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On some joint-life annuities.

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The value of an annuity for a group of lives, say $x, y, z, \ldots$, based on the asumptions that l_x is a continuous function of the age and has therefore at any point in the whole of the chosen range a derivative and is integrable and, that the force of mortality μ_x and the force of interest δ are continuous functions of the time t, is given by the formula

$$\overline{a}(x, y, z, \ldots) = \int_0^\infty e^{-\int_0^t (\Sigma\mu_{x+u} + \delta) du} dt. \tag{1}$$

Using the values of annuities expressed in this manner, it is possible to find values of annuities, the payments under which depend on the time t and the age, of the persons in the given group. When therefore $\varphi(x, y, z, \ldots, t)$ is a continuous function of the time t, if by this function the law of payment of the annuities be determined, the value of the annuity to the given group of persons, can be expressed as follows

$$\overline{a}(x, y, z, \ldots, \varphi) = \int_0^\infty \varphi(x, y, z, \ldots, t)\, e^{-\int_0^t (\Sigma\mu_{x+u} + \delta) du} dt. \tag{2}$$

If we make a convenient choice of the law of payment, we can find expressions for various annuities, in this form several of which we will consider.

We define the reversionary annuity $\overline{a}_{x|y}$ as an annuity, payable from the time interval dt to the person (y), if the person (x) dies in this interval. The value of the annuity, payable from the interval of time dt to the person (y) is $\mu_{x+t}\overline{a}_{y+t}\,dt$ and then the law of payment is $\mu_{x+t}\overline{a}_{y+t}$. Therefore we have

$$\overline{a}_{x|y} = \int_0^\infty \mu_{x+t}\,\overline{a}_{y+t}\, e^{-\int_0^t (\mu_{x+u} + \mu_{y+u} + \delta) du} dt. \tag{3}$$

According to the first theorem for the mean value of a definite integral we have

$$\int_a^b \varphi(t)\,\psi(t)\,dt = \varphi(n)\int_a^b \psi(t)\,dt,$$

where the function $\varphi(t)$ is continuous over the range (a, b) and the value of the coefficient n lies within the range. By application of this theorem for the mean value of the integral to the value (3) we obtain

$$\overline{a}_{xy} = \mu_{x+n}\, \overline{a}_{y+n} \int_0^\infty e^{-\int_0^t (\mu_{x+u}+\mu_{y+u}+\delta)du}\, dt \tag{5}$$

and consequently by (1) it is[1])

$$\overline{a}_{x|y} = \mu_{x+n}\, \overline{a}_{y+n}\, \overline{a}_{xy}. \tag{6}$$

The value of the annuity $\overline{a}_{y|x}$ payable to (x) after (y) will then be

$$\overline{a}_{y|x} = \int_0^\infty \mu_{y+t}\, \overline{a}_{x+t}\, e^{-\int_0^t (\mu_{x+u}+\mu_{y+u}+\delta)du}\, dt.$$

and therefore the value of the annuity payable while exactly one of the lives (x) and (y) survives, will be

$$\overline{a}^{[1]}_{\overline{xy}} = \int_0^\infty (\mu_{y+t}\, \overline{a}_{x+t} + \mu_{x+t}\, \overline{a}_{y+t})\, e^{-\int_0^t (\mu_{x+u}+\mu_{y+u}+\delta)du}\, dt. \tag{7}$$

When we apply the theorem for the mean value of a definite integral to this expression in the manner described and use the value of the life annuity for (x, y) according to formula (1) we obtain

$$\overline{a}^{[1]}_{\overline{xy}} = (\mu_{x+n}\, \overline{a}_{y+n} + \mu_{y+n}\, \overline{a}_{x+n})\, \overline{a}_{xy}. \tag{8}$$

By varying the law of payment, we obtain the following expression for the value of the annuity to continue so long as at least one of the lives (x) and (y) survives

$$a_{\overline{xy}} = \int_0^\infty (1 + \mu_{x+t}\, \overline{a}_{y+t} + \mu_{y+t}\, \overline{a}_{x+t})\, e^{-\int_0^t (\mu_{x+u}+\mu_{y+u}+\delta)du}\, dt. \tag{9}$$

If again we apply the theorem for the mean value of a definite ntegral to this integral we get the expression

$$\overline{a}_{\overline{xy}} = (1 + \mu_{x+n}\, \overline{a}_{y+n} + \mu_{y+n}\, \overline{a}_{x+n})\, \overline{a}_{xy} \tag{10}$$

For the relation (6), it is possible to determine the time n from the formula

$$\mu_{x+n}\, \overline{a}_{y+n} = \frac{\overline{a}_y}{\overline{a}_{xy}} - 1$$

in applying the well-known equation

[1]) Callaway, On the Calculation of contingent assurance values and of compound survivorship annuities, Journal of the Institute of Actuaries, 1932. Evans, On the Calculation of Contingent Assurances and the Compound Survivorship Annuity when Makeham's Law holds, Journal of the Institute of Actuaries, 1925.

$$\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy}.$$

The time n for the relation (6) is for certain cases tabulated according to the H^M 4% table, graduated by Makeham's Law,[2]) in the following table.

x \ y	35	40	45	50	55
35	15,012	14,113	13,133	12,103	11,188
40	13,840	13,106	12,237	11,360	10,216
45	12,618	12,012	11,304	10,467	9,547
50	11,298	10,858	10,297	9,635	8,869
55	9,931	9,600	9,206	8,697	8,088

For three lives (x, y, z) we can proceed in a similar way and find expressions for various annuities. The law of payment for an annuity, which is due to the couple (x, y) so long as the persons y and z are alive, after the death of the person (x), is naturally $\mu_{x+t}\,\bar{a}_{y+t:z+t}$ and therefore the value of such an annuity is

$$\bar{a}_{x|yz} = \int_0^\infty \mu_{x+t}\,\bar{a}_{y+t:z+t}\, e^{-\int_0^t (\mu_{x+u}+\mu_{y+u}+\mu_{z+u}+\delta)du}\, dt. \tag{11}$$

If again we apply to this value the first theorem for the mean value of a definite integral as described above, we obtain, by using the value of the life annuity for three lives (x, y, z) according to the formula (1) the following expression

$$\bar{a}_{x|yz} = \mu_{x+n}\,\bar{a}_{y+n:z+n}\,\bar{a}_{xyz}. \tag{12}$$

In a similar manner, it is possible to find the value of an annuity to be paid while exactly two of the lives (x), (y) and (z) survive thus

$$\bar{a}_{\overline{xyz}}^{[2]} = \int_0^\infty [\Sigma\mu_{x+t}\,\bar{a}_{y+t:z+t}]\, e^{-\int_0^t (\mu_{x+u}+\mu_{y+u}+\mu_{z+u}+\delta)du}\, dt \tag{13}$$

where

$$\Sigma\mu_{x+t}\,\bar{a}_{y+t:z+t} = \mu_{x+t}\,\bar{a}_{y+t:z+t} + \mu_{y+t}\,\bar{a}_{x+t:z+t} + \mu_{z+t}\,\bar{a}_{x+t:y+t}$$

i. e. the sum of the values of the form $\mu_{x+t}\,\bar{a}_{y+t:z+t}$ formed by taking the ages of all three lives in cyclic order. We shall use this shortened notation for other summations of values formed by the cyclic arrangement of the index. If we apply the theorem for the mean value of an integral to the relation (13) we obtain

[2]) Spurgeon, Life contingencies, 1933.

$$a^{[2]}_{\overset{\cdots}{xyz}} = [\Sigma\mu_{x+n}\,\bar{a}_{y+n:z+n}]\,\bar{a}_{xyz}.$$

In the same manner we can express the value of an annuity paid to the person (z) if one of the lives (x) and (y) has died. Let us find first the value of the annuity $\bar{a}^{1}_{xy|z}$ paid to the person (z) after the death of the person (x) before (y). The law of payment for this case is $\mu_{x+t}\bar{a}_{z+t}$ which indicates that the annuity a_{z+t} is due, when the person (x) dies at age $x + t$, (y) being then alive. The value may then be expressed as

$$\bar{a}^{1}_{xy|z} = \int_0^\infty \mu_{x+t}\,\bar{a}_{z+t}\, e^{-\int_0^t (\Sigma\mu_{x+u}+\delta)du}\, dt$$

and by the application of the theorem for the mean value of the integral this becomes

$$\bar{a}^{1}_{xy.z} = \mu_{x+n}\,\bar{a}_{z+n}\,\ddot{a}_{xyz}.$$

The value of the annuity $\bar{a}_{xy|z}$ paid to the person (z), if one of the lives (x) and (y), is dead will be

$$\bar{a}_{xy|z} = \bar{a}^{1}_{xy|z} + \bar{a}_{xy|z}^{\;\;1}$$

and therefore

$$\bar{a}_{xy|z} = \int_0^\infty (\mu_{x+t} + \mu_{y+t})\,\bar{a}_{z+t}\, e^{-\int_0^t (\Sigma\mu_{x+u}+\delta)du}\, dt$$

$$= (\mu_{x+n} + \mu_{y+n})\;\bar{a}_{z+n}\,a_{xyz}.$$

We get analogous values for annuities to be paid to (x) or (y), if one of the other two lives dies. If we desire to find the law of payment for the case, then exactly one determined life, e. g. (z) survives, it is necessary to deduct the values of annuities paid while the other two lives are alive. For the life (z) the law of payment will be

$$(\mu_{x+t} + \mu_{y+t})\,\bar{a}_{z+t} - \mu_{x+t}\,\bar{a}_{y+t:z+t} - \mu_{y+t}\,a_{x+t:z+t}.$$

This law of payment can be written also in the form

$$\mu_{x+t}\,\bar{a}_{y+t|z+t} + \mu_{y+t}\,\bar{a}_{x+t|z+t}$$

for if in interval of time dt the person (x) dies, the annuity becomes a reversionary annuity to be paid to the person (z) if he survives the life (y) and similarly if in the interval of time dt the person (y) dies, the annuity becomes a reversionary annuity to be paid to the person (z) if he survives the life (x).

Following this development, the value of the annuity to be paid when one of three lives (x, y, z) exactly survives, will be

$$\bar{a}^{[1]}_{xyz} = \int_0^\infty [\Sigma (\mu_{y+t} + \mu_{z+t}) \bar{a}_{x+t} - 2\Sigma\mu_{x+t} \bar{a}_{y+t:z+t}] e^{-\int_0^t (\Sigma\mu_{x+u}+\delta)du} dt$$

$$= \int_0^\infty \left[\Sigma \mu_{x+t} \bar{a}^{[1]}_{\overline{y+t:z+t}}\right] e^{-\int_0^t (\Sigma\mu_{x+u}+\delta)du} dt$$

$$= [\Sigma (\mu_{y+n} + \mu_{z+n}) \bar{a}_{x+n} - 2\Sigma \mu_{x+n} \bar{a}_{y+n:z+n}] \bar{a}_{xyz}.$$

For the values of annuities paid when at least two or one of the lives are alive, we have

$$\bar{a}_{\overline{xyz}}^{\,2} = \bar{a}^{[2]}_{\overline{xyz}} + \bar{a}_{xyz},$$

$$\bar{a}_{\overline{xyz}}^{\,1} = \bar{a}^{[1]}_{\overline{xyz}} + \bar{a}^{[2]}_{\overline{xyz}} + \bar{a}_{xyz}$$

and from the values found above, we have

$$\bar{a}_{\overline{xyz}}^{\,2} = \int_0^\infty (1 + \Sigma \mu_{x+t} \bar{a}_{y+t:z+t}) e^{-\int_0^t (\Sigma\mu_{x+u}+\delta)du} dt$$

$$= [1 + \Sigma \mu_{x+n} \bar{a}_{y+n:z+n}] \bar{a}_{xyz},$$

$$\bar{a}_{\overline{xyz}}^{\,1} = \int_0^\infty [1 + \Sigma (\mu_{y+t} + \mu_{z+t}) \bar{a}_{x+t} - \Sigma\mu_{x+t} \bar{a}_{y+t:z+t}] e^{-\int_0^t (\Sigma\mu_{x+u}+\delta)du} dt$$

$$= [1 + \Sigma (\mu_{y+n} + \mu_{z+n}) \bar{a}_{x+n} - \Sigma\mu_{x+n} \bar{a}_{y+n:z+n}] \bar{a}_{xyz}.$$

By analogy with the values for three lives, it is possible, having made an appropriate choice of the law of payment, to find the values of annuities for greater groups of lives. The general form of an annuity paid when out of m lives $(x_1, x_2, x_3, \ldots, x_m)$ exactly r of the lives survive, is

$$\bar{a}^{[r]}_{\overline{x_1:x_2:x_3:\ldots:x_m}} = \int_0^\infty \left\{ \sum_{k=0}^{\frac{m-r-1}{2}} \binom{r+2k}{2k} \sum (\mu_{x_1+t} + \mu_{x_2+t} + \ldots + \mu_{x_{m-r-2k}+t}) \bar{a}_{x_{m-r-2k+1}+t:x_{m-r-2k+2}+t:\ldots:x_m+t} \right\} e^{-\int_0^t (\Sigma\mu_{x_1+u}+\delta)du} dt -$$

$$- \int_0^\infty \left\{ \sum_{k=0}^{\frac{m-r}{2}-1} \binom{r+2k+1}{2k+1} \sum (\mu_{x_1+t} + \mu_{x_2+t} + \ldots + \mu_{x_{m-r-2k-1}+t}) \bar{a}_{x_{m-r-2k}+t:x_{m-r-2k+1}+t:\ldots:x_m+t} \right\} e^{-\int_0^t (\Sigma\mu_{x_1+u}+\delta)du} dt.$$

In the same manner we arrive at the general formula for the value of an annuity payable, when out of m lives $(x_1, x_2, x_3, \ldots, x_m)$ there are alive at least r lives

$$a_{\overline{x_1:x_2:x_3:\ldots:x_m}}^{\;r} = \int_0^\infty \left\{ 1 + \sum_{k=0}^{\frac{m-r-1}{2}} \binom{r+2k-1}{2k} \sum (\mu_{x_1+t} + \mu_{x_2+t} + \ldots + \right.$$

$$\left. + \mu_{x_{m-r-2k}+t})\, a_{x_{m-r-2k+1}+t:x_{m-r-2k+2}+t:\ldots:x_m+t} \right\} e^{-\int_0^t (\Sigma\mu_{x_1+u}+\delta)du}\, dt -$$

$$-\int_0^\infty \left\{ \sum_{k=0}^{\frac{m-r}{2}-1} \binom{r+2k}{2k+1} \sum (\mu_{x_1+t} + \mu_{x_2+t} + \ldots + \right.$$

$$\left. + \mu_{x_{m-r-2k-1}+t})\, a_{x_{m-r-2k}+t:x_{m-r-2k+1}+t:\ldots:x_m+t} \right\} e^{-\int_0^t (\Sigma\mu_{x_1+u}+\delta)du}\, dt,$$

where k can be every integer from 0 to the greatest integer in the upper limit.

Généralisations des formules d'amortissement.

Josef Bílý.

Dans la théorie des opérations financières, il y a deux formules d'amortissement, par lesquelles on peut arriver de la valeur initiale du bilan B_0 après un nombre de n années à une valeur fixée B_n.

Pour la valeur balancée après r années depuis le commencement d'amortissement, on reçoit

a) en cas d'amortissement par décompte annuel $d = \frac{B_0 - B_n}{nB_0}$ de la valeur initiale de bilan B_0:

$$B_r = B_0\,(1 - rd) = \left(1 - \frac{r}{n}\right) B_0 + \frac{r}{n} B_n;$$

b) en cas d'amortissement par décompte annuel

$$d' = 1 - \sqrt[n]{\frac{B_n}{B_0}} \tag{1a}$$

de la valeur dernièrement balancée: