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COPULA-BASED GROUPED RISK AGGREGATION UNDER MIXED OPERATION

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Abstract. This paper deals with the problem of risk measurement under mixed operation. For this purpose, we divide the basic risks into several groups based on the actual situation. First, we calculate the bounds for the subsum of every group of basic risks, then we obtain the bounds for the total sum of all the basic risks. For the dependency relationships between the basic risks in every group and all of the subsums, we give different copulas to describe them. The bounds for the aggregated risk under mixed operation and the algorithm for numerical simulation are given in this paper. In addition, the convergence of the algorithm is proved and some numerical simulations are presented.

Keywords: mixed operation; grouped model; aggregated risk measurement; Value of Risk; numerical simulation

MSC 2010: 91G50, 91G60, 91B30, 62H20, 62E17, 62P99, 65C20

1. Introduction

Risk measurement has attracted researchers' attention since the Mean-Variance Model was built by Markowitz [16] and the risk of a portfolio was represented by its variance in his paper. The research concerning risk measurement became more important after the economic globalization and financial crisis, which led to bankruptcy of many large corporations and then severely impacted the whole country's economy. More and more methods for risk measuring are also put forward, such as the Value at Risk (VaR), which was proposed by J. P. Morgan Company in 1994 for the first time and is now used by researchers as a main measure of risk. In recent years, the research of risk measurement mainly focuses on the aggregated risk, which is defined as $\Psi(X)$, where $X = (X_1, \ldots, X_d)$ denotes a d-dimensional random vector

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composed of d basic risks and a measurable function $\Psi\colon \mathbb{R}^d \to \mathbb{R}$ represents the operation of aggregation. When $\Psi(X) = \sum_{i=1}^{d} X_i$, the aggregated risk can be interpreted as the sum of basic risks, which is just the aggregated risk we study in this paper.

There are many results concerning the problem of the sum of n dependent risks based on VaR, which mainly consider the situation when the marginal distributions of the dependent risks are given but their joint distribution is totally unknown or only partly given. For example, Rüschendorf [19] got the bounds for the sum of n dependent risks for the homogeneous case and the uniform or binomial marginal distribution while $n \geqslant 3$; Denuit et al. [6] and Embrechts et al. [7] gave the so-called standard bounds for the sum of dependent risks; Wang et al. [22] calculated the bounds of the sum of n dependent risks under the condition that all the dependent risks have the same marginal distribution function, which has monotone density on its support and satisfies mean condition; Embrechts et al. [8] used the properties of copula and the methods of mathematical statistics to yield the VaR bounds of the sum of n dependent risks for $n = 2$ and the upper bound for $n \geqslant 3$ in the homogeneous and inhomogeneous cases, respectively. In addition, Wang et al. [21] found the bounds of the sum of n dependent risks for any given marginal distribution and proved the necessary and sufficient condition using the notion of jointly mixable distributions. More details about the sum of dependent risks and other aggregated risks can be found in Junker et al. [14], Joe et al. [13], Heilpern [12], Skoglund et al. [20], Hashorva [11], Bernard et al. [4], among others.

Since 1980's, many countries have successfully implemented mixed operation in financial and banking industries to avoid the financial crisis. At the same time, the study of mixed operation has attracted many scholars' attention and there are many results about it till now. Allen et al. [1] estimated a global cost function for international banks to test for both input and output inefficiencies and suggested that for banks in 15 countries, the prevalence of input X-inefficiencies far outweighed that of output inefficiencies. Moreover, they found that the distribution-free model overestimated the magnitude of X-inefficiencies relative to the stochastic cost frontier approach and large banks in separated banking countries had the largest measure of input inefficiency amounting to 27.5 percent of total costs, as well as significant levels of diseconomies of scale. All other banks have X-inefficiency levels around fifteen percent of total costs with slight economies of scale for small banks; Berger et al. [3] designed a framework for evaluating the causes, consequences, and future implications of financial services industry consolidation, reviewed the extensive research literature within the context of this framework (over 250 references), and then suggested fruitful avenues for future research; Rime et al. [18] examined the performance of Swiss banks from 1996 to 1999 and found evidence of large relative cost and profit inefficiencies in these banks. They also found evidence of economies of scale for small and mid-size banks as well as other similar evidence for large banks and suggested a few obvious benefits from the trend toward larger, universal banks in Switzerland. More similar results are given, e.g., in Chong et al. [5], Fields et al. [9], Frei et al. [10].

As stated above, the results concerning mixed operation mainly focus on the influences of mixed operation on the economic efficiency and the stability of the financial system, or the financial regulation and qualitative risk analysis based on the mixed operation. But there have been only few results about the quantitative risk measurement under mixed operation. Therefore, we propose to measure the aggregated risk faced by a financial body under mixed operation and aim to obtain the bounds of the aggregated risk, i.e., the sum of dependent risks, both in theory and simulation in this paper. As presented in the previous paragraphs, most of the research on the sum of dependent risks is carried out by using certain copula to substitute the joint distribution function of the basic risks and then obtaining the distribution or VaR bounds for the sum based on this copula. Generally, they use one copula to describe the dependency relationship of all basic risks, which may be impractical for the following reasons. Firstly, the aggregated risk faced by a financial body under mixed operation is constituted by a variety of basic risks, which can be divided into several groups according to the different kinds of financial products they belong to. Secondly, the dependency relationship between the basic risks in every group has its own form and all of these forms are not the same. Thirdly, the subsum of every group of basic risks can be seen as a new risk. There is also some dependency relationship between these risks and this dependency relationship is different from others. Consequently, we deal with the problem in this paper differently. For our purpose, a grouped model is built for the aggregated risk as follows. First, the basic risks are divided into several groups based on different kinds of financial products they belong to. Then the bounds for the subsum of every group of basic risks are calculated. Finally, the sum of all basic risks is divided into these subsums and the bounds for it are obtained. That is to say, a copula-based grouped risk aggregation model is specially built for the risk under mixed operation and this model is especially tailored to the mixed operation acquisition for several reasons. In the first place, the risk faced by a financial body in the case of mixed operation is constituted by a great variety of basic risks. Both the quantity and the variety of these basic risks are larger than those in general case. Thus, grouping these basic risks in some way is necessary for more organized and distinct calculation. In the second place, according to the definition of mixed operation, the basic risks come from different kinds of financial products such as stocks and bonds. The dependency relationship between the basic risks which belong to one kind of financial products is different from those belonging to other kinds of financial products. Consequently, it is more reasonable to group all of the basic risks according to the type of financial products they come from and use different copulas to describe the dependency relationships for different groups of basic risks. Then we can divide the sum of all basic risks into several subsums and every subsum is the sum of one group of basic risks. In the third place, there also exists some dependency relationship between different kinds of financial products, and this dependency relationship is different from those mentioned above. For better presentation of this dependency relationship, it is necessary to group the basic risks according to the kind of financial product they belong to and use the copula-based subsum of every group of basic risks to represent the risk variable of every kind of financial products. Based on these subsums and some copula, it can be clearer and more accurate to get the final result of the total risk we are interested in. In conclusion, either from the definition of the mixed operation and the specific characteristic of the basic risks faced by a financial body under mixed operation, or from the theoretical calculation, the copula-based grouped risk aggregation model is especially tailored to the mixed operation acquisition.

This paper is organized as follows. In Section 2.1, we present some necessary facts about copula taken from McNeil et al. [17]. In Section 2.2, the grouped model as well as its structure based on the notion of tree dependence in Arbenz et al. [2] is given. In Section 3, we simply calculate the bound for the distribution function of the total risk variable in different cases based on the known results about the bounds for the distribution function of the sum of dependent variables. In Section 4, the detailed steps of the numerical simulation algorithm inspired by the notion of empirical copula in Algorithm 3.1 from Arbenz et al. [2] are shown. Additionally, the convergence of this algorithm is proved in Theorem 4.3 and the analysis of the simulation results is presented in this section.

2. Model building

2.1 Preliminaries. Let $X = (X_1, \ldots, X_n)$ be a random vector, where $X_1, \ldots, X_n$ represent n basic risks and let $S = X_1 + \ldots + X_n$ be the aggregated risk. Without exception, VaR is used for measuring the aggregated risk in this paper and its definition at the confidence level α is given as

$$\mathrm{VaR}_\alpha(S) = \inf\{s \in \mathbb{R}\colon\ P(S < s) \geqslant \alpha\}. \tag{2.1}$$

Thus for obtaining the VaR bounds of the aggregated risk, we only need to get the bounds for its distribution as VaR of S is the quantile of its distribution function.

Consequently, the following definitions are given:

$$m_+(s) = \inf\{P(S < s)\colon\ X_i \sim F_i,\ i = 1, \dots, n,\ C_X \in \mathcal{C}\}, \tag{2.2}$$

$$M_+(s) = \sup\{P(S < s)\colon\ X_i \sim F_i,\ i = 1, \dots, n,\ C_X \in \mathcal{C}\}, \tag{2.3}$$

where C_X is the copula of $X = (X_1, \dots, X_n)$ and $\mathcal{C}$ is the set of all possible copulas. The values $m_+(s)$ and $M_+(s)$ represent the lower and upper bounds of S, respectively, in the case when the marginal distributions of $X = (X_1, \dots, X_n)$ are known but the dependency structure of $X_1, \dots, X_n$ is unknown. Since the techniques for handling $M_+(s)$ are very similar to those for $m_+(s)$, we focus on $m_+(s)$ in this paper.

As copula is the main tool for dealing with the dependency relationship among dependent random variables, we first give the definition and some of its properties, which are quoted from McNeil et al. [17].

Definition 2.1. A distribution function $C(\mathbf{u}) = C(u_1, \dots, u_d)$ on $[0,1]^d$ with standard uniform marginal distributions is called a d-dimensional copula.

By the Sklar theorem, we know that if F is any joint distribution function with margins $F_1, \dots, F_d$, then there exists a copula $C\colon\ [0,1]^d \to [0,1]$ satisfying

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)) \tag{2.4}$$

for all $x_1, \dots, x_d$ in $\overline{\mathbb{R}} = [-\infty, \infty]$ and the copula C is unique when the margins are continuous. Otherwise it is uniquely determined on $\operatorname{Ran} F_1 \times \operatorname{Ran} F_2 \times \dots \times \operatorname{Ran} F_d$, where $\operatorname{Ran} F_i = F_i(\overline{\mathbb{R}})$ is the range of F_i. Conversely, while C is a copula and $F_1, \dots, F_d$ are univariate distribution functions, the function F defined by the above formula denotes a joint distribution function with margins $F_1, \dots, F_d$.

In addition, the Fréchet bounds of copula tell us that every copula satisfies

$$\max\left\{\sum_{i=1}^{d} u_i + 1 - d, 0\right\} \leqslant C(\mathbf{u}) \leqslant \min\{u_1, \dots, u_d\} \tag{2.5}$$

for every $\mathbf{u} \in [0,1]^d$. The upper and lower bounds are denoted by $M(u_1, \dots, u_d)$ and $W(u_1, \dots, u_d)$, respectively. Here we present several common copulas that will be used in this paper.

(i) The *independent copula* is denoted by $\Pi(u_1, \dots, u_d) = \prod_{i=1}^{d} u_i$. By (2.4) we know that random variables with continuous univariate distribution functions are mutually independent if and only if their dependency structure can be described by this formula.

(ii) The Fréchet upper bound copula from (2.5) is called the *comonotonicity copula*, denoted by $M(u_1, \ldots, u_d) = \min\{u_1, \ldots, u_d\}$. The dependency structure of several random variables can be described by this copula, only if these random variables have continuous univariate distribution functions and they are perfectly positively dependent, i.e., $X_i = T_i(X_1)$, $i = 2, \ldots, d$, where T_i, $i = 2, \ldots, d$ are almost surely strictly increasing functions.

(iii) Similar to the definition of the comonotonicity copula, the Fréchet lower bound copula from (2.5) is called the *countermonotonicity copula* $W(u_1, u_2) = \max\{u_1 + u_2 - 1, 0\}$. But differently from the comonotonicity copula, the *countermonotonicity copula* is only defined when $d = 2$. If the rvs X_1, X_2 have continuous distribution functions and are perfectly negatively dependent, i.e., X_2 is almost surely a strictly decreasing function of X_1, then their copula is denoted by the above formula.

Next, we give some results concerning the lower bound for $S = X_1 + \ldots + X_n$, i.e., the $m_+(s)$ defined in (2.2), which can be found in Wang et al. [21] and are used during the process of solving the problem proposed in this paper.

Denote the sum of conditional means of $\{X_i,\ i = 1, \ldots, n\}$ by

$$\Phi(t) = \sum_{i=1}^{n} \mathbb{E}(X_i\colon\ X_i \geqslant F_i^{-1}(t)), \tag{2.6}$$

where $t \in (0,1)$ and $F^{-1}(t) = \inf\{s \in \mathbb{R}\colon\ F(s) \geqslant t\}$. Let $\Phi(1) = \lim\limits_{t\to 1-} \Phi(t)$, $\Phi(0) = \lim\limits_{t\to 0+} \Phi(t)$. Apparently, $\Phi(t)$ is an increasing continuous function if $\{F_i,\ i = 1, \ldots, n\}$ are continuous. Define

$$\Phi^{-1}(x) = \inf\{t \in [0,1]\colon\ \Phi(t) \geqslant x\} \tag{2.7}$$

for $x \leqslant \Phi(1)$ and $\Phi^{-1}(x) = 1$ for $x > \Phi(1)$. Additionally, let

$$\widetilde{F}_a(x) = \max\{(F(x) - a)/(1 - a), 0\}$$

for $x \in \mathbb{R}$, which denotes the conditional distribution of F on $[F^{-1}(a), \infty)$ for $a \in [0,1)$, and $\widetilde{F}_1(x) = \lim\limits_{a\to 1-} \widetilde{F}_a(x)$ for $a = 1$. The following lemma is Theorem 2.6 in Wang et al. [21]. Before introducing this lemma, we give the definition of jointly mixable functions which will be used in the lemma. Suppose that $F_1, \ldots, F_n$ are univariate distribution functions. If there exist random variables $X_1, \ldots, X_n$ whose distributions are $F_1, \ldots, F_n$, respectively, such that for some constant $C \in \mathbb{R}$,

$$P(X_1 + \ldots + X_n = C) = 1, \tag{2.8}$$

then $F_1, \ldots, F_n$ are *jointly mixable*.

Lemma 2.1. *Suppose the distribution functions $F_1, \dots, F_n$ are continuous, then we have*

$$m_+(s) \geqslant \Phi^{-1}(s) \tag{2.9}$$

for any fixed $s \geqslant \Phi(0)$ and $m_+(s) = \Phi^{-1}(s)$ if and only if the conditional distribution functions $\widetilde{F}_{1,a}, \dots, \widetilde{F}_{n,a}$, where $a = \Phi^{-1}(s)$, are jointly mixable.

2.2 The model. The problem we study in this paper is the measurement of the risk faced by a financial body under mixed operation, i.e., the aggregated risk under mixed operation. The mixed operation here mainly refers to the narrow sense of it, i.e., the business connection between the banking industry and the securities industry. In this sense, as the banking institution and securities institution can engage in business belonging to each other's field, the variety of financial products operated by any of them increases and each kind of financial products contains a great number of basic financial products. So the total risk faced by the finance body consists of many basic risks which have intricate relationships with each other. In order to measure this risk more accurately, we build the model as follows.

In the first place, we group the basic risks according to different kinds of financial products they belong to. Then we sum every group of basic risks to get the subsum of these basic risks and its bound. Finally, we obtain the bounds for the aggregated risk, which now can be represented by the sum of several subsums. In other words, define the aggregated risk S as

$$S = \sum_{i=1}^{N} X_i = \sum_{i=1}^{N} \sum_{k=1}^{n_i} X_{ik}, \tag{2.10}$$

where N is the number of different kinds of financial products that the financial body owns under mixed operation, $X_i = \sum\limits_{k=1}^{n_i} X_{ik}$, $i = 1, \dots, N$, where X_{ik} denotes the k-th basic risk belonging to the i-th kind of financial products. The situation we study here is the same as the most general situation studied by the researchers introduced in Section 1. That is to say, the distribution functions of the basic risks, which are denoted by $\{F_{ik}\colon\ i = 1, \dots, N,\ k = 1, \dots, n_i\}$, are given, but the dependency relationship between these basic risks, which is denoted by copula C_S, is unknown.

However, different from the approach usually used to deal with the dependency relationship between basic risks, we substitute several different lower-dimensional copulas for the high-dimensional copula to describe the dependency relationship between all basic risks. Denote by $\{C_X, C_{X_i},\ i = 1, \dots, N\}$ the copulas of the $N+1$ risk vectors $(X_1, \dots, X_N), (X_{1,1}, \dots, X_{1,n_1}), (X_{2,1}, \dots, X_{2,n_2}), \dots, (X_{N,1}, \dots, X_{N,n_N})$, then the $\left(\sum_{i=1}^{N} n_i\right)$-dimensional copula C_S is substituted by the N-dimensional copula C_X and n_i-dimensional copulas $\{C_{X_i},\ i = 1, \dots, N\}$. As every group of basic risks $\{X_{i,1}, \dots, X_{i,n_1}\}$, $i = 1, \dots, N$, belongs to a specific kind of financial products, the dependency relationships between different groups of basic risks, which are denoted by copulas $\{C_{X_i},\ i = 1, \dots, N\}$, are different. The risks $\{X_1, \dots, X_N\}$ are coming from different kinds of financial products, so the dependency relationship between them is obviously different from that between the basic financial products in one group of financial products. For intuitive understanding of the model, we give the following example. Assume that the financial body under mixed operation owns three kinds of financial products: equities, bonds and funds, the number of which is n_1, n_2, and n_3, respectively. That is to say, $N = 3$. Denote by $X_{11}, \dots, X_{1n_1}, X_{21}, \dots, X_{2n_2}, X_{31}, \dots, X_{3n_3}$ the risk variables of these financial products, which can be seen as their losses, and assume that all the distribution functions of these losses are known but their dependency relationship is unknown. Then the total risk faced by the financial body can be written as $\varrho(S)$, where $S = \sum_{i=1}^{3} \sum_{k=1}^{n_i} X_{ik}$ is the overall loss of the financial products it holds and ϱ is a risk measure such as VaR. According to the model we build, the losses are divided into three groups based on different kinds of financial products they belong to and the three groups of losses are denoted by vectors $(X_{11}, \dots, X_{1n_1})$, $(X_{21}, \dots, X_{2n_2})$, and $(X_{31}, \dots, X_{3n_3})$. Different from the traditional approach to dealing with the aggregated risk of all losses, we first calculate the bound for every group of losses, that is, we calculate three bounds for the losses of all equities, all bonds and all funds, respectively. Then we obtain the bound and risk value of the total risk. During the calculation, we use four lower-dimensional copulas of (X_1, X_2, X_3), $(X_{11}, \dots, X_{1n_1})$, $(X_{21}, \dots, X_{2n_2})$, and $(X_{31}, \dots, X_{3n_3})$ to describe the dependency relationship of all losses rather than one high-dimensional copula of $(X_{11}, \dots, X_{1n_1}, X_{21}, \dots, X_{2n_2}, X_{31}, \dots, X_{3n_3})$. Thus it can be seen that the approach we use in this paper is based on the actual situation we study and can avoid the complicated calculation of high-dimensional copula. In the real world, for the financial products owned by a financial body under mixed operation, the loss distribution functions can be obtained by the methods of curve-fitting or maximum likelihood estimation. For all unknown copulas, we can use empirical copulas to substitute the real ones and then the result can be obtained by a similar process as the simulation given in Section 4. For a more accurate understanding of the model, we give the illustration picture in Figure 1. The variables $X_{11}, \dots, X_{1n_1}, \dots, X_{N1}, \dots, X_{Nn_N}$ have been explained above, $F_{11}, \dots, F_{1n_1}, \dots, F_{N1}, \dots, F_{Nn_N}$ are the corresponding distribution functions. X_i, $i = 1, \dots, N$, is the sum of the variables from the i-th group and C_{X_i}, $i = 1, \dots, N$, is the copula of the variables from the i-th group, i.e., the copula of $(X_{i1}, \dots, X_{in_i})$, $i = 1, \dots, N$. S is the sum of $X_1, \dots, X_N$, it also can be seen as the sum of all basic risk variables $X_{11}, \dots, X_{1n_1}, \dots, X_{N1}, \dots, X_{Nn_N}$ and C_X is the copula of $(X_1, \dots, X_N)$.

$$
\begin{array}{ll}
\text{The first group} & \left.\begin{array}{r} X_{11}(F_{11}) \\ \vdots \\ X_{1n_1}(F_{1n_1}) \end{array}\right\} X_1(C_{X_1}) \\
& \qquad \vdots \qquad\qquad \vdots \\
\text{The } N\text{th group} & \left.\begin{array}{r} X_{N1}(F_{N1}) \\ \vdots \\ X_{Nn_N}(F_{Nn_N}) \end{array}\right\} X_N(C_{X_N})
\end{array}
\left.\vphantom{\begin{array}{c} 1\\1\\1\\1\\1\\1\\1 \end{array}}\right\} S(C_X)
$$

Figure 1. The structure of the grouped model.

3. The bounds for the aggregated risk

As mentioned above, the risk measure in this paper is VaR. Consequently, for the calculation of aggregated risk, we only need to get the bounds of its distribution function, i.e., the $m_+(s)$ and $M_+(s)$. Here we only give the value of $m_+(s)$.

First, assume that $\{F_{ik}\colon\ i = 1, \dots, N,\ k = 1, \dots, n_i\}$, which are the marginal distribution functions of $\sum\limits_{i=1}^{N} n_i$ basic risks, are general continuous distribution functions. Second, during the calculation process we use the method from Wang et al. [21], which is introduced in Subsection 2.1, to obtain the lower bounds for risks $\Big\{X_i = \sum\limits_{k=1}^{n_i} X_{ik},\ i = 1, \dots, N\Big\}$ and then calculate the lower bound for the aggregated risk S, i.e., the sum of risks $\{X_i,\ i = 1, \dots, N\}$. The calculation of the bounds for aggregated risk is shown for two cases: when $X_1, \dots, X_N$ are mutually independent and mutually dependent, respectively. By Lemma 2.1 we know that the distribution function of $X_i,\ i = 1, \dots, N$, satisfies $F_{X_i}(x) = P(X_i \leqslant x) \geqslant \Phi^{-1}_{X_i}(x)$, where

$$\Phi_{X_i}(t) = \sum_{k=1}^{n_i} E[X_{ik}\colon\ X_{ik} \geqslant F^{-1}_{X_{ik}}(t)], \quad \Phi^{-1}_{X_i}(x) = \inf\{t \in [0,1]\colon\ \Phi_{X_i}(t) \geqslant x\}.$$

Based on these facts, we calculate $m_+(s)$ for S as follows.

3.1 Calculation of $m_+(s)$ for $X_1, \dots, X_N$ independent. As $X_1, \dots, X_N$ are mutually independent, for constants $a_1, \dots, a_N$ such that $a_1, \dots, a_N \in (0,1)$ and

$\sum_{i=1}^{N} a_i = 1$, we have

$$\begin{aligned}F_S(s) &= P(X_1 + X_2 + \ldots + X_N \leqslant s)\\ &\geqslant P(X_1 \leqslant a_1 s, X_2 \leqslant a_2 s, \ldots, X_N \leqslant a_N s)\\ &= F_{X_1}(a_1 s) F_{X_2}(a_2 s) \ldots F_{X_N}(a_N s)\\ &\geqslant \Phi_{X_1}^{-1}(a_1 s) \Phi_{X_2}^{-1}(a_2 s) \ldots \Phi_{X_N}^{-1}(a_N s),\end{aligned}$$

where $a_1, \ldots, a_N$ denote the investment proportion that the finance body invests to different kinds of financial products or the relative risk coefficients that the finance body can afford regarding different kinds of financial products. As the above inequality holds for any $a_1, \ldots, a_N \in (0,1)$ satisfying $\sum_{i=1}^{N} a_i = 1$, we get

$$\begin{aligned}F_S(s) &= P(X_1 + X_2 + \ldots + X_N \leqslant s)\\ &\geqslant \sup_{\substack{a_1,\ldots,a_N \in (0,1),\\ \sum_{i=1}^{N} a_i = 1}} \Phi_{X_1}^{-1}(a_1 s) \Phi_{X_2}^{-1}(a_2 s) \ldots \Phi_{X_N}^{-1}(a_N s).\end{aligned}$$

According to the definition of $m_+(s)$, we have

$$m_+(s) \geqslant \sup_{\substack{a_1,\ldots,a_N \in (0,1),\\ \sum_{i=1}^{N} a_i = 1}} \Phi_{X_1}^{-1}(a_1 s) \Phi_{X_2}^{-1}(a_2 s) \ldots \Phi_{X_N}^{-1}(a_N s).$$

3.2 Calculation of $m_+(s)$ for $X_1, \ldots, X_N$ dependent. In this case, we denote by C_S the dependency relationship of $X_1, \ldots, X_N$. Then according to (2.4) and (2.5), we have

$$\begin{aligned}F_S(s) &= P(X_1 + X_2 + \ldots + X_N \leqslant s)\\ &\geqslant P(X_1 \leqslant a_1 s, X_2 \leqslant a_2 s, \ldots, X_N \leqslant a_N s)\\ &= P(F_{X_1}(X_1) \leqslant F_{X_1}(a_1 s), F_{X_2}(X_2) \leqslant F_{X_2}(a_2 s), \ldots, F_{X_N}(X_N) \leqslant F_{X_N}(a_N s))\\ &= C_S(F_{X_1}(a_1 s), F_{X_2}(a_2 s), \ldots, F_{X_N}(a_N s)),\\ &\geqslant \sup_{\substack{a_1,\ldots,a_N \in (0,1),\\ \sum_{i=1}^{N} a_i = 1}} \left\{ \sum_{i=1}^{N} \Phi_{X_i}^{-1}(a_i) + N - 1 \right\},\end{aligned}$$

where $a_1, \ldots, a_N$ are defined as in Subsection 3.1. The second inequality above is obtained by (2.5). Additionally, we can substitute the copula C_S by some specific copula to obtain the lower bound of X. Similarly, by the definition of $m_+(s)$, we have

$$m_+(s) \geqslant \sup_{\substack{a_1,\ldots,a_N \in (0,1),\\ \sum_{i=1}^{N} a_i = 1}} \left\{ \sum_{i=1}^{N} \Phi_{X_i}^{-1}(a_i s) + 1 - N \right\}.$$

4.1 Numerical algorithm. In Arbenz et al. [2], a numerical approximation of a mild tree dependence is given by Algorithm 3.1. In this algorithm they first generate independent samples for basic variables and their copulas and then they define the empirical marginal distribution functions and copulas for these samples. Finally, they get the approximations of the needed joint distribution functions by integrating the empirical copula, which is difficult to complete in practice. In addition, Algorithm 3.1 just gives the train of thought for the numerical approximation but not the concrete steps of the calculating process. Inspired by this algorithm we put forward Algorithm 4.1 in the following, which is a numerical simulation algorithm and different from the above algorithm in several aspects. Firstly, in our algorithm we get the samples of every variable in every group by simulating the samples from given copulas of the variables in every group and then transform them according to the given marginal distribution functions in step 1 and step 2. After obtaining the sample sum of the variables in every group, we get the samples of S by reordering the sample data of $X_1, \dots, X_N$ according to their copula in step 3 to step 7, based on the notion of empirical copula in Algorithm 3.1 in Arbenz et al. [2]. Secondly, unlike Algorithm 3.1, we here describe every step of the simulation process in detail.

Algorithm 4.1. Fix $M \in \mathbb{N}$.

Step 1. For any $i \in \{1, \dots, N\}$, generate M independent samples denoted by $\{(u_{i1}, \dots, u_{in_i})_j,\ j = 1, \dots, M\}$ from the n_i-dimensional random vector $(U_{i1}, \dots, U_{in_i})$ which has the same copula as $(X_{i1}, \dots, X_{in_i})$, i.e., the copula of $(U_{i1}, \dots, U_{in_i})$ is C_{X_i} and $\{U_{i1}, \dots, U_{in_i},\ i = 1, \dots, N\}$ are random variables with uniform distribution on $[0, 1]$.

Step 2. For any $i \in \{1, \dots, N\}$, let $(x_{ik})_j = F_{ik}^{-1}[(u_{ik})_j], k = 1, \dots, n_i,\ j = 1, \dots, M$, where $(u_{ik})_j$ is the value of U_{ik} in the j-th sample of $(U_{i1}, \dots, U_{in_i})$, i.e., the k-th component of $(u_{i1}, \dots, u_{in_i})_j$. Then we get M samples denoted by $\{(x_{i1}, \dots, x_{in_i})_j, j = 1, \dots, M\}$ from the random vectors $(X_{i1}, \dots, X_{in_i})$ whose copula is C_{X_i}.

Step 3. For any $i \in \{1, \dots, N\}$, let $(x_i)_j = \sum_{k=1}^{n_i} (x_{ik})_j,\ j = 1, \dots, M$, where $(x_{ik})_j$ denotes the value of X_{ik} in the j-th sample of $(X_{i1}, \dots, X_{in_i})$, i.e., the k-th component of $(x_{i1}, \dots, x_{in_i})_j$. Then we get M samples of X_i denoted by $\{(x_i)_j, j = 1, \dots, M\}$.

Step 4. Generate M independent samples denoted by $\{(u_1, \dots, u_N)_j,\ j = 1, \dots, M\}$ from the N-dimensional random vector $(U_1, \dots, U_N)$ with copula C_S, which is unknown but can be given according to the situation we are interested in, and $U_i \sim U[0, 1]$, $i = 1, \dots, N$.

Step 5. For any $i \in \{1, \ldots, N\}$, sort $\{(x_i)_j,\ j = 1, \ldots, M\}$ to obtain a new sample sequence $\{(x_i)^j,\ j = 1, \ldots, M\}$, where $(x_i)^j$ denotes the j-th order statistic of $\{(x_i)_j,\ j = 1, \ldots, M\}$.

Step 6. For any $i \in \{1, \ldots, N\}$, let $(x_i')_j = (x_i)^{[M \cdot (u_i)_j + 1]}$, where $(u_i)_j$ is the value of U_i in the j-th sample of $(U_1, \ldots, U_N)$, i.e., the i-th component of $(u_1, \ldots, u_N)_j$, and $[x]$ denotes the integer part of x. Then $\{(x_i')_j,\ j = 1, \ldots, M\}$ is a new sample sequence obtained by reordering the original sample sequence $\{(x_i)_j,\ j = 1, \ldots, M\}$. The new sample sequences $\{(x_1')_j,\ j = 1, \ldots, M\}, \ldots, \{(x_N')_j,\ j = 1, \ldots, M\}$, which are mutually dependent now, form M samples of $(X_1, \ldots, X_N)$, which are denoted by $\{(x_1', \ldots, x_N')_j = ((x_1')_j, \ldots, (x_N')_j),\ j = 1, \ldots, M\}$, and the dependency structure of these new sample sequences is the same as that of $(U_1, \ldots, U_N)$.

Step 7. Let $s_j = \sum\limits_{i=1}^{N} (x_i')_j,\ j = 1, \ldots, M$, then we get M samples of random variable $S = \sum\limits_{i=1}^{N} X_i$, denoted by $\{s_1, \ldots, s_M\}$. Sorting the samples $\{s_j,\ j = 1, \ldots, M\}$ can give us $\mathrm{VaR}_\alpha(S) = s^{([M \cdot \alpha + 1])}$ based on the definition, where $s^{(j)}$ denotes the j-th order statistic of $\{s_j,\ j = 1, \ldots, M\}$ and $[x]$ represents the integer part of x.

4.2 Convergence. In the above algorithm, as the distributions of $\{X_i,\ i = 1, \ldots, N\}$ are unknown, we substitute them by the empirical distribution functions of $\{X_i,\ i = 1, \ldots, N\}$. Therefore, it is necessary to prove the convergence of Algorithm 4.1.

Lemma 4.1. *For $i = 1, \ldots, n$, let Y_i be a random variable with continuous distribution function F_i and independently generate m random numbers $y_i^{(1)}, \ldots, y_i^{(m)}$ from Y_i. The empirical distribution function of these random numbers is denoted by G_{mi}. Then as $m \to \infty$, we have $\sum\limits_{i=1}^{n} G_{mi}^{-1}(F_i(Y_i)) \to \sum\limits_{i=1}^{n} Y_i$ in probability.*

P r o o f. By Mao et al. [15], we know that $G_{mi}^{-1}(x) \to F_i^{-1}(x)$ almost sure for $x \in [0, 1]$. Then we have $G_{mi}^{-1}(x) \to F_i^{-1}(x)$ in probability, i.e., $\lim\limits_{m\to\infty} P[|G_{mi}^{-1}(x) - F_i^{-1}(x)| > \varepsilon] = 0$ for $x \in [0, 1]$.

Let $a_m(x) = P[|G_{mi}^{-1}(x) - F_i^{-1}(x)| > \varepsilon]$. As $|a_m(x)| \leqslant 1$ for $x \in [0, 1]$ and $\lim\limits_{m\to\infty} a_m(x) = 0$, by the dominated convergence theorem, we have

$$\lim_{m\to\infty} \int_0^1 a_m(x)\,\mathrm{d}x = \int_0^1 \lim_{m\to\infty} a_m(x)\,\mathrm{d}x = 0.$$

That is,

$$\lim_{m\to\infty} P[|G_{mi}^{-1}(X) - F_i^{-1}(X)| > \varepsilon] = \lim_{m\to\infty} \int_0^1 P[|G_{mi}^{-1}(x) - F_i^{-1}(x)| > \varepsilon]\,\mathrm{d}x = 0,$$

where X is a uniformly random variable on $[0,1]$, which is independent with $(y_i^{(1)},\dots,y_i^{(m)})$.

Then we have $G_{mi}^{-1}(X) \to F_i^{-1}(X)$ in probability as $m \to \infty$ for standard uniformly distributed X. For $X = F_i(Y_i)$, it holds that $G_{mi}^{-1}(F_i(Y_i)) \to F_i^{-1}(F_i(Y_i))$ in probability. Consequently, we have

$$\sum_{i=1}^{n} G_{mi}^{-1}(F_i(Y_i)) \to \sum_{i=1}^{n} Y_i$$

in probability as $m \to \infty$. □

Lemma 4.2. *Suppose that X is a continuous random variable and $\{X_n\}$ is a sequence of random variables. If $X_n \to X$ in probability, then $\mathrm{VaR}_\alpha(X_n) \to \mathrm{VaR}_\alpha(X)$.*

Proof. Let $a_n = \mathrm{VaR}_\alpha(X_n)$ and $a = \mathrm{VaR}_\alpha(X)$. Argue by reduction to absurdity. If the conclusion does not hold, we can find a subsequence a_{n_k} satisfying $a_{n_k} > a + \varepsilon$ or $a_{n_k} < a - \varepsilon$ for some $\varepsilon > 0$.

In the case when $a_{n_k} > a + \varepsilon$, it can be obtained that

$$\alpha = P(X_{n_k} \leqslant a_{n_k}) > P\Big(X_{n_k} \leqslant a + \frac{\varepsilon}{2}\Big).$$

Letting k tend to ∞, we have

$$\alpha \geqslant P\Big(X \leqslant a + \frac{\varepsilon}{2}\Big),$$

In consideration of the fact that $P(X \leqslant a + \varepsilon/2) > \alpha$, the contradiction exists.

In the case when $a_{n_k} < a - \varepsilon$, we have

$$\alpha \leqslant P\Big(X_{n_k} \leqslant a_{n_k} + \frac{\varepsilon}{4}\Big) < P\Big(X_{n_k} \leqslant a - \frac{\varepsilon}{2}\Big).$$

Letting k tend to ∞ gives

$$\alpha \leqslant P\Big(X < a - \frac{\varepsilon}{2}\Big),$$

which is contradictory with $P(X \leqslant a - \varepsilon/2) < \alpha$. □

Theorem 4.3. *The simulated* VaR *for* $\{s_j,\ j=1,\dots,M\}$ *in Algorithm* 4.1 *converges to the* VaR *of* S, *i.e.,* $\mathrm{VaR}_\alpha(S_M) \to \mathrm{VaR}_\alpha(S)$, *where* $S_M = \{s_j,\ j = 1,\dots,M\}$.

P r o o f. By Lemmas 4.1 and 4.2, we have

$$\mathrm{VaR}_\alpha\left(\sum_{i=1}^{n} G_{mi}^{-1}(F_i(X_i))\right) \to \mathrm{VaR}_\alpha\left(\sum_{i=1}^{n} X_i\right),$$

where $F_i(X_i)$ is the distribution function of X_i. That is to say, $\mathrm{VaR}_\alpha(S_M) \to \mathrm{VaR}_\alpha(S)$. $\square$

4.3 Some results about the numerical simulation and suggestions. We give the simulated results for $N = n_1 = n_2 = 2$ in this subsection. We obtain the simulated VaRfor S in different cases of the distribution functions of X_{11}, X_{12}, X_{21}, X_{22} and the copulas of (X_{11}, X_{12}), (X_{21}, X_{22}), and (X_1, X_2). Assuming that the basic risks are normally distributed and t-distributed, respectively, the copula of (X_{11}, X_{12}), which is denoted by C_{X_1}, is the *Gauss copula* C_ϱ^{Ga} with parameter $\varrho = 0.3$ and the copula of (X_{21}, X_{22}), which is denoted by C_{X_2}, is the *t-copula* $C_{\nu,\varrho}^t$ with parameters $\varrho = 0.5$, $\nu = 4$. For the copula of (X_1, X_2), we give three forms, namely, independent copula, countermonotonicity copula and comonotonicity copula. The confidence level of VaR is 0.95.

Let $M = 10,000$. The results of VaR for the above different cases are shown in Table 1 and the graphs of the empirical distribution functions of the aggregated risk S in these different cases are given in Figure 2.

cases	1	2	3	4
X_{11}	$N(2,1)$	$t(10)$	$t(10)$	$t(10)$
X_{12}	$N(3,4)$	$t(5)$	$t(5)$	$t(5)$
C_{X_1}	C_ϱ^{Ga}	C_ϱ^{Ga}	C_ϱ^{Ga}	C_ϱ^{Ga}
X_{22}	$t(5)$	$N(10,15)$	$N(10,15)$	$N(10,15)$
C_{X_2}	$C_{\nu,\varrho}^t$	$C_{\nu,\varrho}^t$	$C_{\nu,\varrho}^t$	$C_{\nu,\varrho}^t$
C_S	Π	Π	W	M
VaR	41.6757	30.3916	26.3456	32.6293

Table 1. The VaR in different cases of the margins and dependency structures

From the simulated results stated in Table 1, we know that the risk values are quite different while the dependency structures are the same but the margins are different and it also behaves differently when the random basic risk variables with identical margins have different dependency structures. It is obvious that the aggregated risk value is determined by both the marginal distributions and the joint distribution which is substituted by copula here. From the risk values of case 1 and case 2, we know that for the same joint distributions of (X_{11}, X_{12}), (X_{21}, X_{22}), and (X_1, X_2), different marginal distributions of basic risks result in different values of VaR. Though we do not know the specific relationship between the marginal distributions and the final risk value, it does exist and it can be seen from Table 1 that the VaR in case 1 is much different from that in case 2. By comparing the values of VaR in case 2, case 3, and case 4, we find that for the same marginal distributions and joint distributions of (X_{11}, X_{12}) and (X_{21}, X_{22}), different joint distributions of (X_1, X_2) lead to different results. According to the definition of X_1 and X_2 we know that they represent the risk variables of the financial products from different industries, that is to say, while the marginal distributions and the dependency structure of the variables in every group are given, different dependency structures between the two different financial industries lead to different values of VaR. It is obvious from Table 1 that the value of VaR is maximum while the two financial industries are perfectly positively dependent, it is minimum while the two financial industries are perfectly negatively dependent, and it is between these two values while the two financial industries are independent but there is no much difference between these values.

Comparing the results in Table 1 we can see that the marginal distributions, i.e., the distribution functions of the basic variables, as well as the master copula, i.e., the copula of different financial industries, are the reasons that cause different values of VaR and the former has major influence on the resulting value of VaR. Though the latter of the above reasons has less impact on the final result than the former one, its influence cannot be ignored, especially in practice. It has been said before that a financial body under mixed operation owns a great variety of basic financial products which belong to different financial industries such as stocks, funds, etc. Combining this fact with the results and analysis above, we suggest that the financial body should effectively analyze the financial market and various financial products before investment to achieve diversification and invest in different kinds of financial products that are negatively dependent to the greatest extent possible.

Additionally, the curves of the empirical distribution functions of S in the cases 1, 2, 3, and 4, which are stated in Table 1, are presented in Figure 2, where we denote the random variable S by x and the empirical distribution function of S by $F(x)$. Noticing the value ranges of x corresponding to the curves, we can find that the values of the total risk variable S in these different cases are different. From Figure 2 we can easily see that the value range of x is minimum while the two financial industries are perfectly negatively dependent, and the value of x corresponding the left endpoint of the curve in this case is maximum but the value of x corresponding the right endpoint of the curve is minimum. The value range of x is maximum in case 1 and the value of x corresponding the left endpoint of the curve in this case is minimum but the value of x corresponding the right endpoint of the curve is maximum. The value ranges of x as well as the values of x corresponding the left and right endpoints of the curves in other two cases are between the ranges and values in the above two cases. From the shape of every curve in the figure we can see that the shapes in case 2, case 3, and case 4 are nearly the same and obviously different from that in case 1, which implies that the distributions of the basic risk variables have more impact on the resulting VaR than the dependency relationship between different financial industries. By comparing the shape of the curve in case 1 to that in case 2 and the shape in case 2 to that in cases 3 and 4 we can conclude that both of the different marginal distributions and the different master copulas are the reasons which lead to the resulting VaR.

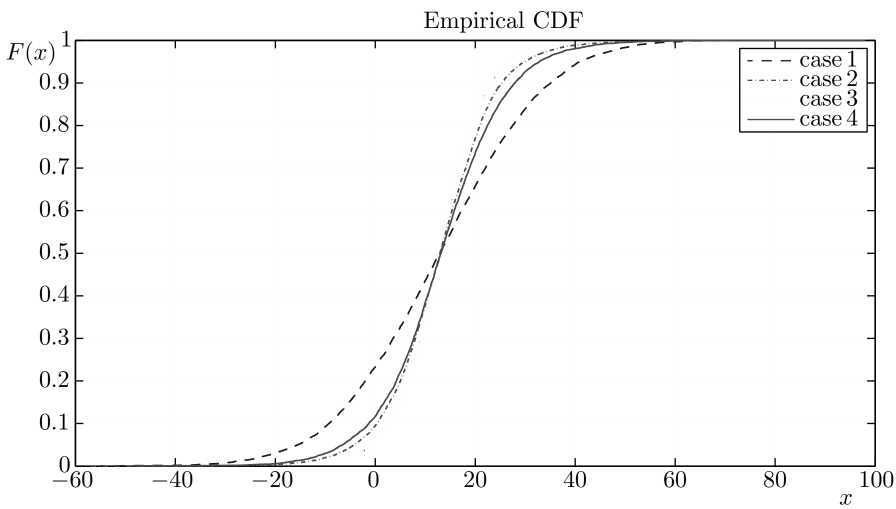

Figure 2. The curves of the empirical CDF corresponding to different margins and dependency structures.

Furthermore, for the management implications, the results and analysis above are helpful too. From the impact of the dependency relationship between different financial industries on the final aggregated risk faced by a financial body, it can be seen that there is a necessity to consider the following things for making more effective management decisions. Firstly, the manager should focus on the market changes and re-assess the portfolio in hand to ensure that all the financial products the body owns have proper dependency relationships, especially for different kinds of financial products, as different dependency relationships cause different values of

VaR. Secondly, for those different kinds of financial products which have a positive dependency relationship to some extent, the manager should pay more attention to any change of them, especially the bad changes, as the risks of these financial products have contagion effect due to their positive dependence. Thirdly, the manager should adjust the proportion of different kinds of financial products contained in the portfolio timely according to the change of the financial market by increasing or reducing the investment amount to some financial products. By this procedure, the investment to the financial products which are positively dependent and to those which are negatively dependent can maintain a good proportion, which can guarantee that the total risk of the portfolio is always affordable.

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