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ON GENERALIZED CS-MODULES

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Abstract. An $\mathscr{S}$-closed submodule of a module M is a submodule N for which M/N is nonsingular. A module M is called a generalized CS-module (or briefly, GCS-module) if any $\mathscr{S}$-closed submodule N of M is a direct summand of M. Any homomorphic image of a GCS-module is also a GCS-module. Any direct sum of a singular (uniform) module and a semi-simple module is a GCS-module. All nonsingular right R-modules are projective if and only if all right R-modules are GCS-modules.

Keywords: direct summand; $\mathscr{S}$-closed submodule; GCS-module; singular submodule

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1. Introduction and preliminaries

In recent years theory of CS-modules and rings has come to play an important role in the theory of rings and modules. A module M is called a CS-*module* if every submodule is essential in a direct summand of M, or equivalently, every closed submodule is a direct summand of M. Although this generalization of injectivity is extremely useful, it does not satisfy some important properties. For example, direct sums of CS-modules need not be a CS-module; also, homomorphic images of CS-modules need not be a CS-module; also, submodules of CS-modules need not be CS-modules. Much work has been done to find necessary and sufficient conditions to ensure that the extending property is preserved under various extensions.

In this paper, we change the condition of CS-modules: "every closed submodule is a direct summand", to the condition that every $\mathscr{S}$-closed submodule is a direct summand. Thus we generalize CS-modules to GCS-modules.

In Section 2, we give the definition of GCS-modules and show that a direct summand of a GCS-module and any image of a GCS-module are all GCS-modules.

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In Section 3, we discuss when a direct sum of GCS-modules is a GCS-module. A direct sum of a singular module and a semi-simple module is a GCS-module.

In Section 4, we investigate rings for which all modules are GCS-modules. All nonsingular right R-modules are projective if and only if all right R-modules are GCS-modules. If R is right nonsingular, then all R modules are GCS-modules if and only if R is (left and right) hereditary Artinian ring.

Throughout this paper, unless otherwise stated, all rings are associative rings with identity and all modules are unitary right R-modules.

A submodule N of M is called an *essential submodule*, denoted by $N \leqslant_e M$, if for any nonzero submodule L of M, $L \cap N \neq 0$. A *closed submodule* N of M, denoted by $N \leqslant_c M$, is a submodule which has no proper essential extension in M. If $L \leqslant_c N$ and $N \leqslant_c M$, then $L \leqslant_c M$ (see [4]).

In [4], a module M is called *singular* if $Z(M) = M$, where $Z(M) = \{m \in M;\ mI = 0$ or some essential right ideal I of $R\}$ and called *nonsingular* if $Z(M) = 0$. A ring R is called right nonsingular if R_R is nonsingular, i.e. $Z_r(R) = 0$. It is well-known that if N is essential in M then M/N is singular. The converse holds if M is nonsingular.

Let M be an R-module, we use $\operatorname{Rad}(M)$ to denote the Jacobson radical of M and $r(m) = \{r \in R;\ mr = 0\}$ the right annihilator of $m \in M$.

2. Generalized CS-modules

Recall from [4] that an $\mathscr{S}$-closed submodule of a module M is a submodule N for which M/N is nonsingular, and we use $L^*(M)$ to denote the collection of all $\mathscr{S}$-closed submodules of M. Note that $L^*(M)$ is closed under arbitrary intersection: For if $\{N_\alpha\} \subseteq L^*(M)$, then $M/(\cap N_\alpha)$ can be embedded in the nonsingular module $\Pi(M/N_\alpha)$. Thus for any $N \leqslant M$, there is a smallest $\mathscr{S}$-closed submodule of M containing N, which is called the $\mathscr{S}$-closure of N in M. Any $\mathscr{S}$-closed submodule is closed but the converse is not true. For example, 0 is closed in a module M, but 0 is an $\mathscr{S}$-closed submodule of M if and only if M is nonsingular. It is easy to see that for any R-module M over a right nonsingular ring R, $L^*(M) = \{M\}$ if and only if M is singular.

Now we collect some results for $\mathscr{S}$-closed submodules as follows.

Lemma 2.1 ([4], Proposition 2.3). *Assume that $Z_r(R) = 0$. Let $N \leqslant M$ be R-modules and K the $\mathscr{S}$-closure of N in M. Then*

(1) *$K/N = Z(M/N)$;*

(2) *K is the only $\mathscr{S}$-closed submodule of M for which $N \leqslant K$ and K/N is singular;*

(3) *if M is nonsingular, then K is the only $\mathscr{S}$-closed submodule of M for which $N \leqslant_e K$.*

Lemma 2.2. *Let M be an R-module and $A \leqslant B \leqslant M$. Then*

(1) *every $\mathscr{S}$-closed submodule of M is closed in M. If M is nonsingular, then every closed submodule of M is $\mathscr{S}$-closed in M;*
(2) *if $A \in L^*(B)$ and $B \in L^*(M)$, then $A \in L^*(M)$;*
(3) *if $A_\alpha \in L^*(B_\alpha)$ for each α in some index set, then $\oplus A_\alpha \in L^*(\oplus B_\alpha)$ and $\Pi A_\alpha \in L^*(\Pi B_\alpha)$;*
(4) *if $f\colon M \to N$ and $K \in L^*(N)$, then $f^{-1}(K) \in L^*(M)$;*
(5) *if M is nonsingular, then $A \leqslant_e B$ if and only if B is contained in the $\mathscr{S}$-closure of A in M.*

Proof. See [4], Proposition 2.4, Exercises 2A.5, 2A.7, 2A.8, 2A.9. □

Now we give the definition of generalized CS-modules, or briefly GCS-modules, which generalizes CS-modules, as follows:

Definition 2.3. A module M is called a *generalized CS-module* (or briefly, GCS-module) if for any nonzero submodule N of M, the $\mathscr{S}$-closure of N in M is a direct summand of M.

Clearly, any CS-module is a GCS-module and any singular module is a GCS-module.

Also any module M satisfying $Z_2(M) = M$ is a GCS-module, where the submodule $Z_2(M)$ of M is defined by $Z_2(M)/Z(M) = Z(M/Z(M))$. In fact, let N be any $\mathscr{S}$-closed submodule of M. Consider the exact sequence $0 \to Z(M) \to M \to M/Z(M) \to 0$, we have $0 \to \mathrm{Hom}_R(M/Z(M), M/N) \to \mathrm{Hom}_R(M, M/N) \to \mathrm{Hom}_R(Z(M), M/N)$. As both $Z(M)$ and $M/Z(M)$ are singular and M/N is nonsingular, we have that $\mathrm{Hom}_R(M/Z(M), M/N) = \mathrm{Hom}_R(Z(M), M/N) = 0$. Hence $\mathrm{Hom}_R(M, M/N) = 0$ and $M = N$.

By [4], Proposition 2.4, a nonsingular module M is a GCS-module if and only if M is a CS-module. By the above definition, we have the following proposition:

Proposition 2.4. (i) *Let R be any ring and M an R-module. Then the following assertions are equivalent:*

(1) *M is a* GCS-*module.*
(2) *Any $\mathscr{S}$-closed submodule is a direct summand.*
(3) *For any homomorphism $f\colon M \to M'$ with M' nonsingular,* $\ker f$ *is a direct summand of M.*
(4) *For any $\mathscr{S}$-closed submodule N of M, the exact sequence:* $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ *splits.*

If R is right nonsingular, then 1–4 are equivalent to:

(5) *For any submodule N of M, there is a direct summand $K \supseteq N$ of M such that K/N is singular and M/K is nonsingular.*

(ii) *Let R be a ring and M a nonsingular R-module. Then the following assertions are equivalent:*

(1) *M is a* CS-*module.*

(2) *M is a* GCS-*module.*

P r o o f. We show (i) only, (ii) is obvious.

(1) $\Rightarrow$ (2). Since the closure of a $\mathscr{S}$-closed submodule N of M is N itself.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (4). By (3) N is a direct summand of M, (4) follows.

(4) $\Rightarrow$ (1). Obvious.

Now assume that R is right nonsingular.

(2) $\Rightarrow$ (5). Let K be the $\mathscr{S}$-closure of N, then M/K is nonsingular and K/N is singular by Lemma 2.1. (5) follows.

(5) $\Rightarrow$ (3). By (5), there is a direct summand K of M such that $K/\ker f$ is singular and M/K is nonsingular. Since $M/\ker f \cong M'$ is nonsingular, hence $K/\ker f$ is both singular and nonsingular, so $K = \ker f$. □

However, in general, a GCS-module need not be extending.

Example 2.5. Let R be any ring and M a singular R-module with unique composition series $M \supset U \supset V \supset 0$. In [6], Corollary 7.4, it is shown that $M \oplus U/V$ is not an CS-module, but is a GCS-module.

Also, a GCS-module need not be singular.

Example 2.6. Let $\mathbb{Z}$ be the ring of all integers. Then $\mathbb{Z}$ is extending and thus it is a GCS-module as a right $\mathbb{Z}$-module. But $\mathbb{Z}$ is not singular.

A closed submodule of an CS-module is a direct summand. But for a GCS-module, a closed submodule need not be a direct summand. For example, let $M = \mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ be a $\mathbb{Z}$-module, for a prime p. Obviously, M is a GCS-module, but $N = \mathbb{Z}(1+\mathbb{Z}_p, p+\mathbb{Z}_{p^3})$ is closed and is not a direct summand of M.

A submodule of a GCS-module need not be a GCS-module, see:

Example 2.7. Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, where $\mathbb{Z}$ is the ring of all integers. By [1], Example 1.3, R is not right extending hence not a GCS-module, since R is right nonsingular. But R is a submodule of its injective hull S, while S is a CS-module and hence a GCS-module.

Recall that a submodule N of M is called a fully invariant submodule if for every $f \in S$ we have $f(N) \subseteq N$, where $S = \mathrm{End}_R(M)$. If $M = K \oplus L$ and N is a fully invariant submodule of M, then we have $N = (N \cap K) \oplus (N \cap L)$ and $M/N = K/(N \cap K) \oplus L/(N \cap L)$. For example, let $p \in \mathbb{N}$ be any prime and consider the Prüfer group $\mathbb{Z}_{p^\infty} = \sum\limits_{n \in \mathbb{N}} \mathbb{Z}\{1/p^n + \mathbb{Z}\} = \bigcup\limits_{n \in \mathbb{Z}} \mathbb{Z}\{1/p^n + \mathbb{Z}\} \subset \mathbb{Q}/\mathbb{Z}$, where $\mathbb{Q}$ is the ring of all rational numbers. Then every submodule of $\mathbb{Z}_{p^\infty}$ is fully invariant (see [7], page 144, 17.13).

Proposition 2.8. *Let R be a right nonsingular ring and M a* GCS*-module. Then any fully invariant submodule is a* GCS*-module.*

Proof. Let N be a fully invariant submodule of M and L a submodule of N. Then, since M is a GCS-module, there are direct summands K, K' of M such that $M = K \oplus K'$, $L \leqslant K$ and that K is the $\mathscr{S}$-closure of L in M, i.e., K/L is singular and M/K is nonsingular. Since N is fully invariant, we have $N = (N \cap K) \oplus (N \cap K')$. Obviously, $L \leqslant N \cap K$ and $(N \cap K)/L \leqslant K/L$ is singular. Since $N/(N \cap K) \cong (N + K)/K \leqslant M/K$ is nonsingular, $N \cap K$ is an $\mathscr{S}$-closed submodule of N. So N is a GCS-module. □

From [5] a decomposition $M = \oplus M_\alpha$ over some collection of submodules of a module M is *deep* if for every submodule N of M we have $N = \oplus(N \cap M_\alpha)$. It is known that for a commutative ring R any decomposition of a cyclic R-module is deep.

Corollary 2.9. *We have the following:*

(1) *Let R be a right nonsingular ring and M a distributive* GCS*-module. Then any submodule is a* GCS*-module.*

(2) *Any submodule of $\mathbb{Z}_{p^\infty}$ is a* GCS*-module, as a $\mathbb{Z}$-module. (Note that every non-zero proper submodule of $\mathbb{Z}_{p^\infty}$ is self-injective but not $\mathbb{Z}$-injective.)*

Suppose that $Z_r(R) = 0$ and any decomposition of module M is deep. If M is a GCS*-module then any submodule of M is a* GCS*-module.*

(3) *If R is a commutative nonsingular ring, then any submodule of a cyclic* GCS*-module is a* GCS*-module.*

A ring R is called a right GCS-*ring* if R_R is a GCS-module. The following proposition shows equivalent conditions for a cyclic submodule of a module to be a GCS-module over a right GCS-ring.

Proposition 2.10. *Let R be a right* GCS*-ring and M a right R-module. Then the following are equivalent:*

(1) *M is nonsingular.*

(2) *Every cyclic submodule of M is projective and a* GCS*-module.*

(3) *Every cyclic submodule of M is projective.*

Proof. (1) $\Rightarrow$ (2). Suppose that M is nonsingular and N is a cyclic submodule of M. Then there is a right ideal I of R such that $N \cong R/I$. Since R is a right GCS-ring and N is nonsingular, I is an $\mathscr{S}$-closed submodule of R_R; hence I is a direct summand of R_R. Thus N is isomorphic to a direct summand of R; hence N is projective and a GCS-module.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). For any $m \in Z(M)$, the module mR is projective and isomorphic to $R/r(m)$, where $r(m)$ is the right annihilator of m. Then $r(m)$ is a direct summand of R. But $m \in Z(M)$ implies that there is an essential right ideal I of R such that $mI = 0$, hence $I \leqslant r(m)$ and $r(m) \leqslant_e R$. Thus $r(m) = R$ and $m = 0$, hence $Z(M) = 0$. $\square$

Example 2.11 ([2], Example 2.3)**.** Let S be the ring of all 3×3 upper triangular matrices over the field of complex numbers and R the sub-ring of S consisting of all elements of S with a real number in the $(2,2)$-position. Then R is a CS-ring. Let e be the element of R with 1 in the $(3,3)$-position and 0 elsewhere, and set $I = Re$. Then I is an ideal of R. But R/I is not a right CS-ring.

This example shows that a homomorphic image of a CS-ring need not be a CS-ring. But any factor module of a singular module is singular; now we will show that any image of a GCS-module is a GCS-module.

A direct summand of an CS-module is also extending (see [6]). For GCS-modules, we first show the following proposition and then show that any direct summand of a GCS-module is a GCS-module.

Proposition 2.12. *Let M be a* GCS*-module. Then any homomorphic image of M is a* GCS*-module. In particular, any direct summand of M is a* GCS*-module.*

Proof. Let $f\colon M \to N$ be an epimorphism and L an $\mathscr{S}$-closed submodule of N. By Lemma 2.2 (4), $f^{-1}(L)$ is an $\mathscr{S}$-closed submodule of M. Since M is a GCS-module, $M = f^{-1}(L) \oplus K$ for some submodule K of M. It is easy to see that $N = L \oplus f(K)$ and so N is a GCS-module. $\square$

Corollary 2.13. *Let R be any ring and M an R-module.*

(1) *Any $\mathscr{S}$-closed submodule of an injective (quasi-injective, extending) module is injective (quasi-injective, extending).*

(2) *If M is an extending module, then any nonsingular homomorphic image is extending.*

Proof. Since any injective module is a GCS-module and any $\mathscr{S}$-closed submodule is closed. $\square$

Since any module is a homomorphic image of some projective module, we have:

Corollary 2.14. (i) *Let R be a ring. Then the following are equivalent:*

(1) *Every R-module is a* GCS-*module.*

(2) *Every projective R-module is a* GCS-*module.*

(ii) *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is a* GCS *ring.*

(2) *Every cyclic R-module is a* GCS-*module.*

It is well known that a ring R is right hereditary if and only if every factor module of an injective right R-module is injective.

Corollary 2.15. *Let R be a ring such that every* GCS-*module is injective. Then R is right hereditary.*

Proposition 2.16. *Let R be a right nonsingular ring and $f\colon M \to M'$ an epimorphism. Suppose that M' is a* GCS-*module and* $\operatorname{Ker} f$ *is singular injective. Then M is a* GCS-*module.*

Proof. Let $N \leqslant M$. First, we assume that $\operatorname{Ker} f \subseteq N \leqslant M$, then $f(N) \leqslant M'$. Since M' is a GCS-module, there is a decomposition of M', $M' = K \oplus H$ such that $K/f(N)$ is singular and M'/K is nonsingular. So $M = f^{-1}(K) + f^{-1}(H)$. Since $\operatorname{Ker} f \leqslant f^{-1}(H)$ and $\operatorname{Ker} f$ is injective, $f^{-1}(H) = T \oplus \operatorname{Ker} f$ for some submodule T of $f^{-1}(H)$. Thus $M = f^{-1}(K) + T$. Since $f^{-1}(K) \cap T \subseteq f^{-1}(K) \cap f^{-1}(H) = \operatorname{Ker} f$ and $f^{-1}(K) \cap T = f^{-1}(K) \cap T \cap T \subseteq \operatorname{Ker} f \cap T = 0$, we have $M = f^{-1}(K) \oplus T$ and $N \leqslant f^{-1}(K)$.

If $x \in f^{-1}(K)$, then $f(x) \in K$ and there is an essential right ideal I of R such that $f(x)I \subseteq f(N)$. It is easy to see that $xI \subseteq N$ and that $f^{-1}(K)/N$ is singular. Also, since $K \in L^*(M')$, we have $f^{-1}(K) \in L^*(M)$ by Lemma 2.2 (4).

Now we assume that $N \not\supseteq \operatorname{Ker} f$. Set $L = N + \operatorname{Ker} f$, then $f(L) = f(N)$. As the case above, there is a decomposition of $M = f^{-1}(K) \oplus T$ such that $f^{-1}(K)/L$ is singular and $f^{-1}(K) \in L^*(M)$. Since $\operatorname{Ker} f$ is singular, we have that $(N + \operatorname{Ker} f)/N \cong$

$\operatorname{Ker} f/(N \cap \operatorname{Ker} f)$ is singular. Since R is right nonsingular, we have that $f^{-1}(K)/N$ is singular.

In either case, M is a GCS-module. □

Proposition 2.17. *Let R be a right nonsingular ring and M a* GCS-*module. Then $M = Z(M) \oplus T$ for some* CS-*submodule T of M. In this case T is $Z(M)$-injective.*

Proof. If $Z(M) = 0$ or $Z(M) = M$, it is trivial.

Suppose that $0 < Z(M) < M$. Since M is a GCS-module, there are direct summands K, T of M such that $M = K \oplus T$, $Z(M) \leqslant K$ and that $K/Z(M)$ is singular and M/K is nonsingular. So K is singular. Since $Z(M) = Z(K) \oplus Z(T) = K \oplus Z(T)$, we have $Z(M) = K$ and T is nonsingular. By Proposition 2.4 (ii), T is extending.

Since for any submodule N of $Z(M)$, $\operatorname{Hom}_R(N, T) = 0$, the module T is $Z(M)$-injective, as required. □

Corollary 2.18. *Let R be a right nonsingular ring and M an R-module. Then:*

(1) *Any* GCS-*module is a direct sum of a* GCS-*submodule and an extending submodule.*
(2) *If every $\mathscr{S}$-closed submodule of M is fully invariant, then M is a* GCS-*module if and only if $M = Z(M) \oplus K$ for some nonsingular extending submodule K of M.*
(3) *Let $M = Z(M) \oplus K$ with K a nonsingular extending submodule of M. Then M is a* GCS-*module if and only if every $\mathscr{S}$-closed submodule N with $N \cap Z(M) = 0$ is a direct summand of M.*
(4) *Let M be a* GCS-*module, then any epimorphism $f\colon M \to N$ with N nonsingular splits.*

Proof. We only show (2) and (3). We first prove (2). The necessity is Proposition 2.17.

Now suppose that $M = Z(M) \oplus K$ for some nonsingular extending submodule K of M. Let N be any $\mathscr{S}$-closed submodule of M, then $N = (N \cap Z(M)) \oplus (N \cap K)$. Since $M/N \cong Z(M)/(N \cap Z(M)) \oplus K/(K \cap N)$ and M/N is nonsingular, we have $Z(M) = N \cap Z(M)$, which implies that $Z(M) \subseteq N$. Since K is a nonsingular extending module and $K \cap N$ is a $\mathscr{S}$-closed submodule of K, we have $K = N \cap K \oplus L$ for some submodule L of K. Thus $M = Z(M) \oplus K = Z(M) \oplus (N \cap K) \oplus L = N \oplus L$ and M is a GCS-module.

Now we prove (3). The necessity is obvious.

Conversely, let N be an $\mathscr{S}$-closed submodule of M with $N \cap Z(M) \neq 0$. Then $Z(M)/(Z(M) \cap N) \cong (Z(M) + N)/N \leqslant M/N$ is both singular and nonsingular;

hence we have $Z(M) + N = N$ and thus $Z(M) \subseteq N$. Now $N = Z(M) \oplus (K \cap N)$ and $K \cap N$ is an $\mathscr{S}$-closed submodule of K. As K is extending, $K \cap N$ is a direct summand of K. Therefore N is a direct summand of M and M is a GCS-module. $\square$

With Proposition 2.17, we get the following well-known result about injective modules:

Corollary 2.19. *Let R be a right nonsingular ring and M an injective module. Then $Z(M)$ is injective.*

Corollary 2.20. *Let R be a right nonsingular ring and M an indecomposable* GCS-*module. Then M is either a singular module or a nonsingular uniform module.*

Recall that a proper submodule N of M is called *small* in M if $N + K = M$ implies that $K = M$. So we have:

Corollary 2.21. *Let R be a right nonsingular ring and M a GCS-module. Suppose that $Z(M)$ is small in M. Then M is nonsingular and extending.*

Lemma 2.22. *Let M be a* GCS-*module and N a nonsingular module. Then for any $f \in \mathrm{Hom}_R(M, N)$, we have $\mathrm{Ker}\, f \in L^*(M)$ and $M \cong \mathrm{Ker}\, f \oplus \mathrm{Im}\, f$.*

Proof. If $0 = f \in \mathrm{Hom}_R(M, N)$, it is obvious. For any $0 \neq f \in \mathrm{Hom}_R(M, N)$, since $\mathrm{Im}\, f \cong M/\mathrm{Ker}\, f$ is a submodule of the nonsingular module N, so $\mathrm{Ker}\, f \in L^*(M)$. Since M is a GCS-module, we have $M = \mathrm{Ker}\, f \oplus T$ for some submodule T of M; obviously, $T \cong \mathrm{Im}\, f$. $\square$

Proposition 2.23. *Let M be a* GCS-*module and N a nonsingular module such that $M \oplus N$ is a* GCS-*module. Suppose that any* GCS-*submodule of N is a direct summand of N. Then for any $K \in L^*(M \oplus N)$, there are decompositions $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$ such that $M \oplus N = K \oplus (M_2 \oplus N_2)$ and $K \cong M_1 \oplus N_1$, $M_2 \oplus N_2 \cong (M \oplus N)/K$.*

Proof. Let $p_1\colon M \oplus N \to M$ and $p_2\colon M \oplus N \to N$ be the canonical projections and set $M_1 = K \cap M$, $N_1 = p_2(K)$, where $K \in L^*(M \oplus N)$. Note that K is a direct summand of $M \oplus N$ and M_1 is the kernel of the restrict projection $p_2\colon K \to N$, hence we have $K \cong M_1 \oplus N_1$ by Lemma 2.22. Thus M_1 and N_1 are GCS-modules. Since $M/M_1 = M/(K \cap M) \cong (M + K)/K \leqslant (M \oplus N)/K$, M/M_1 is nonsingular and M_1 is a direct summand of M. By the hypothesis, we have $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$ for some $M_2 \leqslant M$ and $N_2 \leqslant N$.

Note that $N_2 = (1-p_1)(K) \leqslant K+M$, thus $N \leqslant K+M+N_2$ and $K+M+N_1 = M \oplus N$. We also have $M = M_1 \oplus M_2 \leqslant K+M_2$ and $K+M = K+M_2$. Therefore, $M \oplus N = K + M_2 + N_2$.

Set $L = K \cap (M_2 + N_2)$. Since $p_2(K) = N_1$ and $p_2(M_2+N_2) = N_2$, we have $p_2(L) \leqslant N_1 \cap N_2 = 0$ and $L \leqslant M$.

Since $L \leqslant K \cap M = M_1$ and $p_1(L) = L$, we have $p_1(L) \leqslant p_1(M_2+N_2) = M_2$ and therefore $L \leqslant M_1 \cap M_2 = 0$.

Hence, $M \oplus N = K \oplus (M_2 \oplus N_2)$. □

The following corollary generalizes [4], Lemma 6.12.

Corollary 2.24. *Let M be a* GCS*-module and N a nonsingular* GCS*-module such that $M \oplus N$ is a* GCS*-module. Then for any $K \in L^*(M \oplus N)$, there are decompositions $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$ such that $M \oplus N = K \oplus (M_2 \oplus N_2)$ and $K \cong M_1 \oplus N_1$, $M_2 \oplus N_2 \cong (M \oplus N)/K$.*

Proof. As in the proof of Proposition 2.23, if N is a GCS-module, then N_1 is a direct summand of N, since N/N_1 is nonsingular. The rest of the proof is similar to that of [4], Lemma 6.12, and is omitted. □

3. The direct sum of GCS-modules

A direct sum of singular modules is also singular. But a direct sum of CS-modules need not be extending. For example, if p is a prime, then $\mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ is not extending even though $\mathbb{Z}_p$ and $\mathbb{Z}_{p^3}$ are extending. A direct sum of GCS-modules need not be a GCS-module, i.e., the class of GCS-modules is not closed under direct sum.

Example 3.1 ([2], Example 2.4)**.** Let $R = \mathbb{Z}[x]$, where x is an indeterminate and $\mathbb{Z}$ is the ring of integers. The ring R has no proper closed ideals and is extending, hence is a GCS-module. Let $F = R \oplus R$ and set $C = \{(xr, 2r);\ r \in R\}$. Then C is a closed submodule of F and is not a direct summand of F. Therefore F is not extending. Since $\mathbb{Z}$ is nonsingular, R is nonsingular as an R-module by [4], Exercise 1.D.13, and hence F is nonsingular. So F is not a GCS-module.

This example also shows that the class of GCS-modules is not closed under module extensions. So it is natural to ask when the direct sum of GCS-modules is a GCS-module.

Proposition 3.2. *Let $M = M_1 \oplus M_2$ with each M_i, $(i = 1, 2)$ a* GCS-*module. If M is distributive, then M is a* GCS-*module.*

Proof. Let N be any $\mathscr{S}$-closed submodule of M, then M/N is nonsingular. Since M is distributive, we have $N = (N \cap M_1) \oplus (N \cap M_2)$. As $M_i/(M_i \cap N) \cong (M_i + N)/N \leqslant M/N$ is nonsingular for each i, we obtain that $N \cap M_i$ is an $\mathscr{S}$-closed submodule of M_i for each i. Since each M_i is a GCS-module, there are direct summands H_1 and H_2 of M_1, M_2, respectively, such that $M_i = H_i \oplus (M_i \cap N)$ for $i = 1, 2$. Hence $M = (H_1 \oplus H_2) \oplus ((M_1 \cap N) \oplus (M_2 \cap N)) = ((H_1 \oplus H_2)) \oplus N$. Thus M is a GCS-module. $\square$

Corollary 3.3. *Let M be a distributive module and $M = \bigoplus_{i=1}^{n} M_i$. Then M is a* GCS-*module if and only if M_i is a* GCS-*module for every i.*

In Example 2.5, we have shown that $M \oplus U/V$ is a GCS-module, where M is singular and U/V is simple. In fact, we can generalize this result to the following case:

Theorem 3.4. *Let $M = M_1 \oplus M_2$ with M_1 singular (or uniform) and M_2 semi-simple. Then M is a* GCS-*module.*

Proof. Let N be any $\mathscr{S}$-closed submodule of M. Then $N + M_1 = M_1 \oplus [(N + M_1) \cap M_2]$. Since M_2 is semi-simple, $(N + M_1) \cap M_2$ is a direct summand of M_2 and therefore $N + M_1$ is a direct summand of M. Note that if M_1 is singular (or uniform), then $(N + M_1)/N \cong M_1/(N \cap M_1)$ is both singular and nonsingular. So $N + M_1 = N$ and M is a GCS-module. $\square$

4. Rings in which all modules are GCS-modules

In this section we investigate rings over which all modules are GCS-modules.

Theorem 4.1. *Let R be a right nonsingular ring. Then the following assertion are equivalent:*

(1) *Every nonsingular module is projective.*
(2) *Every projective module is a* GCS-*module.*
(3) *Every module is a* GCS-*module.*
(4) *Every nonsingular module is extending.*

Proof. (1) $\Rightarrow$ (2). Let P be a projective module and N an $\mathscr{S}$-closed submodule of P. Then P/N is nonsingular and projective by (1). Thus N is a direct summand and P is a GCS-module.

(2) $\Rightarrow$ (1). Let M be a nonsingular module. There is a projective module P such that $M \cong P/N$ for some $\mathscr{S}$-closed submodule N of P. Since P is a GCS-module, N is a direct summand of P. Also M is projective if and only if N is a direct summand of P. Hence M is projective.

(3) $\Leftrightarrow$ (2) is Corollary 2.14.

(3) $\Rightarrow$ (4) is Proposition 2.4 (ii).

(4) $\Rightarrow$ (2). Since all projective modules are nonsingular, by (4), all projective modules are extending and GCS-modules. $\square$

A ring R is quasi-Frobenius if and only if every projective right module is injective if and only if every injective module is projective [3], Theorem 24.12. Thus any nonsingular quasi-Frobenius ring satisfies the equivalent conditions of Theorem 4.1. Certainly if R is semi-simple then all nonsingular right R-modules are projective. For the non-semi-simple case, we have the following example:

Example 4.2 ([4], Proposition 5.22)**.** Let T be a semi-simple ring and $n > 1$ a positive integer. If R is the ring of all lower triangular $n \times n$ matrices over T, then R is not semisimple, it is a right and left nonsingular ring and all nonsingular right and left R-modules are projective. Hence all left and right R-modules are GCS-modules.

Consider the following condition for a module M:

(C_2): Every submodule which is isomorphic to a direct summand of M is also a direct summand of M;

and the following condition for a ring R:

(P): For every closed right ideal I, there is $a \in R$ such that $R/I \cong aR$.

The following theorem shows the relation between a GCS-ring and a regular ring. A ring R is regular if and only if every principal right ideal R is generated by an idempotent if and only if every right R-module is flat.

Theorem 4.3. *Let R be a right nonsingular ring satisfying conditions* (C_2) *and* (P)*. Then the following assertions are equivalent:*

(1) *R is a right* GCS*-ring.*

(2) *R is a right extending ring.*

(3) *Every cyclic right R-module is a* GCS*-module.*

(4) *Every cyclic projective R-module is a* GCS*-module.*

(5) *Every nonsingular cyclic right R-module is projective.*

(6) *Every principal right ideal of R is generated by an idempotent.*

(7) *R is (von Neumann) regular.*

(8) *Every right R-module is flat.*

P r o o f. (1) $\Leftrightarrow$ (2) and (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) are obvious. (1) $\Rightarrow$ (3) is Proposition 2.12 and (3) $\Rightarrow$ (4) is obvious. It is sufficient to show that (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (1).

(4) $\Rightarrow$ (5). Let aR be a nonsingular right R-module. Then there is an $\mathscr{S}$-closed right ideal I of R such that $aR \cong R/I$. By (4), R_R is a GCS-module and I is a direct summand of R_R. Thus aR is isomorphic to a direct summand of R_R and is projective.

(5) $\Rightarrow$ (6). Let aR be any principal right ideal of R. Then aR is nonsingular and $aR \cong R/r(a)$, where $r(a)$ is the right annihilator of a. Thus aR is projective by (5) and aR is isomorphic to a direct summand of R. Hence aR is a direct summand of R, i,e., aR is generated by an idempotent of R, since R satisfies condition (C_2).

(6) $\Rightarrow$ (1). Let $0 \neq I$ be any $\mathscr{S}$-closed right ideal of R, then $R/I \cong aR$ for some $a \in R$ by hypothesis. By (6), aR is a direct summand of R_R, aR is projective and so R/I is projective. Therefore I is a direct summand of R_R, and thus R is a GCS-ring. □

Remark 4.4. The condition (C_2) of the theorem above cannot be cancelled. For example, the ring $\mathbb{Z}$ of all integers which does not satisfy (C_2) is a nonsingular GCS-ring, but not a regular ring.

We will now consider the weaker case, namely, rings in which all finitely generated modules are GCS-modules. In fact there is a ring R such that all finitely generated modules are GCS-modules but not all modules are GCS-modules. First we have the following theorem which is similar to Theorem 4.1

Theorem 4.5. *Let R be any right nonsingular ring. Then the following assertions are equivalent:*

(1) *Every finitely generated nonsingular right R-module is projective.*
(2) *Every finitely generated projective right R-module is a* GCS*-module.*
(3) *Every finitely generated right R-module is a* GCS*-module.*
(4) *Every finitely generated nonsingular right R-module is extending.*

P r o o f. Similar to the proof of Theorem 4.1. □

Example 4.6. Let V be an infinite-dimensional vector space over a division ring D and set $R = \mathrm{End}_D(V)$. The ring R is von Neumann regular self-injective by [4], Proposition 2.23. It is shown in [4], Theorem 3.12 that all finitely generated nonsingular right R-modules are projective and hence all finitely generated modules are GCS-modules by Theorem 4.5. However, R is not right artinian and thus [4], Theorem 5.21, shows that not all nonsingular right modules are projective. Thus not all modules are GCS-modules.

If R is a semi-hereditary commutative domain, then all finitely generated non-singular R-modules are projective (see [4], Exercise 5.C.10). But the conditions of Theorem 4.5 are not right-left symmetric. There is a ring R which is both right non-singular and left nonsingular and all finitely generated nonsingular right R-modules are projective, while not all finitely generated nonsingular left R-modules are projective. For example, let F be a field and V an infinite-dimensional vector space over F. Set $R = \mathrm{End}_F(V)$, then R is as required, see [4], 5.C, Exercise 15.

But if R is right nonsingular and the identity of R is a sum of orthogonal primitive idempotents, then the conditions of Theorem 4.5 are left-right symmetric. Combining [2], Theorem 4.1, with Theorem 4.5, we have:

Theorem 4.7. *Let R be a right nonsingular ring such that the identity of R is a sum of orthogonal primitive idempotents. Then the following assertions are equivalent:*

(1) *Every finitely generated nonsingular right R-module is projective.*
(2) *Every finitely generated projective right R-module is a* GCS-*module.*
(3) *Every finitely generated right R-module is a* GCS-*module.*
(4) *Every finitely generated nonsingular right R-module is extending.*
(5) *R is a left and right semi-hereditary left and right extending ring.*
(6) *The left-version of* (1) *through* (4).

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