = \mathcal{P}_2(x_1,x_2) + u\mathcal{Q}_2(x_1,x_2) \end{cases} \tag{4}$$

where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are homogeneous polynomials of degree q and $\mathcal{P}_1$ and $\mathcal{P}_2$ are homogeneous polynomials of degree $2k+1$. The problem is to construct a homogeneous feedback of degree $2k+1-q$ which stabilizes the system (4).

We define the following polynomial functions which paly an important role in our study.

$$\begin{aligned} \mathcal{G}(x_1,x_2) &= det \begin{pmatrix} \mathcal{Q}_1(x_1,x_2) & x_1 \\ \mathcal{Q}_2(x_1,x_2) & x_2 \end{pmatrix} = x_2\mathcal{Q}_1(x_1,x_2) - x_1\mathcal{Q}_2(x_1,x_2); \\ \mathcal{H}(x_1,x_2) &= det \begin{pmatrix} \mathcal{P}_1(x_1,x_2) & x_1 \\ \mathcal{P}_2(x_1,x_2) & x_2 \end{pmatrix} = x_2\mathcal{P}_1(x_1,x_2) - x_1\mathcal{P}_2(x_1,x_2); \\ \mathcal{F}(x_1,x_2) &= det \begin{pmatrix} \mathcal{P}_1(x_1,x_2) & \mathcal{Q}_1(x_1,x_2) \\ \mathcal{P}_2(x_1,x_2) & \mathcal{Q}_2(x_1,x_2) \end{pmatrix} \\ &= \mathcal{P}_1(x_1,x_2)\mathcal{Q}_2(x_1,x_2) - \mathcal{P}_2(x_1,x_2)\mathcal{Q}_1(x_1,x_2). \end{aligned} \tag{5}$$

It is easy to remark that $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{F}$ are homogeneous polynomials of degree $q+1$, $2k+2$ and $2k+q+1$ respectively.

Since $\mathcal{G}$ is a homogeneous polynomial of degree $q+1$, one can write for $x_2 \neq 0$:

$$\mathcal{G}(x_1,x_2) = \mathcal{G}(x_2(\frac{x_1}{x_2},1)) = x_2^{q+1}\mathcal{G}(z,1), \text{ where } z = \frac{x_1}{x_2}.$$

It is clear that $g(z) := \mathcal{G}(z,1)$ is a polynomial function, so one has

$$\mathcal{G}(z,1) = \prod_{i=1}^{p_1}(c_i z - \tilde{c}_i)^{\eta_i} \prod_j (a_j z^2 + \tilde{a}_j z + b_j)^{\mu_j},$$

with $\tilde{a}_j^2 - 4a_j b_j < 0$ and η_i and μ_j lie in $\mathbb{N} \setminus \{0\}$. We get

$$\begin{aligned} \mathcal{G}(x_1,x_2) &= x_2^{q+1} \prod_{i=1}^{p_1} \left(c_i \frac{x_1}{x_2} - \tilde{c}_i \right)^{\eta_i} \prod_j (a_j \left(\frac{x_1}{x_2})^2 + \tilde{a}_j \frac{x_1}{x_2} + b_j \right)^{\mu_j} \\ &= \prod_{i=1}^{p_1}(c_i x_1 - \tilde{c}_i x_2)^{\eta_i} \prod_j (a_j x_1^2 + \tilde{a}_j x_1 x_2 + b_j x_2^2)^{\mu_j}. \end{aligned}$$

As $\mathcal{G}$ has only a finite number of zeros on the unit sphere, we denote these zeros by $C_i = (\tilde{c}_i, c_i) = (\cos\theta_i,\ \sin\theta_i)$ with order of multiplicity η_i, for $i \in I_1 := \{1,\ldots,p_1\}$. Without loss of generality, we can assume that $0 \leq \theta_1 < \theta_2 < \ldots < \theta_{p_1} < \pi$. We denote $S_p\mathcal{G} = \{C_1, C_2, \cdots, C_{p_1}\}$ the set of the zeros of the function $\mathcal{G}$ on the top half of the unit sphere. Since $\mathcal{H}$ is a homogeneous polynomial function of degree $2k+2$, it has a finite number of zeros on the top half of the unit sphere. We denote $(\tilde{d}_j, d_j)$, $j \in I_2 := \{1,\ldots,p_2\}$, the common zeros of the two functions $\mathcal{G}$ and $\mathcal{H}$ on the top half of the unit sphere. So, one can write

$$\mathcal{H}(x_1,x_2)=\left(\prod_{j=1}^{p_2}(d_jx_1-\tilde{d}_jx_2)^{\gamma_j}\right)\widetilde{\mathcal{H}}(x_1,x_2),\text{ with }\tilde{\mathcal{H}}(\tilde{c}_i,c_i)\neq 0\text{ for all }i\in I_1.$$

It is clear that the set $\{(\tilde{d}_j,d_j),\ \ j\in I_2\}$ is a subset of $\{(\tilde{c}_i,c_i),\ \ i\in I_1\}$.

Let $u(x_1,x_2)$ an homogeneous feedback of degree $(2k+1-q)$. The closed loop system (4) by the feedback u can be written as

$$\begin{cases}\dot{x}_1=\mathcal{P}_1(x_1,x_2)+u(x_1,x_2)\mathcal{Q}_1(x_1,x_2)=X_1(x_1,x_2)\\ \dot{x}_2=\mathcal{P}_2(x_1,x_2)+u(x_1,x_2)\mathcal{Q}_2(x_1,x_2)=X_2(x_1,x_2).\end{cases}\tag{6}$$

We recall the function $\Phi(x_1,x_2)=x_2X_1(x_1,x_2)-x_1X_2(x_1,x_2)$.

It is clear that $\Phi(x_1,x_2)=\mathcal{H}(x_1,x_2)+u(x_1,x_2)\mathcal{G}(x_1,x_2)$.

Since $u(x_1,x_2)$ is homogeneous of degree $2k+1-q$, we get the function Φ is homogeneous of degree $2k+2$. From Theorem 2.1, the function Φ plays an important role in the study of the stability of the closed loop system (6). Moreover to determine a feedback $u(x_1,x_2)$ which stabilizes the system (4), we construct a function Φ satisfying the following conditions:

(A_1) The function Φ is C^∞ in $\mathbb{R}^2\setminus\{(0,0)\}$ and homogeneous of degree $2k+2$;

(A_2) The functions $(c_ix_1-\tilde{c}_ix_2)^{\eta_i}$ divide $\Phi(x_1,x_2)-\mathcal{H}(x_1,x_2)$ for all $i\in I_1$;

(A_3) If the set of points $\xi\in\mathbb{R}^2\setminus\{(0,0)\}$ such that $\Phi(\xi)=0$ is non empty and the point $\xi=(\xi_1,\xi_2)$ satisfies $\Phi(\xi)=0$, then $\langle (X_1(\xi),X_2(\xi))^T\mid(\xi)^T\rangle<0$.

Proposition 2.3. (Jerbi and Kharrat [6]) Let $\xi=(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{(0,0)\}$. If $\Phi(\xi_1,\xi_2)=0$ and $\mathcal{F}(\xi_1,\xi_2)\neq 0$, then the subset

$$\Gamma=\{(x_1,x_2)\in\mathbb{R}^2,\ \text{such that}\ \xi_1x_2-\xi_2x_1=0\}$$

is invariant by the closed loop system (6) and one has

$$\langle (X_1(\xi_1,\xi_2),X_2(\xi_1,\xi_2))^T|(\xi_1,\xi_2)^T\rangle=-\frac{\mathcal{F}(\xi_1,\xi_2)}{\mathcal{G}(\xi_1,\xi_2)}(\xi_1^2+\xi_2^2).$$

Remark 2.4. If $\Phi(\xi_1,\xi_2)=0$ and $\mathcal{F}(\xi_1,\xi_2)\neq 0$, then $\mathcal{G}(\xi_1,\xi_2)\neq 0$.

Indeed, if $\Phi(\xi_1,\xi_2)=0$ and $\mathcal{G}(\xi_1,\xi_2)=0$, then $\mathcal{H}(\xi_1,\xi_2)=0$. We get by the definition of the functions $\mathcal{G}$ and $\mathcal{H}$ the family of vectors $\left\{\begin{pmatrix}\mathcal{Q}_1(\xi_1,\xi_2)\\ \mathcal{Q}_2(\xi_1,\xi_2)\end{pmatrix},\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}\right\}$ and $\left\{\begin{pmatrix}\mathcal{P}_1(\xi_1,\xi_2)\\ \mathcal{P}_2(\xi_1,\xi_2)\end{pmatrix},\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}\right\}$ are repectively dependant. This implies $\mathcal{F}(\xi_1,\xi_2)=0$.

The following theorem gives sufficient conditions for the global asymptotic stabilization of the system (4).

Theorem 2.5. If there exists a function Φ satisfying to conditions (A_1), (A_2) and (A_3), then the feedback

$$u(x_1,x_2)=\frac{\Phi(x_1,x_2)-\mathcal{H}(x_1,x_2)}{\mathcal{G}(x_1,x_2)} \text{ if } (x_1,x_2)\neq(0,0)$$

is C^∞ on $\mathbb{R}^2\setminus\{(0,0)\}$ homogeneous of degree $2k+1-q$ and stabilizes the system (4).

Proof. A simple computation gives

$$\Phi(x_1,x_2)=det\begin{pmatrix} X_1(x_1,x_2) & x_1\\ X_1(x_1,x_2) & x_2\end{pmatrix}=\mathcal{H}(x_1,x_2)+u(x_1,x_2)\mathcal{G}(x_1,x_2),$$

where (X_1,X_2) is the vector field of the closed loop system (6). By assumption $(\mathbf{A_3})$, if the point $\xi=(\xi_1,\xi_2)^T\in\mathbb{R}^2\backslash\{(0,0)\}$ satisfies $\Phi(\xi)=0$, then $\langle (X_1(\xi),X_2(\xi))^T \mid (\xi)\rangle<0$.

By Theorem 2.1, the closed loop system (6) is globally asymptotically stable at the origin. □

3. CONSTRUCTION OF THE FUNCTION $\Phi(X_1,X_2)$

We use the numerical data $\mathcal{P}_1(x_1,x_2)$, $\mathcal{P}_2(x_1,x_2)$, $\mathcal{Q}_1(x_1,x_2)$ and $\mathcal{Q}_2(x_1,x_2)$ to compute the functions $\mathcal{G}(x_1,x_2)$, $\mathcal{H}(x_1,x_2)$, $\mathcal{F}(x_1,x_2)$ and their zeros and to construct explicitly the desired functions Φ. We have two cases; in the first one, we treat the case when the function $\mathcal{G}(x_1,x_2)$ is definite i. e. it has no zeros on the unit sphere. In the second one, we suppose that $\mathcal{G}(x_1,x_2)$ has at least a linear factor in its factorization i. e. $\mathcal{G}$ has zeros on the unit sphere.

3.1. Case when the function $\mathcal{G}(x_1,x_2)$ is definite

We recall that $\mathcal{G}(x_1,x_2)=x_2\mathcal{Q}_1(x_1,x_2)-x_1\mathcal{Q}_2(x_1,x_2)$ and $S_p\mathcal{G}$ is the set of the zeros of $\mathcal{G}$ on the top half of the unit sphere.
In this subsection, we consider the case where the function $\mathcal{G}(x_1,x_2)$ is definite and $S_p(\mathcal{G})=\emptyset$.

Theorem 3.1. If there exists a point $M=(m_1,m_2)$ such that $\mathcal{G}(M)\mathcal{F}(M)>0$, then the function

$$\Phi(x_1,x_2)=(m_2x_1-m_1x_2)^{2k+2}$$

satisfies to conditions (A_1), (A_2) and (A_3), and the feedback

$$u(x_1,x_2)=\frac{\Phi(x_1,x_2)-\mathcal{H}(x_1,x_2)}{\mathcal{G}(x_1,x_2)}$$

is C^∞ on $\mathbb{R}^2\setminus\{(0,0)\}$, homogeneous of degree $2k+1-q$ and stabilizes the system (4).

Proof. It is clear that Φ is homogeneous of degree $2k+2$, $\mathcal{G}(x_1,x_2)$ is definite, the point M satisfies $\Phi(M)=0$ and

$$\langle (X_1(\tilde m,m),X_2(\tilde m,m))^T|(\tilde m,m)^T\rangle=-\frac{\mathcal{F}(\tilde m,m)}{\mathcal{G}(\tilde m,m)}(m^2+\tilde m^2)<0.$$

So according to Theorem 2.5, the feedback

$$u(x_1,x_2)=\frac{\Phi(x_1,x_2)-\mathcal{H}(x_1,x_2)}{\mathcal{G}(x_1,x_2)}$$

is C^∞ on $\mathbb{R}^2\setminus\{(0,0)\}$, homogeneous of degree $2k+1-q$ and stabilizes the system (4). $\square$

If $\mathcal{G}(x_1,x_2)\mathcal{F}(x_1,x_2)\leq 0$ for all $(x_1,x_2)\in\mathbb{R}^2\setminus\{(0,0)\}$, then we can not construct a function Φ satisfying to conditions (A_1), (A_2) and (A_3). So we look for a feedback function u such that the closed loop system satisfies to condition (i) of Theorem 2.1. We have the following result.

Theorem 3.2. Suppose that $\mathcal{G}(x_1,x_2)\mathcal{F}(x_1,x_2)\leq 0$ for all $(x_1,x_2)\in\mathbb{R}^2\setminus\{(0,0)\}$. We define

$$I=\lim_{a\to+\infty}\int_{-a}^{a}\frac{(1+s^2)^p\mathcal{Q}_1(1,s)-\sigma\big((1+s^2)^p\mathcal{G}(1,s)\big)'}{(1+s^2)^p\mathcal{G}(1,s)}\,\mathrm{d}s,$$

where $((1+s^2)^p\mathcal{G}(1,s))'$ is the derivative of the function $s\mapsto\big((1+s^2)^p\mathcal{G}(1,s)\big)$, $p=k+\frac{1-q}{2}$ and the constant σ is chosen to satisfy the degree of $\Big[(1+s^2)^p\mathcal{Q}_1(1,s)-\sigma((1+s^2)^p\mathcal{G}(1,s)\big)'\Big]$ is equal to $2p+q-1$ and $\varepsilon=sign(\mathcal{Q}_2(1,0)I)$. One has the two following cases:

i) If $I=0$, then the system (4) is not asymptotically controllable at the origin.

ii) If $I\neq 0$, then for $b=2^n$ large enough, the feedback $u(x_1,x_2)=b\varepsilon(x_1^2+x_2^2)^p$ stabilizes the homogeneous system (4).

Proof. i) If $I=0$, then all the orbits of the equation $(\dot{x}_1,\dot{x}_2)=\mathcal{Q}(x_1,x_2)$ are periodic. So the vector field $\mathcal{P}(x_1,x_2)$ heads towards outside of these orbits, it follows that system (4) is not asymptotically controllable at the origin.
ii) If $I\neq 0$, we choose $b=2^n$ large enough to satisfy

$$\Phi(x_1,x_2)=\mathcal{H}(x_1,x_2)+b\varepsilon(x_1^2+x_2^2)^p\mathcal{G}(x_1,x_2)$$

is a definite function. This implies that the orbits of the closed loop system (6) by the feedback $u(x_1,x_2)$ are spirals. We prove that for n large enough the condition (ii) of Theorem 2.1 is satisfied. We know that the closed loop system (4) by the feedback $u(x_1,x_2)$ is in the form (6), we prove that for n large enough,

$$X_2(1,0)\lim_{a\to+\infty}\int_{-a}^{a}\frac{\mathcal{P}_1(1,s)+u(1,s)\mathcal{Q}_1(1,s)}{\mathcal{H}(1,s)+u(1,s)\mathcal{G}(1,s)}\,\mathrm{d}s<0.$$

Let $a>0$, one has

$$\int_{-a}^{a}\frac{\mathcal{P}_1(1,s)+u(1,s)\mathcal{Q}_1(1,s)}{\mathcal{H}(1,s)+u(1,s)\mathcal{G}(1,s)}\,\mathrm{d}s$$
$$=\int_{-a}^{a}\frac{\mathcal{P}_1(1,s)+u(1,s)\mathcal{Q}_1(1,s)-\sigma\big(\mathcal{H}(1,s)+u(1,s)\mathcal{G}(1,s)\big)'}{\mathcal{H}(1,s)+u(1,s)\mathcal{G}(1,s)}\,\mathrm{d}s,$$

where σ is chosen to satisfy

$$\text{degree}\big(\mathcal{P}_1(1,s) - \sigma\mathcal{H}'(1,s)\big) \leq 2k$$

and

$$\text{degree}\big((1+s^2)^p\mathcal{Q}_1(1,s) - \sigma((1+s^2)^p\mathcal{G}(1,s))'\big) \leq 2k.$$

On the other hand one has for all s,

$$\begin{aligned}
&\frac{\mathcal{P}_1(1,s) + u(1,s)\mathcal{Q}_1(1,s) - \sigma\big(\mathcal{H}(1,s) + u(1,s)\mathcal{G}(1,s)\big)'}{\mathcal{H}(1,s) + u(1,s)\mathcal{G}(1,s)} \\
= \quad & \frac{\mathcal{P}_1(1,s) + 2^n\varepsilon(1+s^2)^p\mathcal{Q}_1(1,s) - \sigma\big(\mathcal{H}(1,s) + 2^n\varepsilon(1+s^2)^p\mathcal{G}(1,s)\big)'}{\mathcal{H}(1,s) + 2^n\varepsilon(1+s^2)^p\mathcal{G}(1,s)} \\
= \quad & \frac{\mathcal{P}_1(1,s) - \sigma\mathcal{H}'(1,s)}{\mathcal{H}(1,s) + 2^n\varepsilon(1+s^2)^p\mathcal{G}(1,s)} + \frac{2^n\varepsilon\left((1+s^2)^p\mathcal{Q}_1(1,s) - \sigma((1+s^2)^p\mathcal{G}(1,s))'\right)}{\mathcal{H}(1,s) + 2^n\varepsilon(1+s^2)^p\mathcal{G}(1,s)}.
\end{aligned}$$

By the limit definition, there exists a positive integer n_0, such that for all $n > n_0$, for all s one has

$$|\mathcal{H}(1,s) + 2^n\varepsilon(1+s^2)^p\mathcal{G}(1,s)| \geq 2^{n_0}(1+s^2)^p|\mathcal{G}(1,s)| - |\mathcal{H}(1,s)| > 0.$$

Indeed, by the hypothesis $\mathcal{G}$ is definite, we get $\mathcal{G}$ can be written as a product of definite quadratic forms. This implies that, there exists a positive real $\gamma > 0$ such that $|\mathcal{G}(x_1,x_2)| \geq \gamma\|x_1^2 + x_2^2\|^{q+1}$, ($\gamma$ is the smallest eigenvalue of the matrices which define the quadratic forms of $|\mathcal{G}|$). In addition $\mathcal{H}$ is a homogeneous polynomial function of degree $2k+2$, so it can be written as a product of quadratic forms. This implies that, there exist two reals δ and μ such that

$$\mu\|x_1^2 + x_2^2\|^{k+1} \leq \mathcal{H}(x_1,x_2) \leq \delta\|x_1^2 + x_2^2\|^{k+1}, \ \forall (x_1,x_2) \in \mathbb{R}^2.$$

We get, for all $s \in \mathbb{R}$

$$\begin{aligned}
|\mathcal{H}(1,s) + 2^n\varepsilon(1+s^2)^p\mathcal{G}(1,s)| \quad & \geq 2^n\varepsilon(1+s^2)^p|\mathcal{G}(1,s)| - |\mathcal{H}(1,s)| \\
& \geq 2^n\gamma(1+s^2)^p(1+s^2)^{\frac{q+1}{2}} - \delta(1+s^2)^{k+1}
\end{aligned}$$

But $2^n\gamma(1+s^2)^p(1+s^2)^{\frac{q+1}{2}} - \delta(1+s^2)^{k+1} = (2^n\gamma - \delta)(1+s^2)^{k+1}$.

By the limit definition, there exists an integer n_0 such that $(2^n\gamma - \delta) > 0$, for all $n > n_0$. We get for all $n > n_0$, for all s

$$\left|\frac{\mathcal{P}_1(1,s) - \sigma\mathcal{H}'(1,s)}{\mathcal{H}(1,s) + 2^n\varepsilon(1+s^2)^p\mathcal{G}(1,s)}\right| \leq \frac{|\mathcal{P}_1(1,s) - \sigma\mathcal{H}'(1,s)|}{2^{n_0}(1+s^2)^p|\mathcal{G}(1,s)| - |\mathcal{H}(1,s)|}.$$

We have also

$$\frac{2^n\varepsilon\big((1+s^2)^p\mathcal{Q}_1(1,s) - \sigma((1+s^2)^p\mathcal{G}(1,s))'\big)}{\mathcal{H}(1,s) + 2^n\varepsilon(1+s^2)^p\mathcal{G}(1,s)} = \frac{\varepsilon\big((1+s^2)^p\mathcal{Q}_1(1,s) - \sigma((1+s^2)^p\mathcal{G}(1,s))'\big)}{2^{-n}\mathcal{H}(1,s) + \varepsilon(1+s^2)^p\mathcal{G}(1,s)}.$$

Using the same arguments as above, there exists a positive integer n_1 such that for all $n > n_1$, for all s

$$|2^{-n}\mathcal{H}(1,s) + \varepsilon(1+s^2)^p\mathcal{G}(1,s)| \geq (1+s^2)^p|\mathcal{G}(1,s)| - 2^{-n_1}|\mathcal{H}(1,s)| > 0.$$

So for all $n > n_1$, for all s, one has

$$\left|\frac{\varepsilon\big((1+s^2)^p\mathcal{Q}_1(1,s) - \sigma((1+s^2)^p\mathcal{G}(1,s))'\big)}{2^{-n}\mathcal{H}(1,s) + \varepsilon(1+s^2)^p\mathcal{G}(1,s)}\right| \leq \frac{|(1+s^2)^p\mathcal{Q}_1(1,s) - \sigma\big((1+s^2)^p\mathcal{G}(1,s)\big)'|}{(1+s^2)^p|\mathcal{G}(1,s)| - 2^{-n_1}|\mathcal{H}(1,s)|}$$

Let $n_2 = \max\{n_0, n_1\}$, we get for all $n > n_2$, for all s

$$\left|\frac{\mathcal{P}_1(1,s) + 2^n\varepsilon(1+s^2)^p\mathcal{Q}_1(1,s) - \sigma\left(\mathcal{H}(1,s) + 2^n\varepsilon(1+s^2)^p\mathcal{G}(1,s)\right)'}{\mathcal{H}(1,s) + 2^n\varepsilon(1+s^2)^p\mathcal{G}(1,s)}\right|$$

$$\leq \frac{|\mathcal{P}_1(1,s) - \sigma\mathcal{H}'(1,s)| + 2^{n_2}|(1+s^2)^p\mathcal{Q}_1(1,s) - \sigma\left((1+s^2)^p\mathcal{G}(1,s)\right)'|}{2^{n_2}(1+s^2)^p|\mathcal{G}(1,s)| - |\mathcal{H}(1,s)|}.$$

Using the dominated convergence theorem, we deduce that

$$\lim_{n\to+\infty}\lim_{a\to+\infty}\int_{-a}^{a}\left(\frac{\mathcal{P}_1(1,s) + 2^n\varepsilon(1+s^2)^p\mathcal{Q}_1(1,s) - \sigma\left(\mathcal{H}(1,s) + 2^n\varepsilon(1+s^2)^p\mathcal{G}(1,s)\right)'}{\mathcal{H}(1,s) + 2^n\varepsilon(1+s^2)^p\mathcal{G}(1,s)}\right)\mathrm{d}s$$

$$= \lim_{a\to+\infty}\int_{-a}^{a}\frac{(1+s^2)^p\mathcal{Q}_1(1,s) - \sigma\left((1+s^2)^p\mathcal{G}(1,s)\right)'}{(1+s^2)^p\mathcal{G}(1,s)}\,\mathrm{d}s.$$

We get for n large enough,

$$X_2(1,0)\lim_{a\to+\infty}\int_{-a}^{a}\frac{X_1(1,s)}{X_2(1,s) - sX_1(1,s)}\,\mathrm{d}s < 0.$$

We conclude that the system (6) is G.A.S. □

Example 3.3. We consider the planar homogeneous system

$$\begin{cases} \dot{x}_1 = \mathcal{P}_1(x_1,x_2) + u(136x_1^3 - 691x_1^2x_2 + 1174x_1x_2^2 - 667x_2^3) \\ \dot{x}_2 = \mathcal{P}_2(x_1,x_2) + u(80x_1^3 - 406x_1^2x_2 + 689x_1x_2^2 - 391x_2^3) \end{cases} \tag{7}$$

with $\begin{cases} \mathcal{P}_1(x_1,x_2) = -110x_1^3 + 574x_1^2x_2 - 999x_1x_2^2 + 580x_2^3, \\ \mathcal{P}_2(x_1,x_2) = -60x_1^3 + 314x_1^2x_2 - 548x_1x_2^2 + 319x_2^3. \end{cases}$

A simple computation gives

$$\mathcal{G}(x_1,x_2) = (10x_1^2 - 34x_1x_2 + 29x_2^2)(8x_1^2 - 27x_1x_2 + 23x_2^2).$$

It is clear that $\mathcal{G}$ is a definite function and $I = 0$, then all the trajectories of $\dot{x} = \mathcal{Q}(x)$ are periodic.

We remark that $\mathcal{F}(2,1) = -1$ and $\mathcal{G}(2,1) = -1$, then we can choose

$$\Phi(x_1, x_2) = (x_1 - 2x_2)^4.$$

Moreover, one has $\mathcal{H}(x_1, x_2) = -424x_2x_1^3 + 1122x_1^2x_2^2 - 1318x_1x_2^3 + 580x_2^4 + 60x_1^4$.
It is easy to verify that the function Φ satisfies to conditions of Theorem 3.1, we deduce that the feedback

$$u(x_1, x_2) = \frac{\Phi(x_1, x_2) - \mathcal{H}(x_1, x_2)}{\mathcal{G}(x_1, x_2)}$$

is C^∞ on $\mathbb{R}^2 \setminus \{(0,0)\}$ and stabilizes the system (7).

3.2. Case when the function $\mathcal{G}(x_1, x_2)$ has a linear factor

In this case, Φ can be constructed by following the steps below.

First, we calculate the function $\mathcal{G}(x_1, x_2) = x_2\mathcal{Q}_1(x_1, x_2) - x_1\mathcal{Q}_2(x_1, x_2)$ and we determine the zeros of $\mathcal{G}$ on the top half of the unit sphere which we denote $C_i = (\tilde{c}_i, c_i) = (\cos\theta_i,\ \sin\theta_i)$ with order of multiplicity η_i, for $i \in I_1 := \{1, \dots, p_1\}$.

Without loss of generality, we can choose and order the θ_i such that $0 \le \theta_1 < \theta_2 < \dots < \theta_{p_1} < \pi$, and denote $\theta_{p_1+1} = \theta_1$. Denote $\mathcal{S}_i = \{r(\cos\theta, \sin\theta),\ \theta_i < \theta < \theta_{i+1},\ \ r \in \mathbb{R}\}$.

Second, we compute $\mathcal{H}(x_1, x_2) = x_2\mathcal{P}_1(x_1, x_2) - x_1\mathcal{P}_2(x_1, x_2)$.

If $\mathbf{S_p}(\mathcal{G}) \cap \mathbf{S_p}(\mathcal{H}) \neq \emptyset$:

We determine the common zeros of $\mathcal{G}$ and $\mathcal{H}$ on the unit sphere which we denote

$$D_j = (\tilde{d}_j, d_j) = (\cos\theta_j,\ \sin\theta_j), \quad \text{for } j \in I_2 := \{1, \dots, p_2\}.$$

So, one can write

$$\mathcal{H}(x_1, x_2) = \prod_{j=1}^{p_2} (d_j x_1 - \tilde{d}_j x_2)^{\gamma_j} \widetilde{\mathcal{H}}(x_1, x_2),$$

with $\widetilde{\mathcal{H}}(C_i) \neq 0$ for all $i \in I_1$.

Now, we introduce the following notations, for $j \in I_2$,

$$\beta_j = \langle \mathcal{P}(D_j) | D_j \rangle,\ \alpha_j = \langle \mathcal{Q}(D_j) | D_j \rangle.$$

Remark 3.4. All straight line $\langle D_j | (x_1, x_2) \rangle = 0$ is invariant by the open loop system (4) and in the case when $\alpha_j = 0$ and $\beta_j \ge 0$, the system (4) is not asymptotically controllable at the origin.

Remark 3.5. The origin of the homogeneous system (4) and the system described by

$$(\dot{x}_1, \dot{x}_2)^T = (x_1^2 + x_2^2)^p \mathcal{P}(x_1, x_2) + u\mathcal{Q}(x_1, x_2), \tag{8}$$

where $(x_1, x_2) \in \mathbb{R}^2$ and p is a positive integer, are of the same nature. Indeed, if $u(x_1, x_2)$ is a homogeneous feedback of degree $2k + 2p + 1 - q$, which is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$ and stabilizes the system (8), then $v(x_1,x_2) = \dfrac{1}{(x_1^2+x_2^2)^p}u(x_1,x_2)$ is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$ and stabilizes the system (4) and inversely the same result holds. So without loss of generality we can suppose that $2k+1 > q$.

Proposition 3.6. (Jerbi and Ould Maaloum [5]) Let $\{(\tilde{d}_j, d_j),\ j \in I_2\}$ the set of the common zeros of $\mathcal{G}$ and $\mathcal{H}$ on the top half of the unit sphere. If the subset $\{j \in I_2$ such that $\alpha_j = 0$ and $\beta_j \geq 0\}$ is empty, then there exists an homogeneous function f of degree $2k+1-q$ such that $(\widetilde{\mathcal{P}}_1, \widetilde{\mathcal{P}}_2) = (\mathcal{P}_1, \mathcal{P}_2) + f\ (\mathcal{Q}_1, \mathcal{Q}_2)$ satisfies

$$\langle (\widetilde{\mathcal{P}}_1(\tilde{d}_j, d_j), \widetilde{\mathcal{P}}_2(\tilde{d}_j, d_j))^T | (\tilde{d}_j, d_j)^T \rangle < 0, \text{ for all } j \in I_2.$$

P r o o f. Let

$$\begin{array}{rll} f: & \mathbb{R}^2 \setminus \{(0,0)\} & \to \mathbb{R} \\ & (x_1,x_2) & \mapsto -a(x_1^2+x_2^2)^{k-q}(x_1\mathcal{Q}_1(x_1,x_2) + x_2\mathcal{Q}_2(x_1,x_2)) \end{array}$$

and $f(0,0) = 0$, where a is a positive real which will be chosen later. It is clear that f is homogeneous of degree $2k+1-q$.

Let $j \in I_2$, since $\mathcal{G}(D_j) = 0$ and $\mathcal{H}(D_j) = 0$, we get $\mathcal{Q}(D_j) = \alpha_j D_j$ and $\mathcal{P}(D_j) = \beta_j D_j$. So

$$\begin{array}{rl} & \langle \mathcal{P}(D_j) + f(D_j)\mathcal{Q}(D_j) | D_j \rangle \\ = & \langle \beta_j D_j - a(\widetilde{d}_j^2 + d_j^2)^{k-q} \dfrac{\widetilde{d}_j \mathcal{Q}_1(\widetilde{d}_j, d_j) + d_j \mathcal{Q}_2(\widetilde{d}_j, d_j)}{\widetilde{d}_j^2 + d_j^2} \alpha_j D_j | D_j \rangle \\ = & \beta_j - a\alpha_j^2. \end{array}$$

Finally, if we choose $a = (\sup\limits_{j\in I_2 \alpha_j \neq 0} \dfrac{\beta_j}{\alpha_j^2}) + 1$, we get

$$\left\langle \begin{pmatrix} \mathcal{P}_1(\widetilde{d}_j, d_j) + f(\widetilde{d}_j, d_j)\widetilde{\mathcal{Q}}_1(\widetilde{d}_j, d_j) \\ \mathcal{P}_2(\widetilde{d}_j, d_j) + f(\widetilde{d}_j, d_j)\widetilde{\mathcal{Q}}_2(\widetilde{d}_j, d_j) \end{pmatrix} | \begin{pmatrix} \widetilde{d}_j \\ d_j \end{pmatrix} \right\rangle < 0,$$

for all $j \in I_2$. □

Remark 3.7. If $\mathcal{G} = 0$ and $\mathcal{H} \neq 0$ (resp. $\mathcal{G} \neq 0$ and $\mathcal{H} = 0$), then all zeros of $\mathcal{H}$ (resp. $\mathcal{G}$) are the common zeros of $\mathcal{H}$ and $\mathcal{G}$. In this case, if the subset $\{j \in I_2$ *such that* $\alpha_j = 0$ *and* $\beta_j \geq 0\}$ is empty, then the function f defined in the proof of Proposition 3.6 is the stabilizing feedback of the system (4).

If $\mathcal{G} = 0$ and $\mathcal{H} = 0$, then there exist homogeneous polynomial functions P_1 and Q_1 of degree $2k$ and $q-1$ respectively such that

$$(\mathcal{P}_1, \mathcal{P}_2)(x_1,x_2) = P_1(x_1,x_2)(x_1,x_2), \text{ and } (\mathcal{Q}_1, \mathcal{Q}_2)(x_1,x_2) = Q_1(x_1,x_2)(x_1,x_2).$$

Proposition 3.8. If $\mathcal{G} = 0$ and $\mathcal{H} = 0$, then the system (4) is globally asymptotically stable by a homogeneous feedback of degree $2k+1-q$ if and only if the following is satisfied: for all $(x_1,x_2) \in S^1$, one has $\{Q_1(x_1,x_2) = 0 \Rightarrow P_1(x_1,x_2) < 0\}$.

P r o o f. If $\mathcal{G}(x_1,x_2)=0$ and $\mathcal{H}(x_1,x_2)=0$ for all $(x_1,x_2)\in\mathbb{R}^2$, then for all $u(x_1,x_2)$ a homogeneous feedback of degree $2k+1-q$, the straight line passing through any point $(x_1,x_2)\in\mathbb{R}^2\setminus\{(0,0)\}$ is invariant by the closed loop system (6). According to Theorem 2.1, a necessary condition of stability is

$$P_1(x_1,x_2)+u(x_1,x_2)Q_1(x_1,x_2)<0, \text{ for all } (x_1,x_2)\in\mathbb{R}^2\setminus\{(0,0)\}.$$

We have two cases.

i) If there exists $(x_1,x_2)\in S^1$ such that $Q_1(x_1,x_2)=0$ and $P_1(x_1,x_2)\geq 0$, then the system (4) can not be asymptotically stabilizable by a homogeneous feedback of degree $2k+1-q$.

ii) If for all $(x_1,x_2)\in S^1$, one has $\{Q_1(x_1,x_2)=0\Rightarrow P_1(x_1,x_2)<0\}$, then the feedback function defined by

$$u(x_1,x_2)=-a(x_1^2+x_2^2)^{k-q+2}Q_1(x_1,x_2),$$

where a is a real which will be chosen later, stabilizes the system (4).

Denote $\Delta=\{(x_1,x_2)\in S^1 \text{ such that } Q_1(x_1,x_2)=0\}$. We know that Q_1 is homogeneous of degree $q-1$, we deduce that the set Δ contains a finite number of points $N_i=(\cos\theta_i,\sin\theta_i)$ in S^1, $i\in\{1,\ldots,d\}$; These points can be ordered as follow $0\leq\theta_1<\theta_2<\ldots<\theta_d<2\pi$ and $\theta_{d+1}=\theta_1$.

Let $i\in\{1,\ldots,d\}$; By continuity of P_1 and the fact that $P_1(N_i)<0$, there exists $\delta_i>0$ such that $\theta_{i-1}<\theta_i-\delta_i<\theta_i+\delta_i<\theta_{i+1}$ and $P_1(x_1,x_2)<0$ for all $(x_1,x_2)\in\{r(\cos\theta,\sin\theta),\ r>0,\ \theta\in]\theta_i-\delta_i,\theta_i+\delta_i[\}$.

Denote $\Lambda=\{(\cos\theta,\sin\theta),\ \ \theta\notin]\theta_i-\delta_i,\theta_i+\delta_i[, \text{ for all } i\}$. Λ is a compact set and for all $(x_1,x_2)\in\Lambda$, one has $Q_1(x_1,x_2)\neq 0$. So, we can choose

$$a=\sup_{(x_1,x_2)\in\Lambda}\left(\frac{P_1(x_1,x_2)}{Q_1^2(x_1,x_2)}\right)$$

We can easily verify that

$$P_1(x_1,x_2)+u(x_1,x_2)Q_1(x_1,x_2)<0, \text{ for all } (x_1,x_2)\in\mathbb{R}^2\setminus\{(0,0)\}.$$

We conclude that the closed loop system (6) is globally asymptotically stable at the origin. □

Now to simplify the notations, we redefine

$$u:=u+f \text{ and } (\mathcal{P}_1,\mathcal{P}_2):=(\mathcal{P}_1,\mathcal{P}_2)+f\ (\mathcal{Q}_1,\mathcal{Q}_2),$$

where f is the function introduced in the proof of Proposition 3.6. The system (4) becomes

$$\begin{cases}\dot{x}_1=\mathcal{P}_1(x_1,x_2)+u\mathcal{Q}_1(x_1,x_2)\\ \dot{x}_2=\mathcal{P}_2(x_1,x_2)+u\mathcal{Q}_2(x_1,x_2)\end{cases}\tag{9}$$

which satisfies to the following condition

$$\left\langle \begin{pmatrix} \mathcal{P}_1(\widetilde{d}_j, d_j) \\ \mathcal{P}_2(\widetilde{d}_j, d_j) \end{pmatrix} | \begin{pmatrix} \widetilde{d}_j \\ d_j \end{pmatrix} \right\rangle < 0, \text{ for all } j \in I_2. \tag{10}$$

Denote $\mathcal{D}(x_1, x_2) := \prod_{j=1}^{p_2}(d_j x - \tilde{d}_j y)^{\gamma_j}$, we get $\mathcal{H}(x_1, x_2) := \mathcal{D}(x_1, x_2)\widetilde{\mathcal{H}}(x_1, x_2)$.
Let $\lambda_i = \widetilde{\mathcal{H}}(\tilde{c}_i, c_i) \neq 0, \quad for\ i \in \{1, \cdots, p_1)$ and $\lambda_{p_1+1} = \widetilde{\mathcal{H}}(-\tilde{c}_1, -c_1) = \varsigma\lambda_1$ (where $\varsigma = (-1)^{degree(\widetilde{\mathcal{H}})}$). If we choose the feedback law $u(x_1, x_2)$ such that $u(D_j) = 0$, then the restriction of the system (9) on the straight line $\langle (x_1, x_2)|D_j\rangle = 0$ is asymptotically stable. So we can choose the function u, which verify $u(D_j) = 0$, in the following form

$$u(x_1, x_2) = \mathcal{D}(x_1, x_2)\widetilde{u}(x_1, x_2).$$

The function Φ becomes

$$\Phi(x_1, x_2) = \mathcal{D}(x_1, x_2)\left(\widetilde{\mathcal{H}}(x_1, x_2) + \widetilde{u}(x_1, x_2)\mathcal{G}(x_1, x_2)\right).$$

We denote

$$\widetilde{\Phi}(x_1, x_2) = \widetilde{\mathcal{H}}(x_1, x_2) + \widetilde{u}(x_1, x_2)\mathcal{G}(x_1, x_2). \tag{11}$$

Remark 3.9. i) If $S_p(\mathcal{G}) \cap S_p(\mathcal{H}) = \emptyset$, then $\widetilde{\Phi} = \Phi$.

ii) The function $\widetilde{\Phi}$ must be chosen to satisfy $\mathcal{G}$ divide $\widetilde{\Phi} - \widetilde{\mathcal{H}}$.

In the following, we give the necessary steps for the construction of the zeros of the function $\widetilde{\Phi}$.

Lemma 3.10. Let $i \in \{1, \cdots, p_1\}$. If $\widetilde{\mathcal{H}}(C_i)\widetilde{\mathcal{H}}(C_{i+1}) < 0$, then there exists a point $M_i = (\tilde{m}_i, m_i)$ lie in the sector $\mathcal{S}_i$ such that $\widetilde{\Phi}(M_i) = 0$.

P r o o f. Since $\widetilde{\Phi}(C_i) = \widetilde{\mathcal{H}}(C_i)$ for all $i \in I_1$, one has: if $\widetilde{\mathcal{H}}(C_i)\widetilde{\mathcal{H}}(C_{i+1}) < 0$, then there exits a point $M_i = (\tilde{m}_i, m_i)$ lie in the sector $\mathcal{S}_i$ such that $\widetilde{\Phi}(M_i) = 0$. This is equivalent to the assumption $\Phi(M_i) = 0$. □

In order to construct a function Φ satisfying to condition (A_3), we must choose a point M_i in the sector $\mathcal{S}_i$ such that $\mathcal{F}(\tilde{m}_i, m_i)\mathcal{G}(\tilde{m}_i, m_i) > 0$.

We recall that $\lambda_i = \widetilde{\mathcal{H}}(\tilde{c}_i, c_i) \neq 0, \quad for\ i \in \{1, \cdots, p_1\}$.

Proposition 3.11. Suppose that the system (9) is stabilizable by a homogeneous feedback of degree $2k+1-q$. If $\lambda_i\lambda_{i+1} < 0$, then there exists a point $M = (\tilde{m}, m) \in S^1 \cap \mathcal{S}_i$ in the top half of the unit sphere such that $\mathcal{F}(M)\mathcal{G}(M) > 0$.

P r o o f. Let $u(x_1, x_2)$ be a homogeneous feedback of degree $(2k + 1 - q)$ which stabilizes the control system (9). The closed loop system (9) by the homogeneous feedback $u(x_1, x_2)$ can be written as:

$$\begin{cases} \dot{x}_1 = \mathcal{P}_1(x_1, x_2) + u(x_1, x_2)\mathcal{Q}_1(x_1, x_2) = X_1(x_1, x_2) \\ \dot{x}_2 = \mathcal{P}_2(x_1, x_2) + u(x_1, x_2)\mathcal{Q}_2(x_1, x_2) = X_2(x_1, x_2). \end{cases} \tag{12}$$

We recall that $\Phi(x_1, x_2) = x_2 X_1(x_1, x_2) - x_1 X_2(x_1, x_2)$.
It is clear that

$$\begin{aligned} \Phi(x_1, x_2) &= \mathcal{H}(x_1, x_2) + u(x_1, x_2)\mathcal{G}(x_1, x_2) \\ &= \mathcal{D}(x_1, x_2)\left(\widetilde{\mathcal{H}}(x_1, x_2) + \widetilde{u}(x_1, x_2)\mathcal{G}(x_1, x_2)\right). \end{aligned}$$

Since $u(x_1, x_2)$ is homogeneous of degree $2k + 1 - q$, it follows that Φ is homogeneous of degree $2k + 2$ and $\widetilde{\Phi}$ is homogeneous of degree $2k + 2 - \text{degree}(\mathcal{D})$. Using the fact that $\mathcal{G}(C_i) = 0$ and $\mathcal{G}(C_{i+1}) = 0$, it follows that $\widetilde{\Phi}(C_i)\widetilde{\Phi}(C_{i+1}) = \lambda_i \lambda_{i+1} < 0$. So there exists $M = (\tilde{m}, m) \in S^1 \cap \mathcal{S}_i$ on the top half of the unit sphere such that $\widetilde{\Phi}(M) = 0$, which implies $\Phi(M) = 0$. From the form of the function Φ, one has $X(M) = \nu M$ where $\nu = -\mathcal{F}(M)\mathcal{G}(M)$. By hypothesis, the closed loop system (12) is asymptotically stable, so $\left\langle \begin{pmatrix} X_1(\tilde{m}, m) \\ X_2(\tilde{m}, m) \end{pmatrix} \mid \begin{pmatrix} \tilde{m} \\ m \end{pmatrix} \right\rangle < 0$, which implies $\mathcal{F}(M)\mathcal{G}(M) > 0$. $\square$

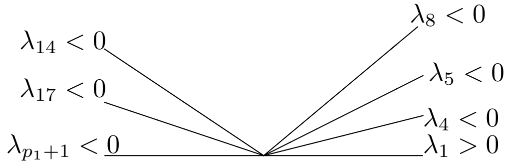
In the following, we denote ℓ the number of the sectors $\mathcal{S}_i$ satisfying to condition $\lambda_i \lambda_{i+1} < 0$.

3.2.1. Case when $\ell \neq 0$:

We have $\ell \leq p_1$. In all of these sectors, we choose a point M on the unit sphere such that $\mathcal{F}(M)\mathcal{G}(M) > 0$. We get ℓ points $M_i = (\cos\varphi_i, \sin\varphi_i)$, which we put in order $M_1, \cdots, M_\ell$ such that $0 \leq \varphi_1 < \varphi_2 < \ldots < \varphi_\ell < \pi$.

Remark 3.12. We recall that $\lambda_1 = \widetilde{\mathcal{H}}(\tilde{c}_1, c_1)$ and $\lambda_{p_1+1} = \widetilde{\mathcal{H}}(-\tilde{c}_1, -c_1) = \varsigma\lambda_1$, where $\varsigma = (-1)^{degree(\widetilde{\mathcal{H}})}$. Since $\mathcal{H}$ is homogeneous of degree $2k + 2$, then in the case where the degree of $\mathcal{D}$ is even, the degree of $\widetilde{\mathcal{H}}$ is even, $\varsigma = 1$ and the number of points M_i is also even and in the case where the degree of $\mathcal{D}$ is odd, the degree of $\widetilde{\mathcal{H}}$ is odd, $\varsigma = -1$ and the number of points M_i is also odd. We can deduce that in all these cases the degree of the homogeneous function $\mathcal{D}(x_1, x_2)\mathcal{Z}(x_1, x_2)$ is even, where $\mathcal{Z}(x_1, x_2) := \prod_{j=1}^{\ell}(m_j x - \tilde{m}_j y)$.

Example 3.13. Suppose that degree of $\widetilde{\mathcal{H}}$ is odd. We recall that $\lambda_i = \widetilde{\mathcal{H}}(\tilde{c}_i, c_i) \neq 0$, for $i \in \{1, \cdots, p_1\}$ and $\lambda_{p_1+1} = \widetilde{\mathcal{H}}(-\tilde{c}_1, -c_1) = \varsigma\lambda_1 = -\lambda_1$, where $\varsigma = (-1)^{degree(\widetilde{\mathcal{H}})}$. We suppose more that $\lambda_i > 0$ for $i \notin \{4, 5, 8, 14, 17, p_1 + 1\}$.



The point M_i defined below lies in the sector $\mathcal{S}_i = \{r(\cos\theta, \sin\theta),\ \theta_i < \theta < \theta_{i+1},\ \ r \in \mathbb{R}\}$, when $\lambda_i \lambda_{i+1} < 0$. In this situation, we have 9 points M_i, $i \in$

$\{3,6,7,8,13,14,16,17,p_1\}$, which is odd. In the case when degree of $\widetilde{\mathcal{H}}$ is even, necessarily one has $\lambda_{p_1+1} = \lambda_1 > 0$. If we deal with the same situation that $\lambda_i > 0$ for $i \notin \{4,5,8,14,17\}$, then one has 8 points M_i, $\quad i \in \{3,6,7,8,13,14,16,17\}$, which is even.

Now, we introduce the function

$$\overline{\mathcal{H}}(x_1,x_2) = \frac{\widetilde{\mathcal{H}}(x_1,x_2)}{\displaystyle\prod_{j=1}^{\ell}(m_j x_1 - \tilde{m}_j x_2)}.$$

It is clear that $\overline{\mathcal{H}}(\tilde{c}_i, c_i) > 0$ for all $i \in I_1$, and this is equivalent to $\widetilde{\mathcal{H}}(\tilde{c}_i, c_i)\mathcal{Z}(\tilde{c}_i, c_i) > 0$ for all $i \in I_1$. Without loss of generality, we can suppose that $\mathcal{G}(1,0) \neq 0$.

Let the following polynomial functions. For $s \in \mathbb{R}$,

$$g(s) := \mathcal{G}(s,1),\ \widetilde{h}(s) := \widetilde{\mathcal{H}}(s,1),\ Z(s) := \mathcal{Z}(s,1).$$

Remark 3.14. We can easily remark that Z and g are relatively prime polynomials, then by the Bézout's identity, there exist Polynomials U and V in $\mathbb{R}[X]$ such that $U(s)Z(s) + V(s)g(s) = 1$, for all $s \in \mathbb{R}$. We get, for all $s \in \mathbb{R}$,

$$\tilde{h}(s)U(s)Z(s) + \tilde{h}(s)V(s)g(s) = \tilde{h}(s). \tag{13}$$

Lemma 3.15. There exists a real $a > 0$ large enough such that $\tilde{h}(s)U(s) + ag(s)^{2\mu} > 0$ for all $s \in \mathbb{R}$, where μ is an integer chosen to satisfy the degree of the polynomial $g(s)^{2\mu}$ is greater than the degree of $\tilde{h}(s)U(s)$.

Proof. Denote $\{s_i,\ i \in I_1\}$ the set of the zeros of $g(s)$. One has $Z(s_i)\tilde{h}(s_i) > 0$ for all $i \in I_1$; By the fact that $U(s)Z(s) + V(s)g(s) = 1$, one has $Z(s_i)U(s_i) = 1$ and $U(s_i)\tilde{h}(s_i) > 0$ for all $i \in I_1$. But the degree of the polynomial function $P(s) = \tilde{h}(s)U(s) + g(s)^{2\mu}$ is even, so $\lim_{s\to\infty} P(s) = +\infty$.

Let $p_1 < p_2 < \cdots < p_m$ the zeros of $P(s)$. It is clear that $P(s) > 0$, for all $s \notin [p_1, p_m]$. In addition, the set $\mathcal{K} = \{s \in [p_1,p_m]$ *such that* $P(s) \leq 0\}$ is compact, so $a_1 = \min_{s\in\mathcal{K}} g(s)^{2\mu} > 0$.

Let $a_2 = \min_{s\in[p_1,p_m]} P(s)$, we get

$$P(s) + (a-1)g(s)^{2\mu} \geq a_2 + (a-1)a_1, \text{ for all } s \in \mathcal{K}.$$

We conclude by choosing $a = 1 + \frac{a_1 - a_2}{a_1}$. $\square$

From Remark 3.5, it is sufficient to construct a feedback law which stabilizes the homogeneous system

$$\begin{cases} \dot{x}_1 = (x_1^2 + x_2^2)^p \mathcal{P}_1(x_1,x_2) + u\mathcal{Q}_1(x_1,x_2) \\ \dot{x}_2 = (x_1^2 + x_2^2)^p \mathcal{P}_2(x_1,x_2) + u\mathcal{Q}_2(x_1,x_2) \end{cases} \tag{14}$$

instead of the system (4), where p is a positive integer chosen to satisfy the conditions of the following theorem.

Theorem 3.16. Let $a > 0$ large enough and μ an integer chosen to satisfy

$$\tilde{h}(s)U(s) + ag(s)^{2\mu} > 0 \text{ for all } s \in \mathbb{R}.$$

We define the function

$$\varphi(s) = \frac{Z(s)D(s)((1+s^2)^p \tilde{h}(s)U(s) + ag(s)^{2\mu})}{1+g(s)^{2\nu}},$$

where p is an integer satisfying

$$\begin{cases} 0 < p \leq q+1, \\ \text{degree}(Z(s)D(s)) + 2(q+1)\mu - 2(q+1)\nu = 2k+2+2p. \end{cases}$$

The homogeneous polynomial function

$$\Phi(x_1, x_2) = x_2^{2k+2+2p} \varphi(\frac{x_1}{x_2})$$

satisfies to conditions of Theorem 2.5.

Proof. From remark 3.12, it is clear that degree of $(Z(s)D(s))$ is even, then using the Euclidean division of $k+1-\dfrac{\text{degree}(Z(s)D(s))}{2}$ by $q+1$ we can write

$$k+1-\frac{\text{degree}(Z(s)D(s))}{2} = (q+1)r + p_1 = (q+1)(r+1) - p,$$

where p is an integer satisfying $0 < p \leq q+1$. Next, we construct a homogeneous feedback for the homogeneous system (14). If we choose μ an integer such that $\mu > r+1$ and the degree of the polynomial $g(s)^{2\mu}$ is greater than the degree of $\tilde{h}(s)U(s)$, then there exists a real $a > 0$ large enough such that $\tilde{h}(s)U(s) + ag(s)^{2\mu} > 0$ for all $s \in \mathbb{R}$. Let

$$\varphi(s) = \frac{Z(s)D(s)((1+s^2)^p \tilde{h}(s)U(s) + ag(s)^{2\mu})}{1+g(s)^{2\nu}},$$

where $\nu = \mu - r - 1$. A simple computation gives

$$(1+g(s)^{2\nu})\varphi(s) = Z(s)D(s)((1+s^2)^p \tilde{h}(s)U(s) + ag(s)^{2\mu}).$$

According to the equation (13), one has

$$(1+s^2)^p h(s)U(s)Z(s) + (1+s^2)^p h(s)V(s)g(s) = (1+s^2)^p h(s).$$

Then

$$(1+g(s)^{2\nu})\varphi(s) = (1+s^2)^p h(s) - (1+s^2)^p h(s)V(s)g(s) + aZ(s)D(s)g(s)^{2\mu}.$$

Finally

$$\varphi(s) - (1+s^2)^p h(s) = g(s)\left(aZ(s)D(s)g(s)^{2\mu-1} - g(s)^{2\nu-1}\varphi(s) - (1+s^2)^p h(s)V(s)\right)$$

and the function $g(s)$ divide $\varphi(s) - h(s)(1+s^2)^p$. Using the condition

$$\text{degree}(Z(s)D(s)) + 2(q+1)\mu - 2(q+1)\nu = 2k+2+2p,$$

we deduce $\Phi(x_1,x_2) = x_2^{2k+2+2p}\varphi(\frac{x_1}{x_2})$ is homogeneous of degree $2k+2+2p$. We can easily remark that $g(s) := \mathcal{G}(1,s)$ divide $\varphi(s) - h(s)(1+s^2)^p := \Phi(1,s) - (1+s^2)^p\mathcal{H}(1,s)$ and $\mathcal{G}(x_1,x_2)$ divide $\Phi(x_1,x_2) - (x_1^2+x_2^2)^p\mathcal{H}(x_1,x_2)$. Moreover, the function Φ is constructed to satisfy $\Phi(x_1,x_2) = 0$ if and only if $\mathcal{D}(x_1,x_2) = 0$ and $\mathcal{Z}(x_1,x_2) = 0$.

If $\mathcal{D}(x_1,x_2) = 0$, then $u(x_1,x_2) = 0$; This implies, by the inequality (10),

$$\left\langle \begin{pmatrix} X_1(x_1,x_2) \\ X_2(x_1,x_2) \end{pmatrix} | \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \mathcal{P}_1(x_1,x_2) \\ \mathcal{P}_2(x_1,x_2) \end{pmatrix} | \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle < 0.$$

□

3.2.2. Case when $\ell = 0$:

If $\ell = 0$, then $\lambda_i\lambda_{i+1} > 0$ for all i. We deal with two cases;

1) $S_p(\mathcal{G}) \cap S_p(\mathcal{H}) \neq \emptyset$. In this case we choose $Z(x_1,x_2) = 1$ and $U(x_1,x_2) = 1$ and the stabilizing feedback of the system (14) can be computed using Theorem 3.16 as

$$\Phi(x_1,x_2) = \mathcal{D}(x_1,x_2) \times \left(\frac{(x_1^2+x_2^2)^p\tilde{\mathcal{H}}(x_1,x_2) + a\mathcal{G}(x_1,x_2)^{2\mu}}{x_2^{2(q+1)\nu} + \mathcal{G}(x_1,x_2)^{2\nu}} \right)$$

with $\text{degree}(D(s)) + 2(q+1)(\mu-\nu) = 2k+2+2p$.

2) $S_p(\mathcal{G}) \cap S_p(\mathcal{H}) = \emptyset$. We deal with two subcases;

a) If there exists a point M such that $\mathcal{F}(M)\mathcal{G}(M) > 0$, it follows that the system (9) is G.A.S. by a homogeneous feedback and it can be computed using Theorem 3.16 with the following choice

$$Z(x_1,x_2) = (mx_1 - \tilde{m}x_2)^2 \text{ and } D(x_1,x_2) = 1.$$

b) If for all $M \in \mathbb{R}^2$, one has $\mathcal{F}(M)\mathcal{G}(M) \leq 0$, we denote $\alpha_i = \langle \mathcal{Q}(C_i)|C_i\rangle$. We deal with the following cases:

• If there exist $i,\ j \in I_1$ such that $\alpha_i\alpha_j < 0$, then the system (9) is not asymptotically controllable at the origin (see [6]).

• If there exist $i,\ j \in I_1$ such that $\alpha_i \neq 0$ and $\alpha_j = 0$. Since $\mathcal{F}(x_1,x_2)\mathcal{G}(x_1,x_2) \leq 0$ and $\mathcal{F}(C_i) = \alpha_i\lambda_i \neq 0$, one can write

$$\mathcal{G}(x_1,x_2) = (c_ix_1 - \tilde{c}_ix_2)^2\mathcal{G}_1(x_1,x_2).$$

According to the condition $S_p(\mathcal{G}) \cap S_p(\mathcal{H}) \neq \emptyset$, with $\mathcal{D}(x_1,x_2) = (c_ix_1 - \tilde{c}_ix_2)^2$, we can compute the homogeneous feedback $v(x_1,x_2)$ for the system

$$\begin{cases} \dot{x}_1 = (c_ix_1 - \tilde{c}_ix_2)^2\mathcal{P}_1(x_1,x_2) + v(x_1,x_2)\mathcal{Q}_1(x_1,x_2) \\ \dot{x}_2 = (c_ix_1 - \tilde{c}_ix_2)^2\mathcal{P}_2(x_1,x_2) + v(x_1,x_2)\mathcal{Q}_2(x_1,x_2). \end{cases}$$

The set of the homogeneous vectors fields which are GAS at the origin is an open set, then for $\varepsilon > 0$ small enough one has the homogeneous system

$$\begin{cases} \dot{x}_1 = (c_i x_1 - \tilde{c}_i x_2)^2 \mathcal{P}_1(x_1, x_2) + v(x_1, x_2)\mathcal{Q}_1(x_1, x_2) + \varepsilon(x_1^2 + x_2^2)\mathcal{P}_1(x_1, x_2) \\ \dot{x}_2 = (c_i x_1 - \tilde{c}_i x_2)^2 \mathcal{P}_2(x_1, x_2) + v(x_1, x_2)\mathcal{Q}_2(x_1, x_2) + \varepsilon(x_1^2 + x_2^2)\mathcal{P}_2(x_1, x_2) \end{cases}$$

is GAS at the origin. Finally, the system

$$\begin{cases} \dot{x}_1 = \mathcal{P}_1(x_1, x_2) + \frac{v(x_1,x_2)}{(c_i x_1 - \tilde{c}_i x_2)^2 + \varepsilon(x_1^2 + x_2^2)} \mathcal{Q}_1(x_1, x_2) \\ \dot{x}_2 = \mathcal{P}_2(x_1, x_2) + \frac{v(x_1,x_2)}{(c_i x_1 - \tilde{c}_i x_2)^2 + \varepsilon(x_1^2 + x_2^2)} \mathcal{Q}_2(x_1, x_2) \end{cases}$$

is GAS at the origin.

- If $\alpha_i = 0$ for all i, we compute

$$I = \lim_{a \to +\infty} \int_{-a}^{a} \frac{\mathcal{Q}_1(1, s)}{\mathcal{G}(1, s)} \, \mathrm{d}s.$$

i) If $I = 0$, then all the orbits of the system $(\dot{x}_1, \dot{x}_2) = \mathcal{Q}(x_1, x_2)$ are periodic and the vector fields $\mathcal{P}(x_1, x_2)$ head towards outside of these orbits. It follows that the system (9) is not asymptotically controllable at the origin.

ii) In the case where $I \neq 0$ and $\mathcal{G}(x_1, x_2)\mathcal{F}(x_1, x_2) \leq 0$ for all $(x_1, x_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$, without loss of generality we can suppose $\mathcal{G}(x_1, x_2) \geq 0$.

If $\mathcal{H}(C_i)I \leq 0$, then the system (9) is not asymptotically controllable at the origin.
If $\mathcal{H}(C_i)I > 0$, according to the hypothesis that for all $(x_1, x_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$ the function $\mathcal{G}(x_1, x_2) \geq 0$, one has q is odd and for b large enough we get

$$\Phi(x_1, x_2) = \mathcal{H}(x_1, x_2) + b\mathcal{H}(C_i)(x_1^2 + x_2^2)^p \mathcal{G}(x_1, x_2)$$

with $2p + q = 2k + 1$, is a definite function. From Theorem 1, the orbits of the closed loop system (14) by the feedback $u(x_1, x_2)$ are spirals and for $b = 2^n$ large enough one has

$$J = \int_{-\infty}^{+\infty} \frac{\mathcal{P}_1(1, s) + b\mathcal{H}(C_i)(1 + s^2)^p \mathcal{G}(1, s)\mathcal{Q}_1(1, s)}{\Phi(1, s)} \, \mathrm{d}s \sim \int_{-\infty}^{+\infty} \frac{\mathcal{Q}_1(1, s)}{\mathcal{G}(1, s)} \, \mathrm{d}s = I.$$

The closed loop system (14) by the proposed feedback $u(x_1, x_2) = 2^n \mathcal{H}(C_i)(x_1^2 + x_2^2)^p$ yields

$$X_2(1, 0) = \mathcal{P}_2(1, 0) + 2^n \mathcal{H}(C_i)\mathcal{Q}_2(1, 0) = \mathcal{P}_2(1, 0) - 2^n \mathcal{H}(C_i)\mathcal{G}(1, 0)$$

Moreover, for n large enough

$$X_2(1, 0)J \sim -2^n \mathcal{H}(C_i)\mathcal{G}(1, 0)I < 0$$

The condition (i) of theorem (1) is satisfied and the closed loop system (14) is G.A.S.

Example 3.17. We consider the planar homogeneous system

$$\left\{\begin{array}{l} \dot{x}_1 = -2x_1 - 6x_2 + ux_1(2x_1^3 + 5x_1^2x_2 - 13x_1x_2^2 + 4x_2^3) \\ \dot{x}_2 = -2x_1 - 3x_2 + ux_1(-x_1^3 + 2x_1^2x_2 + 10x_1x_2^2 - 13x_2^3) \end{array}\right. \tag{15}$$

Denote $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2)$ and $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$, where $\mathcal{P}_1(x_1, x_2) = -2x_1 - 6x_2$, $\mathcal{P}_2(x_1, x_2) = -2x_1 - 3x_2$, $\mathcal{Q}_1(x_1, x_2) = x_1(2x_1^3 + 5x_1^2x_2 - 13x_1x_2^2 + 4x_2^3)$ and $\mathcal{Q}_2(x_1, x_2) = x_1(-x_1^3 + 2x_1^2x_2 + 10x_1x_2^2 - 13x_2^3)$.

It is clear that

$$\left\{\begin{array}{l} \mathcal{G}(x_1, x_2) = x_1(x_1 - x_2)(x_1 + x_2)(x_1 - 2x_2)(x_1 + 2x_2) \\ \mathcal{H}(x_1, x_2) = (x_1 + 2x_2)(2x_1 - 3x_2). \end{array}\right.$$

All the points in $S_p\mathcal{G}$ are also equilibrium points of the system $(\dot{x}_1, \dot{x}_2) = \mathcal{Q}(x_1, x_2)$. A simple computation gives

$$S_p\mathcal{G} = \left\{C_1 = \frac{1}{\sqrt{5}}(2,1),\ C_2 = \frac{1}{\sqrt{2}}(1,1),\ C_3 = (0,1),\ C_4 = \frac{1}{\sqrt{2}}(-1,1),\ C_5 = \frac{1}{\sqrt{5}}(-2,1)\right\}$$

and

$$S_p\mathcal{H} = \left\{\frac{1}{\sqrt{13}}(3,2), \frac{1}{\sqrt{5}}(-2,1)\right\}.$$

The common zeros of $\mathcal{G}$ and $\mathcal{H}$ are $D = \frac{1}{\sqrt{5}}(-2,1)$.

To construct the stabilizing feedback or the function Φ, we follow the steps below:

• We compute $\beta = \langle(\mathcal{P}(D))^T|D^T\rangle = 1/5 > 0$. By the fact that β is positive and D is also a zero of Φ, we consider a change of feedback law $u = u_1 + f$. The new system becomes

$$\begin{array}{ll} (\dot{x}_1, \dot{x}_2) & = (x_1^2 + x_2^2)^2\mathcal{P}(x_1, x_2) + f(x_1, x_2)\mathcal{Q}(x_1, x_2) + u_1(x_1, x_2)\mathcal{Q}(x_1, x_2) \\ & = \widetilde{\mathcal{P}}(x_1, x_2) + u_1(x_1, x_2)\mathcal{Q}(x_1, x_2). \end{array} \tag{16}$$

It is clear that $f(x_1, x_2) = x_1$ is a homogeneous function of degree one and satisfies $\langle(\widetilde{\mathcal{P}}(D))^T|D^T\rangle = -\frac{43}{125} < 0$.

• We deal with the new system (16). We recall

$\widetilde{\mathcal{P}}_1(x_1, x_2) = -x_2x_1^4 - 17x_2^2x_1^3 - 8x_2^3x_1^2 - 2x_2^4x_1 - 6x_2^5$ and
$\widetilde{\mathcal{P}}_2(x_1, x_2) = -3x_1^5 - x_2x_1^4 + 6x_2^2x_1^3 - 19x_2^3x_1^2 - 2x_2^4x_1 - 3x_2^5$.

For the new system one has

$$\begin{array}{rcl} \mathcal{H}(x_1, x_2) & = & x_2\widetilde{\mathcal{P}}_1(x_1, x_2) - x_1\widetilde{\mathcal{P}}_2(x_1, x_2) \\ & = & (x_1 + 2x_2)(3x_1^5 - 5x_2x_1^4 + 3x_2^2x_1^3 - 4x_2^3x_1^2 + 2x_2^4x_1 - 3x_2^5) \\ & = & (x_1 + 2x_2)\widetilde{\mathcal{H}}(x_1, x_2); \\ \mathcal{G}(x_1, x_2) & = & x_1(x_1 - x_2)(x_1 + x_2)(x_1 - 2x_2); \\ \Phi(x_1, x_2) & = & \mathcal{H}(x_1, x_2) + u_1(x_1, x_2)\mathcal{G}(x_1, x_2). \end{array}$$

We compute $\lambda_i = \widetilde{\mathcal{H}}(C_i)$, for $i \in \{1, \cdots, 4\}$, we find

$$\lambda_1 = \frac{1}{\sqrt{5}} > 0,\ \lambda_2 = -\frac{1}{\sqrt{2}} < 0,\ \lambda_3 = -3 < 0 \text{ and } \lambda_4 = -\frac{5}{\sqrt{2}} < 0.$$

Since $\lambda_1\lambda_2 < 0$, then from Proposition 3.11 the system (16) is G.A.S. by an homogeneous feedback of degree $2k+1-q$ if and only if there exists $M = (\tilde{m}, m) \in S^1 \cap \mathcal{S}_1$ in the top half of the unit sphere such that $\mathcal{F}(M)\mathcal{G}(M) > 0$. We can choose $M = \frac{1}{\sqrt{13}}(3,2)$ because $\mathcal{F}(M) = \frac{6}{\sqrt{13}}\mathcal{G}(M)$; it follows that $\mathcal{Z}(x_1,x_2) = 2x_1 - 3x_2$. We can compute the following polynomials $g(s) := \mathcal{G}(s,1) = s^5 - 5s^3 + 4s$, $\widetilde{h}(s) := \widetilde{\mathcal{H}}(s,1) = 3s^5 - 5s^4 + 3s^3 - 4s^2 + 2s - 3$ and $Z(s) := \mathcal{Z}(s,1) = 2s - 3$.

We can easily remark that Z and g are relatively prime polynomials, then by the Bézout's identity, there exist polynomials U and V in $\mathbb{R}[X]$ such that $U(s)Z(s) + V(s)g(s) = 1$.

We get $U(s) = \dfrac{16}{105}s^4 + \dfrac{24}{105}s^3 - \dfrac{44}{105}s^2 - \dfrac{66}{105}s - \dfrac{35}{105}$, and $V(s) = -\dfrac{32}{105}$.

A simple computation gives $\mu = 2$, $p = 3$, $\nu = 1$ and $a = 7$. These constants satisfies

$$(s^2+1)^3\tilde{h}(s)U(s) + 7g(s)^4 > 0, \text{ for all } s \in \mathbb{R}.$$

So, we get

$$\varphi(s) = \Phi(s,1) = \frac{Z(s)D(s)((1+s^2)^3\tilde{h}(s)U(s) + 7g(s)^4)}{1+g(s)^2}.$$

Finally, the function Φ is given by

$$\Phi(x_1,x_2) = \frac{(2x_1 - 3x_2)(x_1 + 2x_2)(x_2^5(x_1^2+x_2^2)^3\widetilde{\mathcal{H}}(x_1,x_2)U(x_1,x_2) + 7\mathcal{G}(x_1,x_2)^4)}{(x_1^2+x_2^2)^3(x_2^{10} + \mathcal{G}(x_1,x_2)^2)}$$

with $U(x_1,x_2) = \frac{16}{105}x_1^4 + \frac{24}{105}x_1^3x_2 - \frac{44}{105}x_1^2x_2^2 - \frac{66}{105}x_1x_2^3 - \frac{35}{105}x_2^4$, is homogeneous of degree 6. This let us conclude that the feedback function defined by

$$u(x_1,x_2) = \frac{\Phi(x_1,x_2) - \mathcal{H}(x_1,x_2)}{\mathcal{G}(x_1,x_2)}$$

stabilizes the system (16).

4. CONCLUSION

In this paper, we study the problem of stabilization of nonlinear control homogeneous polynomial systems in the plane. We focus on a homogeneous feedback law which preserve the homogeneity of the closed loop system. Our study is based on Theorem (2.1) of Hahn, which gives a complete classification of the stability of homogeneous systems in the plane. Our study is divided in two parts. In the first one, the function $\mathcal{G}$, defined in (5), is definite i. e. $\mathcal{G}$ has no zeros on the unit sphere. In the second one, the function $\mathcal{G}$ has a finite number of zeros on the unit sphere. In each case we study the possibility of constructing a function Φ which satisfies to conditions (A_1) (A_2) and (A_3). The construction of such a function Φ allow us to determine a stabilizing feedback for the system (4).

(Received June 4, 2015)

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