Ravi P. Agarwal; Alexander Domoshnitsky; Abraham Maghakyan
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ON EXPONENTIAL STABILITY OF SECOND ORDER DELAY DIFFERENTIAL EQUATIONS

RAVI P. AGARWAL, Kingsville,
ALEXANDER DOMOSHNITSKY, ABRAHAM MAGHAKYAN, Ariel

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Abstract. We propose a new method for studying stability of second order delay differential equations. Results we obtained are of the form: the exponential stability of ordinary differential equation implies the exponential stability of the corresponding delay differential equation if the delays are small enough. We estimate this smallness through the coefficients of this delay equation. Examples demonstrate that our tests of the exponential stability are essentially better than the known ones. This method works not only for autonomous equations but also for equations with variable coefficients and delays.

Keywords: delay equations; uniform exponential stability; exponential estimates of solutions; Cauchy function

MSC 2010: 34K20

1. INTRODUCTION

The main object of this paper is the second order delay differential equation

$$x''(t)+\sum_{i=1}^{m}a_i(t)x'(t-\theta_i(t))+\sum_{i=1}^{m}b_i(t)x(t-\tau_i(t))=f(t),\quad t\in[0,\infty), \tag{1.1}$$

with a corresponding initial function defining what should be set in the equation instead of $x(t-\tau_i(t))$ when $t-\tau_i(t)<0$ or $x'(t-\theta_i(t))$ when $t-\theta_i(t)<0$. For simplicity and without loss of generality we can consider the zero initial function

$$x(\xi)=x'(\xi)=0,\quad \text{for } \xi<0. \tag{1.2}$$

Concerning the coefficients, delays and the function f we assume that f, a_i, b_i, τ_i, θ_i $(i=1,\dots,m)$ are measurable essentially bounded functions $[0,\infty)\to(-\infty,\infty)$, and $\tau_i(t)\geqslant 0$, $\theta_i(t)\geqslant 0$ for $t\geqslant 0$.

Various aspects of oscillation and the asymptotic behavior of solutions to this equation were studied in the known books [13], [18], [20]. Various applications of equation (1.1) and its generalizations can be found, for example, in the theory of self-excited oscillations, in oscillation processes in a vacuum tube, in dynamics of an autogenerator, in description of processes of infeed grinding and cutting (see the book [17]), in position control in mechanical engineering (for example, the model of container crane: it is important the crane to move rapidly, the payload may sway and, as a result, the crane operator can lose control of the payload), in electromechanical systems, in combustion engines [14]. It was noted in [7], [8] that the equation

$$x''(t) + a_1 x'(t) + a_2 x'(t-\tau) + b_1 x(t) + b_2 x(t-\tau) = 0, \tag{1.3}$$

in the case $b_1 b_2 < 0$, is of interest in machine tool analysis, in biology in explaining self-balancing of the human body and in robotics in constructing biped robots [15] (see bibliography in [7], [8]). The problem of stabilizing the rolling of a ship by the activated tanks method in which ballast water is pumped from one position to another was reduced in [19] to the analysis of stability of the second order equation (1.1).

Stability of delay equations was studied in the book [17]. Note the results on stability of autonomous equations obtained there. Stability and instability of second order autonomous delay differential equation (1.3) with constant coefficients and delays were studied in [7], [8]. These results were based on Pontryagin's technique for the analysis of roots of quasi-polynomials [22]. Results on stability of the equation

$$x''(t) + a x'(t) + b x(t-\tau) = 0, \quad a > 0,\ b > 0, \tag{1.4}$$

were obtained in [4] by the method of Lyapunov's functions. It was proven by Burton (see [4]) that the simple inequality $b\tau < a$ implies the exponential stability of equation (1.4). Other results obtained by the method of Lyapunov's functions were presented in the papers [6], [23]. In [5] the technique of fixed point theorems was used for the analysis of stability of equation (1.1). Quite different results for stability of equation (1.4) by a development of the fixed point method were obtained in [3]. First results on the exponential stability of the equation

$$x''(t) + a(t) x'(t-\theta(t)) + b(t) x(t-\tau(t)) = 0, \quad a(t) > 0,\ b(t) > 0, \tag{1.5}$$

without the assumption $\theta(t) \equiv 0$, as far as we know, were obtained in [11] and then developed in [2]. Asymptotic properties of equation (1.5) without damping term (i.e.; in the case $a(t) \equiv 0$ for $t \in [0,\infty)$) were studied in ([20], Chapter III, Section 16, pages 105–106), where instability of the equation

$$x''(t) + b x(t-\tau) = 0$$

for every pair of positive constants b and τ was obtained. Conditions of instability of the equation

$$(1.6) \qquad x''(t) + \sum_{i=1}^{m} b_i(t)x(t-\tau_i(t)) = 0, \quad b_i(t) > 0,\ \tau_i(t) > 0,\ t \in [0,\infty)$$

with variable coefficients and delays were obtained in [10].

The condition $\int_0^\infty \tau(t)\,\mathrm{d}t < \infty$ is necessary and sufficient for boundedness of all solutions to the equation

$$x''(t) + bx(t-\tau(t)) = 0$$

(see [10]). Results about boundedness of solutions for vanishing delays ($\tau_i(t) \to 0$ for $t \to \infty$) and about asymptotic representations of solutions were obtained in [16], [21], see also ([20], Chapter III, Section 16). Boundedness of solutions for equations with advanced arguments ($\tau_i(t) \leqslant 0$) was studied in [12]. First results on the exponential stability of the equation $x''(t) + ax(t) - bx(t-\tau) = 0$ with constant coefficients and delay were obtained in [7], [8]. First results on the exponential stability of the second order equation (1.6) without damping term and with variable coefficients and delays were obtained in the recent paper [9].

Let us try to imagine situations in which variable delays and coefficients arising in the delayed feedback control may be important: 1) the case of control for missiles, where the delay depends on their distance from the controller and is variable; 2) spending of fuel implies the change of the mass of the missiles that leads to variable coefficients in the delay system.

We understand a solution of equation (1.1), (1.2) as a function $x\colon [0,\infty) \to (-\infty,\infty)$ with absolutely continuous on every finite interval derivative x' and essentially bounded second derivative x'' which satisfies this equation almost everywhere.

The general solution of equation (1.1), (1.2) can be represented in the form [1]

$$(1.7) \qquad x(t) = \int_0^t C(t,s)f(s)\,\mathrm{d}s + x_1(t)x(0) + x_2(t)x'(0),$$

where $x_1(t), x_2(t)$ are two solutions of the homogeneous equation (1.8), (1.2), where

$$(1.8) \qquad x''(t) + \sum_{i=1}^{m} a_i(t)x'(t-\theta_i(t)) + \sum_{i=1}^{m} b_i(t)x(t-\tau_i(t)) = 0, \quad t \in [0,\infty),$$

satisfying the conditions

$$(1.9) \qquad x_1(0) = 1,\ x_1'(0) = 0,\ x_2(0) = 0,\ x_2'(0) = 1;$$

the kernel $C(t,s)$ in this representation is called the Cauchy function (fundamental function in other terminology) of equation (1.1).

Note that in the classical books on delay differential equations [13], [18], [20], the homogeneous equations are considered as equation (1.8) with absolutely continuous initial function

$$x(\xi)=\varphi(\xi),\ x'(\xi)=\varphi'(\xi) \quad \text{for } \xi<0. \tag{1.10}$$

Let us formulate several definitions concerning stability.

Definition 1.1. Equation (1.8), (1.10) is uniformly exponentially stable if there exist $N>0$ and $\alpha>0$ such that the solution of (1.8), (1.10), where

$$x(\xi)=\varphi(\xi),\ x'(\xi)=\varphi'(\xi), \quad \xi<t_0, \quad x(t_0)=x_0,\ x'(t_0)=x_0',$$

satisfies the estimate

$$|x(t)|\leqslant N\mathrm{e}^{-\alpha(t-t_0)},\ |x'(t)|\leqslant N\mathrm{e}^{-\alpha(t-t_0)}, \quad 0\leqslant t<\infty, \tag{1.11}$$

where N and α do not depend on t_0.

Definition 1.2. The Cauchy function $C(t,s)$ of equation (1.1) satisfies the exponential estimate if there exist positive N and α such that

$$|C(t,s)|\leqslant N\mathrm{e}^{-\alpha(t-s)},\ |C_t'(t,s)|\leqslant N\mathrm{e}^{-\alpha(t-s)}, \quad 0\leqslant s\leqslant t<\infty. \tag{1.12}$$

It is known that for equation (1.1) with bounded delays these two definitions are equivalent [1].

In this paper we develop the approach of the paper [2] and improve essentially its results on the exponential stability. In the corresponding cases we improve the noted above Burton's result [4] for equation (1.4) (see Remark 2.4 below). Our technique in the study of the exponential stability is based on the Bohl-Perron theorem: for equation (1.1) with bounded delays, the exponential estimate of the Cauchy function is equivalent to the fact that for every bounded right hand side f, the solution x and its derivative x' are bounded [1].

The paper is built as follows. In the first section we describe known results on asymptotic properties of second order delay equations. In Section 2, we formulate the main results of the paper and compare them with known results. Auxiliary assertions can be found in Section 3. Proofs of the assertions, formulated in Section 2, can be found in Section 4. An open problem is formulated in Section 5.

2. Formulation of main results

Let us consider the following ordinary differential equation:

$$x''(t) + Ax'(t) + Bx(t) = z(t), \quad t \in [0, \infty),\ A > 0,\ B > 0, \tag{2.1}$$

where z is an essentially bounded measurable function with constant positive coefficients A and B. Denote by $W(t,s)$ the Cauchy function of equation (2.1). It is known that for every fixed s the function $W(t,s)$, as a function of the variable t, satisfies the homogeneous equation

$$x''(t) + Ax'(t) + Bx(t) = 0, \quad t \in [s, \infty), \tag{2.2}$$

and the initial conditions

$$x(s) = 0, \quad x'(s) = 1. \tag{2.3}$$

It is known that the solution of the equation (2.1) which satisfies the initial conditions

$$x(0) = 0, \quad x'(0) = 0 \tag{2.4}$$

can be written in the form

$$x(t) = \int_0^t W(t,s) z(s)\,\mathrm{d}s. \tag{2.5}$$

Its derivatives are

$$x'(t) = \int_0^t W'_t(t,s) z(s)\,\mathrm{d}s, \quad x''(t) = \int_0^t W''_{tt}(t,s) z(s)\,\mathrm{d}s + z(t). \tag{2.6}$$

Let us denote

$$|W| = \lim_{t\to\infty} \sup_{t\geqslant 0} \int_0^t |W(t,s)|\,\mathrm{d}s, \tag{2.7}$$

$$|W'_t| = \lim_{t\to\infty} \sup_{t\geqslant 0} \int_0^t |W'_t(t,s)|\,\mathrm{d}s, \quad |W''_{tt}| = \lim_{t\to\infty} \sup_{t\geqslant 0} \int_0^t |W''_{tt}(t,s)|\,\mathrm{d}s. \tag{2.8}$$

Denote by A_i and B_i the average values of the coefficients $a_i(t)$ and $b_i(t)$ in equation (1.1), respectively. To connect equations (1.1) and (2.1) we suppose below that the coefficients A and B in equation (2.1) are the sums $A = \sum_{i=1}^{m} A_i$, $B = \sum_{i=1}^{m} B_i$. Denote also $\Delta a_i(t) = A_i - a_i(t)$, $|\Delta a_i|^* = \operatorname*{ess\,sup}_{t\geqslant 0} |\Delta a_i(t)|$, $\Delta b_i(t) = B_i - b_i(t)$, $|\Delta b_i|^* = \operatorname*{ess\,sup}_{t\geqslant 0} |\Delta b_i(t)|$, $\theta_i^* = \operatorname*{ess\,sup}_{t\geqslant 0} \theta_i(t)$, $\theta^* = \max_{i=1,\dots,m} \theta_i^*$, $\tau_i^* = \operatorname*{ess\,sup}_{t\geqslant 0} \tau_i(t)$, $\tau^* = \max_{i=1,\dots,m} \tau_i^*$.

Theorem 2.1. *Let $A > 0$, $B > 0$ and let the following inequality be fulfilled:*

$$\sum_{i=1}^{m} |A_i|\theta_i^*\{|W_{tt}''| + 1\} + \sum_{i=1}^{m} |\Delta a_i|^* |W_t'| + \sum_{i=1}^{m} |B_i|\tau_i^* |W_t'| + \sum_{i=1}^{m} |\Delta b_i|^* |W| < 1. \tag{2.9}$$

Then the Cauchy function $C(t,s)$ of equation (1.1) *and the fundamental system $x_1(t), x_2(t)$ of equation* (1.8), (1.2) *satisfy the exponential estimate.*

Remark 2.1. Denoting $P = |W|$, $Q = |W_t'|$, $R = |W_{tt}''|$, we have a simple geometrical interpretation of this result. Let as define the coordinates: $X = \sum_{i=1}^{m} |A_i|\theta_i^*$, $Y = \sum_{i=1}^{m} \{|\Delta a_i|^* + |B_i|\tau_i^*\}$, $Z = \sum_{i=1}^{m} |\Delta b_i|^*$. Condition (2.9) is fulfilled for every interior point (X,Y,Z) of the pyramid bounded by the planes $(1+R)X + QY + PZ = 1$, $X = 0$, $Y = 0$, $Z = 0$.

Remark 2.2. It is clear from inequality (2.9) that in the case, when the coefficients $a_i(t)$ and $b_i(t)$ $(i = 1, \ldots, m)$ are close to constants, the second and fourth terms are small, and in the case of small delays $\theta_i(t)$ and $\tau_i(t)$ $(i = 1, \ldots, m)$, the first and third terms are small. We can make the conclusion that in this case equation (1.1) preserves the property of the exponential stability of equation (2.1).

Remark 2.3. Below in Section 3 we compute the exact values of $|W|$, $|W_t'|$ and $|W_{tt}''|$. It is clear from inequality (2.9) that in the case of exact $|W|$, $|W_t'|$ and $|W_{tt}''|$, we can obtain better tests of the exponential stability. Let us compare exact values of $|W|$, $|W_t'|$ with their estimates obtained in [2]. For $A = 3$, $B = 1.25$, we have the case $A^2 - 4B = 4 > 0$ and $|W| = 0.8$ in both works in this situation. Concerning $|W_t'|$ we have $|W_t'| = 3$, according to [2], and $|W_t'| \approx 0.53499$, according to our result. For $A = 3, B = 2$, we have also the case $A^2 - 4B = 1 > 0$. In both cases $|W| = 0.5$. Concerning $|W_t'|$ we have $|W_t'| = 6$, according to [2], and $|W_t'| = 0.5$, according to our result. For $A = 1, B = 1$, we are in the case $A^2 < 4B$. We have $|W| = 2$, according to [2], and $|W| \approx 1.6044$, according to our result, and $|W_t'| = 3.15$, according to [2], and $|W_t'| \approx 0.551$, according to our result. For the case $A^2 = 4B$, we have in both cases $|W| = 1/B$. Concerning $|W_t'|$ we have $|W_t'| = 2/\sqrt{B}$, according to [2], and $|W_t'| = 2/(\mathrm{e}\sqrt{B})$, according to our result.

Theorem 2.2. *Let $A>0$, $B>0$, $A^2>4B$ and let the following inequality be fulfilled:*

$$\begin{aligned}(2.10)\qquad &\sum_{i=1}^{m}|A_i|\theta_i^*\Big\{\frac{2}{A+\sqrt{A^2-4B}}\Big\{\frac{A-\sqrt{A^2-4B}}{A+\sqrt{A^2-4B}}\Big\}^{(A-\sqrt{A^2-4B})/\sqrt{A^2-4B}}+1\Big\}\\ &+\sum_{i=1}^{m}|\Delta a_i|^*\frac{4}{A+\sqrt{A^2-4B}}\Big\{\frac{A-\sqrt{A^2-4B}}{A+\sqrt{A^2-4B}}\Big\}^{(A-\sqrt{A^2-4B})/(2\sqrt{A^2-4B})}\\ &+\sum_{i=1}^{m}|B_i|\tau_i^*\frac{4}{A+\sqrt{A^2-4B}}\Big\{\frac{A-\sqrt{A^2-4B}}{A+\sqrt{A^2-4B}}\Big\}^{(A-\sqrt{A^2-4B})/(2\sqrt{A^2-4B})}\\ &+\sum_{i=1}^{m}|\Delta b_i|^*\frac{1}{B}<1.\end{aligned}$$

Then the Cauchy function $C(t,s)$ of equation (1.1) *and the fundamental system $x_1(t),x_2(t)$ of equation* (1.8), (1.2) *satisfy the exponential estimate.*

Theorem 2.3. *Let $A>0$, $B>0$, $A^2=4B$ and let the following inequality be fulfilled:*

$$\begin{aligned}(2.11)\qquad &\sum_{i=1}^{m}|A_i|\theta_i^*\Big\{2+\frac{A}{4}-\Big(1-\frac{A}{4}\Big)\frac{1}{\mathrm{e}^2}\Big\}+\sum_{i=1}^{m}|\Delta a_i|^*\frac{4}{A\mathrm{e}}\\ &+\sum_{i=1}^{m}|B_i|\tau_i^*\frac{4}{A\mathrm{e}}+\sum_{i=1}^{m}|\Delta b_i|^*\frac{1}{B}<1.\end{aligned}$$

Then the Cauchy function $C(t,s)$ of equation (1.1) *and the fundamental system $x_1(t),x_2(t)$ of equation* (1.8), (1.2) *satisfy the exponential estimate.*

Theorem 2.4. *Let $A>0$, $B>0$, $A^2<4B$ and let the following inequality be fulfilled:*

$$\begin{aligned}(2.12)\qquad &\sum_{i=1}^{m}|A_i|\theta_i^*\Big\{\frac{2B}{A\sqrt{4B-A^2}}+1\Big\}\\ &+\sum_{i=1}^{m}|\Delta a_i|^*\frac{2}{\sqrt{B}}\frac{\exp\big[-\frac{A}{\sqrt{4B-A^2}}\big(\pi+\operatorname{arctg}\frac{\sqrt{4B-A^2}}{A}\big)\big]}{1-\exp\big[-\frac{A}{\sqrt{4B-A^2}}\pi\big]}\\ &+\sum_{i=1}^{m}|B_i|\tau_i^*\frac{2}{\sqrt{B}}\frac{\exp\big[-\frac{A}{\sqrt{4B-A^2}}\big(\pi+\operatorname{arctg}\frac{\sqrt{4B-A^2}}{A}\big)\big]}{1-\exp\big[-\frac{A}{\sqrt{4B-A^2}}\pi\big]}\\ &+\sum_{i=1}^{m}|\Delta b_i|^*\frac{1}{B}\frac{1+\exp\big(-\frac{A}{\sqrt{4B-A^2}}\pi\big)}{1-\exp\big(-\frac{A}{\sqrt{4B-A^2}}\pi\big)}<1.\end{aligned}$$

Then the Cauchy function $C(t,s)$ of equation (1.1) *and the fundamental system* $x_1(t), x_2(t)$ *of equation* (1.8), (1.2) *satisfy the exponential estimate.*

Let us formulate corollaries for the equation

$$x''(t) + Ax'(t-\theta(t)) + Bx(t-\tau(t)) = f(t), \quad t \in [0,\infty), \tag{2.13}$$

where

$$x(\xi) = 0, \; x'(\xi) = 0 \quad \text{for } \xi < 0, \tag{2.14}$$

A, B are constants and $\theta(t), \tau(t), f(t)$ are measurable essentially bounded functions. Denote $\theta^* = \operatorname*{ess\,sup}_{t\geqslant 0} \theta(t)$, $\tau^* = \operatorname*{ess\,sup}_{t\geqslant 0} \tau(t)$. All the corollaries are results of substitution of the values $|W|$, $|W'_t|$ and $|W''_{tt}|$ in Theorems 2.2–2.4.

Corollary 2.1. *Let* $A > 0$, $B > 0$, $A^2 > 4B$,

$$\begin{aligned} A\theta^*\Bigl\{1 + \frac{2}{A+\sqrt{A^2-4B}}\Bigl\{\frac{A-\sqrt{A^2-4B}}{A+\sqrt{A^2-4B}}\Bigr\}^{(A-\sqrt{A^2-4B})/\sqrt{A^2-4B}}\Bigr\}& \\ + B\tau^*\frac{4}{A+\sqrt{A^2-4B}}\Bigl\{\frac{A-\sqrt{A^2-4B}}{A+\sqrt{A^2-4B}}\Bigr\}^{(A-\sqrt{A^2-4B})/(2\sqrt{A^2-4B})} &< 1. \end{aligned} \tag{2.15}$$

Then the Cauchy function $C(t,s)$ of equation (2.13) *and the fundamental system* $x_1(t)$, $x_2(t)$ *of equation* (2.13), (2.14) *satisfy the exponential estimate.*

Corollary 2.2. *Let* $A > 0$, $B > 0$, $A^2 = 4B$,

$$A\theta^*\Bigl\{1 + \frac{A}{4} - \Bigl(1 - \frac{A}{4}\Bigr)\frac{1}{\mathrm{e}^2}\Bigr\} + \tau^*\frac{4B}{A\mathrm{e}} < 1. \tag{2.16}$$

Then the Cauchy function $C(t,s)$ of equation (2.13) *and the fundamental system* $x_1(t), x_2(t)$ *of equation* (2.13), (2.14) *satisfy the exponential estimate.*

Corollary 2.3. *Let* $A > 0$, $B > 0$, $A^2 < 4B$,

$$A\theta^* + \frac{2B\theta^*}{\sqrt{4B-A^2}} + 2\sqrt{B}\tau^*\frac{\exp\bigl[-\frac{A}{\sqrt{4B-A^2}}\bigl(\pi + \operatorname{arctg}\frac{\sqrt{4B-A^2}}{A}\bigr)\bigr]}{1-\exp\bigl[-\frac{A}{\sqrt{4B-A^2}}\pi\bigr]} < 1. \tag{2.17}$$

Then the Cauchy function $C(t,s)$ of equation (2.13) *and the fundamental system* $x_1(t), x_2(t)$ *of equation* (2.13), (2.14) *satisfy the exponential estimate.*

Remark 2.4. For the case $a = 1$, $b = 1$, the result $b\tau < a$ by Burton [4] leads us to the condition $\tau < 1$ for the exponential stability of equation (1.4). Corollary 2.3 claims that for $\tau < 4$ equation (1.4) is exponentially stable.

3. Values of integrals of the modulus of Cauchy functions for auxiliary equations

Consider all possible cases: 1) $A^2 > 4B$, 2) $A^2 = 4B$, 3) $A^2 < 4B$.

Lemma 3.1. *Let $A > 0$, $B > 0$, $A^2 > 4B$. Then*

$$\lim_{t\to\infty} \sup_{t\geqslant 0} \int_0^t |W(t,s)|\,\mathrm{d}s = \frac{1}{B}, \tag{3.1}$$

$$\begin{aligned}\lim_{t\to\infty} \sup_{t\geqslant 0} \int_0^t |W_t'(t,s)|\,\mathrm{d}s \\ = \frac{4}{A+\sqrt{A^2-4B}}\Big\{\frac{A-\sqrt{A^2-4B}}{A+\sqrt{A^2-4B}}\Big\}^{(A-\sqrt{A^2-4B})/(2\sqrt{A^2-4B})},\end{aligned} \tag{3.2}$$

$$\begin{aligned}\lim_{t\to\infty} \sup_{t\geqslant 0} \int_0^t |W_{tt}''(t,s)|\,\mathrm{d}s \\ = 1+\frac{2}{A+\sqrt{A^2-4B}}\Big\{\frac{A-\sqrt{A^2-4B}}{A+\sqrt{A^2-4B}}\Big\}^{(A-\sqrt{A^2-4B})/\sqrt{A^2-4B}}.\end{aligned} \tag{3.3}$$

Lemma 3.2. *Let $A > 0$, $B > 0$, $A^2 = 4B$. Then*

$$\lim_{t\to\infty} \sup_{t\geqslant 0} \int_0^t |W(t,s)|\,\mathrm{d}s = \frac{1}{B}, \tag{3.4}$$

$$\lim_{t\to\infty} \sup_{t\geqslant 0} \int_0^t |W_t'(t,s)|\,\mathrm{d}s = \frac{4}{A\mathrm{e}}, \tag{3.5}$$

$$\lim_{t\to\infty} \sup_{t\geqslant 0} \int_0^t |W_{tt}''(t,s)|\,\mathrm{d}s = 1+\frac{A}{4}-\Big(1-\frac{A}{4}\Big)\frac{1}{\mathrm{e}^2}. \tag{3.6}$$

Lemma 3.3. *Let $A > 0$, $B > 0$, $A^2 < 4B$. Then*

$$\lim_{t\to\infty} \sup_{t\geqslant 0} \int_0^t \big|W(t,s)\big|\,\mathrm{d}s = \frac{1}{B}\frac{1+\exp\big(-\frac{A}{\sqrt{4B-A^2}}\pi\big)}{1-\exp\big(-\frac{A}{\sqrt{4B-A^2}}\pi\big)}, \tag{3.7}$$

$$\lim_{t\to\infty} \sup_{t\geqslant 0} \int_0^t \big|W_t'(t,s)\big|\,\mathrm{d}s = \frac{2}{\sqrt{B}}\frac{\exp\big[-\frac{A}{\sqrt{4B-A^2}}\big(\pi+\operatorname{arctg}\frac{\sqrt{4B-A^2}}{A}\big)\big]}{1-\exp\big[-\frac{A}{\sqrt{4B-A^2}}\pi\big]}, \tag{3.8}$$

$$\lim_{t\to\infty} \sup_{t\geqslant 0} \int_0^t |W_{tt}''(t,s)|\,\mathrm{d}s = \frac{2B}{A\sqrt{4B-A^2}}. \tag{3.9}$$

P r o o f of Lemma 3.1. The characteristic equation for (2.2) is

$$k^2 + Ak + B = 0. \tag{3.10}$$

Solving the characteristic equation (3.10), we obtain

$$k_1 = \frac{-A - \sqrt{A^2 - 4B}}{2}, \quad k_2 = \frac{-A + \sqrt{A^2 - 4B}}{2}. \tag{3.11}$$

Substituting the initial conditions (2.3), we get

$$\begin{aligned} W(t,s) = \frac{1}{\sqrt{A^2-4B}} \Big\{ & \exp\Big[\frac{-A+\sqrt{A^2-4B}}{2}(t-s)\Big] \\ & - \exp\Big[\frac{-A-\sqrt{A^2-4B}}{2}(t-s)\Big]\Big\}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} W'_t(t,s) = \frac{1}{\sqrt{A^2-4B}} \Big\{ & \frac{-A+\sqrt{A^2-4B}}{2} \exp\Big[\frac{-A+\sqrt{A^2-4B}}{2}(t-s)\Big] \\ & - \frac{(-A-\sqrt{A^2-4B})}{2} \exp\Big[\frac{-A-\sqrt{A^2-4B}}{2}(t-s)\Big]\Big\}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} W''_{tt}(t,s) = \frac{1}{\sqrt{A^2-4B}} \Big\{ & \Big(\frac{-A+\sqrt{A^2-4B}}{2}\Big)^2 \exp\Big[\frac{-A+\sqrt{A^2-4B}}{2}(t-s)\Big] \\ & - \Big(\frac{-A-\sqrt{A^2-4B}}{2}\Big)^2 \exp\Big[\frac{-A-\sqrt{A^2-4B}}{2}(t-s)\Big]\Big\}. \end{aligned} \tag{3.14}$$

The proof of (3.1) follows from [11].

Let us find the points where the derivative $W'_t(t,s)$ as a function of t for fixed s changes its sign. We have

$$t - s = \frac{1}{\sqrt{A^2-4B}} \ln \frac{A+\sqrt{A^2-4B}}{A-\sqrt{A^2-4B}}. \tag{3.15}$$

Denoting

$$t^* = \frac{1}{\sqrt{A^2-4B}} \ln \frac{A+\sqrt{A^2-4B}}{A-\sqrt{A^2-4B}}, \tag{3.16}$$

we see from formula (3.13) that $W'_t(t,s) \geqslant 0$ for $t-s \leqslant t^*$ and $W'_t(t,s) \leqslant 0$ for $t-s > t^*$. We can compute for $t > t^*$ the integrals

$$
\begin{aligned}
&\int_0^t |W'_t(t,s)|\,\mathrm{d}s = -\int_0^{t-t^*} W'_t(t,s)\,\mathrm{d}s + \int_{t-t^*}^t W'_t(t,s)\,\mathrm{d}s \\
&= \frac{1}{\sqrt{A^2-4B}}\left\{\exp\left[\frac{-A+\sqrt{A^2-4B}}{2}t^*\right] - \exp\left[\frac{-A-\sqrt{A^2-4B}}{2}t^*\right]\right\} \\
&\quad + \frac{1}{\sqrt{A^2-4B}}\left\{-\exp\left[\frac{-A+\sqrt{A^2-4B}}{2}t\right] + \exp\left[\frac{-A-\sqrt{A^2-4B}}{2}t\right]\right\} \\
&\quad + \frac{1}{\sqrt{A^2-4B}}\left\{-1+\exp\left[\frac{-A+\sqrt{A^2-4B}}{2}t^*\right] + 1 - \exp\left[\frac{-A-\sqrt{A^2-4B}}{2}t^*\right]\right\} \\
&= \frac{1}{\sqrt{A^2-4B}}\left\{2\exp\left[\frac{-A+\sqrt{A^2-4B}}{2}t^*\right] - 2\exp\left[\frac{-A-\sqrt{A^2-4B}}{2}t^*\right]\right\} \\
&\quad + \frac{1}{\sqrt{A^2-4B}}\left\{-\exp\left[\frac{-A+\sqrt{A^2-4B}}{2}t\right] + \exp\left[\frac{-A-\sqrt{A^2-4B}}{2}t\right]\right\} \\
&= \frac{2}{\sqrt{A^2-4B}}\left\{\left[\frac{A+\sqrt{A^2-4B}}{A-\sqrt{A^2-4B}}\right]^{(-A+\sqrt{A^2-4B})/(2\sqrt{A^2-4B})}\right. \\
&\quad \left. - \left[\frac{A+\sqrt{A^2-4B}}{A-\sqrt{A^2-4B}}\right]^{(-A-\sqrt{A^2-4B})/(2\sqrt{A^2-4B})}\right\} \\
&\quad - \frac{1}{\sqrt{A^2-4B}}\left\{\exp\left[\frac{-A+\sqrt{A^2-4B}}{2}t\right] - \exp\left[\frac{-A-\sqrt{A^2-4B}}{2}t\right]\right\} \\
&= \frac{2}{\sqrt{A^2-4B}}\left\{\left[\frac{A+\sqrt{A^2-4B}}{A-\sqrt{A^2-4B}}\right]^{(-A+\sqrt{A^2-4B})/(2\sqrt{A^2-4B})} \left[1 - \left(\frac{A-\sqrt{A^2-4B}}{A+\sqrt{A^2-4B}}\right)\right]\right\} \\
&\quad - \frac{1}{\sqrt{A^2-4B}}\left\{\exp\left[\frac{-A+\sqrt{A^2-4B}}{2}t\right] - \exp\left[\frac{-A-\sqrt{A^2-4B}}{2}t\right]\right\} \\
&= \frac{4}{A+\sqrt{A^2-4B}}\left\{\left[\frac{A+\sqrt{A^2-4B}}{A-\sqrt{A^2-4B}}\right]^{(-A+\sqrt{A^2-4B})/(2\sqrt{A^2-4B})}\right\} \\
&\quad + \frac{1}{\sqrt{A^2-4B}}\left\{\exp\left[\frac{-A+\sqrt{A^2-4B}}{2}t\right] - \exp\left[\frac{-A-\sqrt{A^2-4B}}{2}t\right]\right\}.
\end{aligned}
$$

After the passage to the limit, we get the equality

$$
\begin{aligned}
(3.17)\qquad &\lim_{t\to\infty}\sup_{t\geqslant 0}\int_0^t |W'_t(t,s)|\,\mathrm{d}s \\
&= \frac{4}{A+\sqrt{A^2-4B}}\left[\frac{A-\sqrt{A^2-4B}}{A+\sqrt{A^2-4B}}\right]^{(A-\sqrt{A^2-4B})/(2\sqrt{A^2-4B})}.
\end{aligned}
$$

Let us find the points where the derivative $W''_{tt}(t,s)$ as a function of t for fixed s changes its sign. We have

$$t-s=\frac{2}{\sqrt{A^2-4B}}\ln\frac{A+\sqrt{A^2-4B}}{A-\sqrt{A^2-4B}}. \tag{3.18}$$

Denoting

$$t^{**}=\frac{2}{\sqrt{A^2-4B}}\ln\frac{A+\sqrt{A^2-4B}}{A-\sqrt{A^2-4B}}, \tag{3.19}$$

we see from formula (3.16) that $W''_{tt}(t,s)\leqslant 0$ for $t-s\leqslant t^*$ and $W''_{tt}(t,s)\geqslant 0$ for $t-s>t^{**}$. Using the roots (3.11) of the characteristic equation (3.10) we can compute for $t>t^{**}$ the integrals

$$\begin{aligned}\int_0^t |W''_{tt}(t,s)|\,\mathrm{d}s &= -\int_0^{t-t^{**}} W''_{tt}(t,s)\,\mathrm{d}s+\int_{t-t^{**}}^t W''_{tt}(t,s)\,\mathrm{d}s\\ &= -\frac{1}{k_2-k_1}\int_0^{t-t^{**}}[k_1^2\mathrm{e}^{k_1(t-s)}-k_2^2\mathrm{e}^{k_2(t-s)}]\,\mathrm{d}s\\ &\quad+\frac{1}{k_2-k_1}\int_{t-t^{**}}^t[k_1^2\mathrm{e}^{k_1(t-s)}-k_2^2\mathrm{e}^{k_2(t-s)}]\,\mathrm{d}s.\end{aligned}$$

Denote

$$I_1=-\frac{1}{k_2-k_1}\int_0^{t-t^{**}}[k_1^2\mathrm{e}^{k_1(t-s)}-k_2^2\mathrm{e}^{k_2(t-s)}]\,\mathrm{d}s, \tag{3.20}$$

$$I_2=\frac{1}{k_2-k_1}\int_{t-t^{**}}^t[k_1^2\mathrm{e}^{k_1(t-s)}-k_2^2\mathrm{e}^{k_2(t-s)}]\,\mathrm{d}s. \tag{3.21}$$

We obtain

$$\begin{aligned}I_1 &= -\frac{1}{k_2-k_1}[k_1\mathrm{e}^{k_1t}-k_2\mathrm{e}^{k_2t}-k_1\mathrm{e}^{k_1t^{**}}+k_2\mathrm{e}^{k_2t^{**}}]\\ &= -\frac{1}{\sqrt{A^2-4B}}\Bigg[\frac{A+\sqrt{A^2-4B}}{2}\exp\Bigg(\frac{-A-\sqrt{A^2-4B}}{\sqrt{A^2-4B}}\ln\frac{A+\sqrt{A^2-4B}}{A-\sqrt{A^2-4B}}\Bigg)\Bigg]\\ &\quad-\frac{1}{\sqrt{A^2-4B}}\Bigg[\frac{-A+\sqrt{A^2-4B}}{2}\exp\Bigg(\frac{-A+\sqrt{A^2-4B}}{\sqrt{A^2-4B}}\ln\frac{A+\sqrt{A^2-4B}}{A-\sqrt{A^2-4B}}\Bigg)\Bigg]\\ &\quad-\frac{k_1}{k_2-k_1}\mathrm{e}^{k_1t}+\frac{k_2}{k_2-k_1}\mathrm{e}^{k_2t}\\ &= -\frac{1}{\sqrt{A^2-4B}}\Bigg[\frac{A+\sqrt{A^2-4B}}{2}\Bigg(\frac{A+\sqrt{A^2-4B}}{A-\sqrt{A^2-4B}}\Bigg)^{(-A-\sqrt{A^2-4B})/\sqrt{A^2-4B}}\Bigg]\\ &\quad-\frac{1}{\sqrt{A^2-4B}}\Bigg[\frac{-A+\sqrt{A^2-4B}}{2}\Bigg(\frac{A+\sqrt{A^2-4B}}{A-\sqrt{A^2-4B}}\Bigg)^{(-A+\sqrt{A^2-4B})/\sqrt{A^2-4B}}\Bigg]\\ &\quad-\frac{k_1}{k_2-k_1}\mathrm{e}^{k_1t}+\frac{k_2}{k_2-k_1}\mathrm{e}^{k_2t};\end{aligned}$$

$$
\begin{aligned}
I_2 &= \frac{1}{k_2-k_1}[-k_1+k_1\mathrm{e}^{k_1t^{**}}+k_2-k_2\mathrm{e}^{k_2t^{**}}]\\
&= 1+\frac{-A-\sqrt{A^2-4B}}{2\sqrt{A^2-4B}}\left[\frac{A+\sqrt{A^2-4B}}{A-\sqrt{A^2-4B}}\right]^{(-A-\sqrt{A^2-4B})/\sqrt{A^2-4B}}\\
&\quad+\frac{A-\sqrt{A^2-4B}}{2\sqrt{A^2-4B}}\left[\frac{A+\sqrt{A^2-4B}}{A-\sqrt{A^2-4B}}\right]^{(-A+\sqrt{A^2-4B})/\sqrt{A^2-4B}}\\
&= 1+\left[\frac{A-\sqrt{A^2-4B}}{A+\sqrt{A^2-4B}}\right]^{(A-\sqrt{A^2-4B})/\sqrt{A^2-4B}}\frac{1}{A+\sqrt{A^2-4B}};\\
I_1+I_2 &= 1-\frac{1}{2\sqrt{A^2-4B}}(A+\sqrt{A^2-4B})\exp\Big(\frac{-A-\sqrt{A^2-4B}}{2}t\Big)\\
&\quad-\frac{1}{2\sqrt{A^2-4B}}(A-\sqrt{A^2-4B})\exp\Big(\frac{-A+\sqrt{A^2-4B}}{2}t\Big)\\
&\quad+\frac{2}{A+\sqrt{A^2-4B}}\left[\frac{A-\sqrt{A^2-4B}}{A+\sqrt{A^2-4B}}\right]^{(A-\sqrt{A^2-4B})/\sqrt{A^2-4B}}.
\end{aligned}
$$

After the passage to the limit, we get the equality

$$
\begin{aligned}
\lim_{t\to\infty}\sup_{t\geqslant 0}&\int_0^t |W''_{tt}(t,s)|\,\mathrm{d}s\\
&= 1+\frac{2}{A+\sqrt{A^2-4B}}\left[\frac{A-\sqrt{A^2-4B}}{A+\sqrt{A^2-4B}}\right]^{(A-\sqrt{A^2-4B})/\sqrt{A^2-4B}}.
\end{aligned}
\tag{3.22}
$$

Lemma 3.1 has been proved. $\square$

P r o o f of Lemma 3.3. Solving characteristic equation (3.10), we get

$$
k_1=\frac{-A}{2}+\mathrm{i}\frac{\sqrt{4B-A^2}}{2},\quad k_2=\frac{-A}{2}-\mathrm{i}\frac{\sqrt{4B-A^2}}{2},
\tag{3.23}
$$

and substituting initial conditions (2.3), we get

$$
W(t,s)=m\exp[-\alpha(t-s)]\sin\beta(t-s),
\tag{3.24}
$$

where

$$
\alpha=\frac{A}{2},\quad \beta=\frac{\sqrt{4B-A^2}}{2},\quad m=\frac{2}{\sqrt{4B-A^2}};
\tag{3.25}
$$

$$
W'_t(t,s)=m\{-\alpha\exp[-\alpha(t-s)]\sin\beta(t-s)+\beta\exp[-\alpha(t-s)]\cos\beta(t-s)\},
\tag{3.26}
$$

which can be rewritten in the form

$$
W'_t(t,s)=-m\sqrt{\alpha^2+\beta^2}\exp[-\alpha(t-s)]\sin(\beta(t-s)-\varphi_0),
\tag{3.27}
$$

where $\varphi_0 = \operatorname{arctg}(\sqrt{4B - A^2}/A)$.

$$
\begin{aligned}
&(3.28) \qquad W''_{tt}(t,s) = m\sqrt{\alpha^2+\beta^2}\exp[-\alpha(t-s)]\sin(\beta(t-s)-2\varphi_0),\\
&\qquad\qquad \int_0^t |W(t,s)|\,\mathrm{d}s = \int_0^t m\exp[-\alpha(t-s)]|\sin(\beta(t-s))|\,\mathrm{d}s.
\end{aligned}
$$

Denoting $t - s = \zeta$, $\mathrm{d}s = -\,\mathrm{d}\zeta$, $n = \lfloor \beta t/\pi \rfloor$, where $\lfloor q \rfloor$ means the floor integer part of q, we can write

$$
\begin{aligned}
&\int_0^t m\exp[-\alpha\zeta]|\sin(\beta\zeta)|\,\mathrm{d}\zeta\\
&\quad = \sum_{k=0}^{n-1}\int_{k\pi/\beta}^{(k+1)\pi/\beta} m(-1)^k\exp[-\alpha\zeta]\sin(\beta\zeta)\,\mathrm{d}\zeta + \int_{n\pi/\beta}^{t} m(-1)^n\exp[-\alpha\zeta]\sin(\beta\zeta)\,\mathrm{d}\zeta\\
&\quad = \sum_{k=0}^{n-1} m(-1)^k\exp[-\alpha\zeta]\Big[-\frac{\beta\cos\beta\zeta+\alpha\sin\beta\zeta}{\alpha^2+\beta^2}\Big]\Big|_{k\pi/\beta}^{(k+1)\pi/\beta}\\
&\qquad + m(-1)^n\Big\{\exp[-\alpha\zeta]\Big[-\frac{\beta\cos\beta\zeta+\alpha\sin\beta\zeta}{\alpha^2+\beta^2}\Big]\Big\}\Big|_{n\pi/\beta}^{t}\\
&\quad = \frac{m\beta}{\alpha^2+\beta^2}\sum_{k=0}^{n-1}(-1)^k\Big[(-1)^k\exp\Big(-\frac{k\alpha\pi}{\beta}\Big)-(-1)^{k+1}\exp\Big(-\frac{k\alpha\pi}{\beta}-\frac{\pi\alpha}{\beta}\Big)\Big]\\
&\qquad + \frac{m}{\alpha^2+\beta^2}m(-1)^{n+1}\{\exp[-\alpha\zeta][\beta\cos\beta\zeta+\alpha\sin\beta\zeta]\}|_{n\pi/\beta}^{t}\\
&\quad = \frac{m\beta}{\alpha^2+\beta^2}\sum_{k=0}^{n-1}\exp\Big(-\frac{k\alpha\pi}{\beta}\Big)\Big(1+\exp\Big(-\frac{\pi\alpha}{\beta}\Big)\Big)\\
&\qquad + \frac{m\beta}{\alpha^2+\beta^2}(-1)^n\Big[\exp\Big(-\frac{n\alpha\pi}{\beta}\Big)\cos n\pi - \exp(-\alpha t)(\beta\cos\beta t+\alpha\sin\beta t)\Big]\\
&\quad = \frac{m\beta}{\alpha^2+\beta^2}\frac{\big[1-\exp(-n\alpha\pi/\beta)\big]\big(1+\exp(-\pi\alpha/\beta)\big)}{1-\exp(-\pi\alpha/\beta)}\\
&\qquad + \frac{m}{\alpha^2+\beta^2}\Big[\beta\exp\Big(-\frac{n\alpha\pi}{\beta}\Big)-\exp(-\alpha t)(\beta\cos\beta t+\alpha\sin\beta t)\Big].
\end{aligned}
$$

After the passage to the limit, we get the equality

$$
\lim_{t\to\infty}\sup_{t\geqslant 0}\int_0^t |W(t,s)|\,\mathrm{d}s = \frac{m\beta}{\alpha^2+\beta^2}\frac{1+\exp(-\pi\alpha/\beta)}{1-\exp(-\pi\alpha/\beta)},
$$

and substituting α, β and m from formulas (3.28) we obtain

$$
\lim_{t\to\infty}\sup_{t\geqslant 0}\int_0^t |W(t,s)|\,\mathrm{d}s = \frac{1}{B}\frac{1+\exp(-\pi A/\sqrt{4B-A^2})}{1-\exp(-\pi A/\sqrt{4B-A^2})}.
$$

Now we obtain

$$
\begin{aligned}
I(t) = {} & \int_0^t |W_t'(t,s)|\,\mathrm{d}s = m\sqrt{\alpha^2+\beta^2}\int_0^t \exp[-\alpha(t-s)]|\sin(\beta(t-s)-\varphi_0)|\,\mathrm{d}s \\
& = m\sqrt{\alpha^2+\beta^2}\int_0^t \exp(-\alpha\tau)|\sin(\beta\tau-\varphi_0)|\,\mathrm{d}\tau \\
& = m\sqrt{\alpha^2+\beta^2}\Big[-\int_0^{\varphi_0/\beta} \exp(-\alpha\tau)\sin(\beta\tau-\varphi_0)\,\mathrm{d}\tau \\
& \quad + \sum_{k=0}^{n-1}(-1)^k\int_{k\pi/\beta+\varphi_0/\beta}^{(k+1)\pi/\beta+\varphi_0/\beta} \exp(-\alpha\tau)\sin(\beta\tau-\varphi_0)\,\mathrm{d}\tau \\
& \quad + (-1)^n\int_{n\pi/\beta+\varphi_0/\beta}^{t} \exp(-\alpha\tau)\sin(\beta\tau-\varphi_0)\,\mathrm{d}\tau\Big],
\end{aligned}
$$

where $n = \lfloor (t\beta-\varphi_0)/\pi \rfloor$. Continuing this computing, we obtain

$$
\begin{aligned}
I(t) = {} & -\frac{m}{\sqrt{\alpha^2+\beta^2}}\exp(-\alpha\tau)[\alpha\sin(\beta\tau-\varphi_0)+\beta\cos(\beta\tau-\varphi_0)]|_0^{\varphi_0/\beta} \\
& -\frac{m}{\sqrt{\alpha^2+\beta^2}}\sum_{k=0}^{n-1}(-1)^k\exp(-\alpha\tau)[\alpha\sin(\beta\tau-\varphi_0)+\beta\cos(\beta\tau-\varphi_0)]|_{k\pi/\beta+\varphi_0/\beta}^{(k+1)\pi/\beta+\varphi_0/\beta} \\
& -\frac{m}{\sqrt{\alpha^2+\beta^2}}(-1)^n\exp(-\alpha\tau)[\alpha\sin(\beta\tau-\varphi_0)+\beta\cos(\beta\tau-\varphi_0)]|_{n\pi/\beta+\varphi_0/\beta}^{t} \\
= {} & -\frac{m}{\sqrt{\alpha^2+\beta^2}}\beta\exp\Big(-\frac{\alpha}{\beta}\varphi_0\Big)[\alpha\sin(-\varphi_0)+\beta\cos(-\varphi_0)] \\
& -\frac{m}{\sqrt{\alpha^2+\beta^2}}\sum_{k=0}^{n-1}(-1)^k\Big[\exp\Big(-\alpha\Big(\frac{\pi}{\beta}(k+1)+\frac{\varphi_0}{\beta}\Big)\Big)(-1)^{k+1}\beta \\
& \qquad - \exp\Big(-\alpha\Big(\frac{\pi}{\beta}k+\frac{\varphi_0}{\beta}\Big)\Big)(-1)^k\beta\Big] \\
& -\frac{m}{\sqrt{\alpha^2+\beta^2}}(-1)^n\Big[\exp(-\alpha t)\big(\alpha\sin(\beta t-\varphi_0)+\beta\cos(\beta t-\varphi_0)\big) \\
& \qquad - \exp\Big(-\alpha\Big(\frac{\pi}{\beta}n+\frac{\varphi_0}{\beta}\Big)\Big)\beta(-1)^n\Big] \\
= {} & \frac{m}{\sqrt{\alpha^2+\beta^2}}\Big[\beta\cos\varphi_0-\alpha\sin\varphi_0-\exp\Big(-\frac{\alpha}{\beta}\varphi_0\Big)\beta\Big] \\
& + \frac{m}{\sqrt{\alpha^2+\beta^2}}\sum_{k=0}^{n-1}\Big[\beta\exp\Big(-\alpha\frac{\pi}{\beta}(k+1)-\alpha\frac{\varphi_0}{\beta}\Big)+\exp\Big(-\alpha\frac{\pi}{\beta}k-\alpha\frac{\varphi_0}{\beta}\Big)\Big] \\
& + \frac{m}{\sqrt{\alpha^2+\beta^2}}\Big[(-1)^{n+1}(\alpha\sin(\beta t-\varphi_0)+\beta\cos(\beta t-\varphi_0))\exp(-\alpha t) \\
& \qquad + \beta\exp\Big(-\alpha\Big(\frac{\pi}{\beta}n+\frac{\varphi_0}{\beta}\Big)\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{m}{\sqrt{\alpha^2+\beta^2}}\Big[-\beta\exp\Big(-\frac{\alpha}{\beta}\varphi_0\Big)+\beta\exp\Big(-\frac{\alpha}{\beta}\varphi_0\Big)\sum_{k=0}^{n-1}\Big(1+\exp\Big(-\frac{\alpha}{\beta}\pi\Big)\Big)\exp\Big(-\frac{\alpha}{\beta}\pi k\Big)\Big]\\
&\quad+\frac{m}{\sqrt{\alpha^2+\beta^2}}\Big[(-1)^{n+1}(\alpha\sin(\beta t-\varphi_0)+\beta\cos(\beta t-\varphi_0))\exp(-\alpha t))\\
&\qquad\qquad+\beta\exp\Big(-\alpha\Big(\frac{\pi}{\beta}n+\frac{\varphi_0}{\beta}\Big)\Big)\Big]\\
&= \frac{m}{\sqrt{\alpha^2+\beta^2}}\Big[\beta\exp\Big(-\frac{\alpha}{\beta}\varphi_0\Big)\Big(1+\exp\Big(-\frac{\alpha}{\beta}\pi\Big)\Big)\frac{1-\exp\big(-\frac{\alpha}{\beta}\pi n\big)}{1-\exp\big(-\frac{\alpha}{\beta}\pi\big)}-\beta\exp\Big(-\frac{\alpha}{\beta}\varphi_0\Big)\Big]\\
&\quad+\frac{m}{\sqrt{\alpha^2+\beta^2}}\Big[(-1)^{n+1}(\alpha\sin(\beta t-\varphi_0)+\beta\cos(\beta t-\varphi_0))\exp(-\alpha t))\\
&\qquad\qquad+\beta\exp\Big(-\alpha\Big(\frac{\pi}{\beta}n+\frac{\varphi_0}{\beta}\Big)\Big)\Big].
\end{aligned}
$$

After the passage to the limit, we obtain

$$
\begin{aligned}
\lim_{t\to\infty} I(t) &= \frac{m}{\sqrt{\alpha^2+\beta^2}}\Big[\beta\exp\Big(-\frac{\alpha}{\beta}\varphi_0\Big)\frac{1+\exp\big(-\frac{\alpha}{\beta}\pi\big)}{1-\exp\big(-\frac{\alpha}{\beta}\pi\big)}-\beta\exp\Big(-\frac{\alpha}{\beta}\varphi_0\Big)\Big]\\
&= \frac{m\beta}{\sqrt{\alpha^2+\beta^2}}\frac{2\exp\big(-\frac{\alpha}{\beta}\pi\big)\exp\big(-\frac{\alpha}{\beta}\varphi_0\big)}{1-\exp\big(-\frac{\alpha}{\beta}\pi\big)}\\
&= \frac{2}{\sqrt{B}}\frac{\exp\big[-\frac{A}{\sqrt{4B-A^2}}(\pi+\varphi_0)\big]}{1-\exp\big(-\frac{A}{\sqrt{4B-A^2}}\pi\big)},
\end{aligned}
$$

where $\varphi_0 = \operatorname{arctg}(\sqrt{4B-A^2}/A)$.

Now we obtain

$$
\begin{aligned}
J(t) &= \int_0^t |W''_{tt}(t,s)|\,\mathrm{d}s = m(\alpha^2+\beta^2)\int_0^t \exp[-\alpha(t-s)]|\sin(\beta(t-s)-2\varphi_0)|\,\mathrm{d}s\\
&= m(\alpha^2+\beta^2)\int_0^t \exp(-\alpha\tau)|\sin(\beta\tau-2\varphi_0)|\,\mathrm{d}\tau,
\end{aligned}
$$

and denoting $\varphi_1 = 2\varphi_0$, we obtain

$$
J(t) = m(\alpha^2+\beta^2)\int_0^t \exp(-\alpha\tau)|\sin(\beta\tau-\varphi_1)|\,\mathrm{d}\tau
$$

and using the methods developed in computing $I(t)$ in the previous part, we obtain

$$
\begin{aligned}
J(t) &= m\Big[\beta\exp\Big(-\frac{\alpha}{\beta}\varphi_1\Big)\frac{1+\exp\big(-\frac{\alpha}{\beta}\pi\big)}{1-\exp\big(-\frac{\alpha}{\beta}\pi\big)}-\beta\exp\Big(-\frac{\alpha}{\beta}\varphi_1\Big)\Big]\\
&\quad+m\Big[(-1)^{n+1}(\alpha\sin(\beta t-\varphi_1)+\beta\cos(\beta t-\varphi_1))\exp(-\alpha t))\\
&\qquad\qquad+\beta\exp\Big(-\alpha\Big(\frac{\pi}{\beta}n+\frac{\varphi_1}{\beta}\Big)\Big)\Big].
\end{aligned}
$$

After the passage to the limit, we obtain

$$\begin{aligned}\lim_{t\to\infty} J(t) &= m\Big[\beta\exp\Big(-\frac{\alpha}{\beta}\varphi_1\Big)\frac{1+\exp\big(-\frac{\alpha}{\beta}\pi\big)}{1-\exp\big(-\frac{\alpha}{\beta}\pi\big)} - \beta\exp\Big(-\frac{\alpha}{\beta}\varphi_1\Big)\Big]\\ &= m\beta\exp\Big(-\frac{\alpha}{\beta}\varphi_1\Big)\Big[\frac{1+\exp\big(-\frac{\alpha}{\beta}\pi\big)}{1-\exp\big(-\frac{\alpha}{\beta}\pi\big)} - 1\Big]\\ &= 2\exp\Big(-\frac{2A}{\sqrt{4B-A^2}}\varphi_0\Big)\frac{\exp\big(-\frac{A}{\sqrt{4B-A^2}}\pi\big)}{1-\exp\big(-\frac{A}{\sqrt{4B-A^2}}\pi\big)}.\end{aligned}$$

Thus we obtain

$$\lim_{t\to\infty} J(t) = \frac{2\exp\big(-\frac{A}{\sqrt{4B-A^2}}(2\varphi_0+\pi)\big)}{1-\exp\big(\frac{A}{\sqrt{4B-A^2}}\pi\big)}.$$

□

P r o o f of Lemma 3.2. Solving characteristic equation (3.10), we get

$$k_1 = k_2 = \frac{-A}{2},$$

and substituting the initial conditions (2.3), we get

$$\begin{aligned}W(t,s) &= (t-s)\exp\Big[-\frac{A}{2}(t-s)\Big],\\ W_t'(t,s) &= \Big(1-\frac{A}{2}(t-s)\Big)\exp\Big[-\frac{A}{2}(t-s)\Big],\\ W_{tt}''(t,s) &= \frac{A}{4}(A(t-s)-4)\exp\Big[-\frac{A}{2}(t-s)\Big].\end{aligned}$$

Let us compute the integrals $\int_0^t |W(t,s)|\,\mathrm{d}s$, $\int_0^t |W_t'(t,s)|\,\mathrm{d}s$ and $\int_0^t |W_{tt}''(t,s)|\,\mathrm{d}s$:

$$\int_0^t |W(t,s)|\,\mathrm{d}s = \int_0^t (t-s)\exp\Big[\frac{-A}{2}(t-s)\Big]\,\mathrm{d}s = \frac{4}{A^2} - \Big(\frac{2t}{A}+\frac{4}{A^2}\Big)\exp\Big(-\frac{A}{2}t\Big),$$

and after the passage to the limit we get

$$\lim_{t\to\infty}\int_0^t |W(t,s)|\,\mathrm{d}s = \frac{4}{A^2} = \frac{1}{B}.$$

Let us compute the integral

$$\int_0^t |W_t'(t,s)|\,\mathrm{d}s = \int_0^t \Big|\Big(1-\frac{A}{2}(t-s)\Big)\exp\Big[-\frac{A}{2}(t-s)\Big]\Big|\,\mathrm{d}s$$

for sufficiently large t.

We can find the point $t^* = 2/A$, such that the derivative $W'_t(t,s)$ changes its sign at the point $t - s = t^*$. We have

$$\begin{aligned}\int_0^t |W'_t(t,s)|\,\mathrm{d}s &= -\int_0^{t-t^*} \Big(1 - \frac{A}{2}(t-s)\Big)\exp\Big[-\frac{A}{2}(t-s)\Big]\,\mathrm{d}s \\ &\quad + \int_{t-t^*}^t \Big(1 - \frac{A}{2}(t-s)\Big)\exp\Big[-\frac{A}{2}(t-s)\Big]\,\mathrm{d}s \\ &= \frac{4}{A\mathrm{e}} - t\exp\Big(-\frac{A}{2}t\Big).\end{aligned}$$

After the passage to the limit we get the inequality

$$\lim_{t\to\infty}\int_0^t |W'_t(t,s)|\,\mathrm{d}s = \frac{4}{A\mathrm{e}}.$$

Let us compute

$$\int_0^t |W''_{tt}(t,s)|\,\mathrm{d}s = \int_0^t \Big|\frac{A}{4}(A(t-s)-4)\exp\Big[-\frac{A}{2}(t-s)\Big]\Big|\,\mathrm{d}s.$$

We can find the point $t^{**} = 4/A$ such that the derivative $W''_{tt}(t,s)$ changes its sign at the point $t - s = t^{**}$. Then

$$\begin{aligned}\int_0^t |W''_{tt}(t,s)|\,\mathrm{d}s &= \int_0^t \Big|\frac{A}{4}(A(t-s)-4)\exp\Big[-\frac{A}{2}(t-s)\Big]\Big|\,\mathrm{d}s, \\ \int_0^t |W''_{tt}(t,s)|\,\mathrm{d}s &= \int_0^{t-t^{**}} \frac{A}{4}[A(t-s)-4]\exp\Big[-\frac{A}{2}(t-s)\Big]\,\mathrm{d}s \\ &\quad - \int_{t-t^{**}}^t \frac{A}{4}[A(t-s)-4]\exp\Big[-\frac{A}{2}(t-s)\Big]\,\mathrm{d}s \\ &= 1 - \frac{1}{\mathrm{e}^2} + \frac{A}{4}\Big(1 + \frac{1}{\mathrm{e}^2}\Big) + \Big(2 - \frac{A}{2}t\Big)\exp\Big(-\frac{A}{2}t\Big).\end{aligned}$$

After the passage to the limit, we obtain

$$\lim_{t\to\infty}\int_0^t |W''_{tt}(t,s)|\,\mathrm{d}s = 1 - \frac{1}{\mathrm{e}^2} + \frac{A}{4}\Big(1 + \frac{1}{\mathrm{e}^2}\Big).$$

$\square$

4. Proofs of main theorems

Consider the equation

$$x''(t)+\sum_{i=1}^{m}a_i(t)x'(t-\theta_i(t))+\sum_{i=1}^{m}b_i(t)x(t-\tau_i(t))=f(t),\quad t\in[0,\infty), \tag{4.1}$$

$$x(\xi)=x'(\xi)=0,\quad \text{for } \xi<0. \tag{4.2}$$

It is known [1] that in the analysis of stability, we can consider only the zero initial conditions

$$x(0)=0,\quad x'(0)=0. \tag{4.3}$$

Let us write equation (1.1) in the form

$$\begin{aligned}x''(t)+Ax'(t)-Ax'(t)+\sum_{i=1}^{m}a_i(t)x'(t-\theta_i(t))+Bx(t)-Bx(t)&\\+\sum_{i=1}^{m}b_i(t)x(t-\tau_i(t))=f(t),&\end{aligned}$$

and

$$\begin{aligned}x''(t)+Ax'(t)+Bx(t)-\sum_{i=1}^{m}A_i\int_{t-\theta_i(t)}^{t}x''(s)\,\mathrm{d}s-\sum_{i=1}^{m}\Delta a_i(t)x'(t-\theta_i(t))&\\-\sum_{i=1}^{m}B_i\int_{t-\tau_i(t)}^{t}x'(s)\,\mathrm{d}s-\sum_{i=1}^{m}\Delta b_i(t)x(t-\tau_i(t))=f(t),\quad t\in[0,\infty).&\end{aligned}$$

Let us make the so called W-transform [1], substituting

$$x(t)=\int_0^t W(t,s)z(s)\,\mathrm{d}s, \tag{4.4}$$

where $z\in L_\infty$ (L_∞ is the space of essentially bounded functions $z\colon[0,\infty)\to(-\infty,\infty)$), into the last equation. It is clear that

$$x'(t)=\int_0^t W_t'(t,s)z(s)\,\mathrm{d}s,\ x''(t)=\int_0^t W_{tt}''(t,s)z(s)\,\mathrm{d}s+z(t). \tag{4.5}$$

We get the equation

$$z(t)=(Kz)(t)+f(t), \tag{4.6}$$

where the operator $K\colon L_\infty \to L_\infty$ is defined by the equality

$$(4.7)\qquad (Kz)(t) = \begin{cases} \displaystyle\sum_{i=1}^{m} A_i\sigma(t-\theta_i(t)) \int_{t-\theta_i(t)}^{t} \Big\{ \int_0^s W''_{ss}(s,\xi)z(\xi)\,\mathrm{d}\xi + z(s) \Big\}\,\mathrm{d}s \\ \displaystyle + \sum_{i=1}^{m} \Delta a_i(t)\sigma(t-\theta_i(t)) \int_0^{t-\theta_i(t)} W'_t(t-\theta_i(t),s)z(s)\,\mathrm{d}s(t) \\ \displaystyle + \sum_{i=1}^{m} B_i\sigma(t-\tau_i(t)) \int_{t-\tau_i(t)}^{t} \int_0^s W'_s(s,\xi)z(\xi)\,\mathrm{d}\xi\,\mathrm{d}s \\ \displaystyle + \sum_{i=1}^{m} \Delta b_i(t)\sigma(t-\tau_i(t)) \int_0^{t-\tau_i(t)} W(t,s)z(s)\,\mathrm{d}s, \end{cases}$$

and

$$(4.8)\qquad \sigma(t) = \begin{cases} 1, & t \geqslant 0, \\ 0, & t < 0. \end{cases}$$

Let us denote $|K| = \lim\limits_{t\to\infty} \sup_{t\geqslant 0}(K1)(t)$. The inequality (2.9) implies that $|K|$ is less than one. There exists a bounded operator $(I-K)^{-1}\colon L_\infty \to L_\infty$. For every bounded right hand side f, the solution z of equation (4.6) is bounded.

In the case $A > 0$ and $B > 0$, the Cauchy function $W(t,s)$ and its derivative $W'_t(t,s)$ satisfy the exponential estimates. The boundedness of the solution x of equation (1.1) and its derivative x' follow now from the boundedness of z. According to Bohl-Perron theorem [1], the Cauchy function $C(t,s)$ of equation (1.1) and the solutions x_1 and x_2 satisfy the exponential estimate.

To prove Theorems 2.2–2.4 we set the values of $|W|$, $|W'_t|$ and $|W''_{tt}|$, obtained in Lemmas 3.1–3.3, into (2.9).

5. Open problem

In the previous works on the exponential stability of equation (1.1), it was assumed that $a_i > 0$ for all $i = 1,\dots,m$ (see [2], [4], [3], [11]) and $a_i = 0$ for all $i = 1,\dots,m$ in the paper [9]. Only for a special case $m = 2$, $a_1 > 0, \tau_1 = 0$, $a_2 < 0$ and $\tau_2 = \text{const}$, stability of equation (1.1) is studied in the papers [7], [8]. It should be stressed that assertions on stability of our paper could be true for equation (1.1) also in the case when several among the coefficients a_i are negative, but the sum $A = \sum\limits_{i=1}^{m} A_i$ of all average values A_i of the coefficients a_i is positive. Is it possible to obtain the exponential stability of equation (1.1) in the case of negativity of all coefficients $a_i < 0$ for all $i = 1,\dots,m$, and consequently $A < 0$? Such results are considered impossible, but, in our opinion, assertions of this sort will be proved in a future.

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Authors' addresses: Ravi P. Agarwal, Texas A&M University, 401 Joe Routt Blvd, Kingsville, TX 78363-8202, Texas, USA, e-mail: Agarwal@tamuk.edu; Alexander Domoshnitsky, Abraham Maghakyan, Department of Mathematics and Computer Science, Ariel University, Qiryat University, 40700 Ariel, Israel, e-mail: adom@ariel.ac.il, mabraham@ariel.ac.il.