Ladislav Truksa

Hypergeometric orthogonal systems of polynomials. III

Aktuárské vědy, Vol. 2 (1931), No. 4, 177–203

Persistent URL: http://dml.cz/dmlcz/144552

Hypergeometric orthogonal systems of polynomials.

By Dr. *L. Truksa.*

Let n, m increase to ∞, so that the quotients

$$\frac{n+1}{n+m+2} = p, \quad \frac{m+1}{n+m+2} = q, \quad \frac{n+1}{m+1} = \frac{p}{q}. \tag{74}$$

remain finite. In this case it is

$$\begin{aligned}
&\lim_{n,\, m \to \infty} \varphi_\lambda(n, m, x) = \\
&= \lim \frac{\left(\frac{s-1}{m+1} q + 1 - \frac{x}{m+1\,\omega}, \lambda/\frac{1}{m+1}\right) \left(\overline{s-1}\, p - \lambda + 1 + \frac{x}{\omega}, \lambda\right)}{\left(\frac{p}{q}, \lambda/\frac{1}{m+1}\right) \left(1, \lambda/\frac{1}{m+1}\right) \left(1 + \frac{p}{q} + \frac{s-1}{m+1}, \lambda/\frac{1}{m+1}\right) \left(s - \lambda, \lambda\right)} \times \\
&\times \left(\frac{p}{q} + 1, 2\lambda/\frac{1}{m+1}\right) = \frac{\left(\overline{s-1}\, p - \lambda + 1 + \frac{x}{\omega}, \lambda\right) \left(\frac{p}{q} + 1\right)^{2\lambda}}{\left(\frac{p}{q}\right)^{\lambda} \left(1 + \frac{p}{q}\right)^{\lambda} \left(s - \lambda, \lambda\right)} = \\
&= \frac{\left(s - 1\, p - \lambda + 1 + \frac{x}{\omega}, \lambda\right)}{(s - \lambda, \lambda)\, p^{\lambda}}.
\end{aligned} \tag{75}$$

To deduce the limiting form of the function $\Phi_0(n, m, x)$ we use the well-known relation

$$\lim_{x \to \infty} \frac{\Gamma(x+h)}{\Gamma(x)\,.\,x^h} = 1, \qquad \text{or } \lim \frac{\Gamma(ax+\alpha)}{\Gamma(ax+\beta)} \backsim (ax)^{\alpha-\beta} \tag{76}$$

and obtain

$$\begin{aligned}
&\lim \Phi_0(n, m, \bar{x}) = \\
&= \lim \frac{\Gamma\left(n + 1 + p\,\overline{s-1} + \frac{x}{\omega}\right) \Gamma\left(\overline{s-1}\, q + m + 1 - \frac{x}{\omega}\right)}{\omega \Gamma(n+1)\, \Gamma(m+1)\, \Gamma\left(p\,\overline{s-1} + 1 + \frac{x}{\omega}\right)} \times \\
&\times \frac{\Gamma\left(\frac{n+1}{p}\right) \Gamma(s)}{\Gamma\left(q\,\overline{s-1} + 1 - \frac{x}{\omega}\right) \Gamma\left(\frac{n+1}{p} + s - 1\right)} =
\end{aligned} \tag{77}$$

$$= \frac{\Gamma(s)\, p^{p\,\overline{s-1}+x/\omega}\, q^{q\,\overline{s-1}-x/\omega}}{\omega\Gamma\left(p\,\overline{s-1}+1+\frac{x}{\omega}\right)\Gamma\left(\overline{s-1}\,q+1-\frac{x}{\omega}\right)}.$$

This function may be called the generalized Laplace frequency function, as we shall recognise later on from the corresponding limiting passage for $s\to\infty$, $\omega\to 0$. Essentially this is the binomial frequency function

$$\varkappa\begin{pmatrix} s-1 \\ p\,\overline{s-1}+\frac{x}{\omega}\end{pmatrix} p^{x/\omega}(1-p)^{\overline{s-1}-x/\omega}.$$

The function $\Phi_\lambda(n, m, x)$ is reduced to the function

$$\frac{\left(\overline{s-1}\,p-\lambda+1+\frac{x}{\omega},\lambda\right) p^{p\overline{s-1}+x/\omega-\lambda}\,\Gamma(s)\, q^{q\overline{s-1}-x/\omega}}{(s-\lambda,\lambda)\,\omega\,\Gamma\left(p\,\overline{s-1}+1+\frac{x}{\omega}\right)\Gamma\left(\overline{s-1}\,q+1-\frac{x}{\omega}\right)} =$$

$$= \frac{\Gamma(s-\lambda)\, p^{x/\omega-\lambda+p\,\overline{s-1}}\, q^{\overline{s-1}\,q-x/\omega}}{\omega\Gamma\left(\overline{s-1}\,p-\lambda+1+\frac{x}{\omega}\right)\Gamma\left(\overline{s-1}\,q+1-\frac{x}{\omega}\right)} \qquad (78)$$

in the limiting case under consideration.

b) The generalized Poisson function.

We obtain very important special cases of the degeneration of the function $\Phi_\lambda(m, n, x)$ using the expression $\Phi_\lambda(n, m, z)$

$$z = x+\frac{s-1}{2}\,\omega,$$

so that

$$\Phi_0(n, m, z) = \frac{\Gamma\left(n+1+\frac{z}{\omega}\right)\Gamma\left(s+m-\frac{z}{\omega}\right)\Gamma(s)\,\Gamma(n+m+2)}{\omega\Gamma(n+1)\,\Gamma(m+1)\,\Gamma\left(1+\frac{z}{\omega}\right)\Gamma\left(s-\frac{z}{\omega}\right)\Gamma(n+m+s+1)}$$

$$\varphi_\lambda(n, m, z) =$$

$$= \frac{\left(\overline{s-1}+m+1-\frac{z}{\omega},\lambda\right)\left(-\lambda+1+\frac{z}{\omega},\lambda\right)(n+m+2\lambda, 2\lambda)}{(n+1,\lambda)\,(m+1,\lambda)\,(n+m+s+1,\lambda)\,(s-\lambda,\lambda)}$$

and supposing, that the quotient $d = \dfrac{s-1}{n+m+2}$ remains constant, if s, m approaches infinity.

On this supposition

$$\lim \varphi_\lambda(m, n, z) =$$

$$= \lim \frac{\left(1 + d + \frac{n+1}{m+1} d - \frac{z}{\omega(m+1)}, \lambda / \frac{1}{m+1}\right) \left(-\lambda + 1 + \frac{z}{\omega}, \lambda\right)}{(n+1, \lambda) \left(1, \lambda / \frac{1}{m+1}\right) \left(\frac{n+1}{m+1} \overline{1+d} + 1 + d, \lambda / \frac{1}{m+1}\right)} \times$$

$$\times \frac{\left(1 + \frac{n+1}{m+1}, 2\lambda / \frac{1}{m+1}\right)}{\left(d + \frac{n+1}{m+1} d + \frac{1-\lambda}{m+1}, \lambda / \frac{1}{m+1}\right)} = \frac{\left(-\lambda + 1 + \frac{z}{\omega}, \lambda\right)}{(n+1, \lambda)\, d^\lambda} \tag{80}$$

$$\lim \Phi_0(n, m, z) = \lim \frac{\Gamma\left(n + 1 + \frac{z}{\omega}\right) \Gamma\left(\overline{m+1}\, \overline{1+d} + \overline{n+1}\, d - \frac{z}{\omega}\right)}{\omega \Gamma(m+1)\, \Gamma(n+1)\, \Gamma\left(1 + \frac{z}{\omega}\right)} \times$$

$$\times \frac{\Gamma(\overline{m+1}\, d + \overline{n+1}\, d + 1)\, \Gamma(m + 1 + n + 1)}{\Gamma\left(\overline{m+1}\, d + \overline{n+1}\, d + 1 - \frac{z}{\omega}\right) \Gamma(\overline{m+1}\ \overline{1+d} + \overline{n+1}\ \overline{1+d})}.$$

By applying the formula (76) we obtain the value

$$\frac{\Gamma\left(n + 1 + \frac{z}{\omega}\right)}{\omega \Gamma(n+1)\, \Gamma\left(1 + \frac{z}{\omega}\right)} (1+d)^{-n-1-\frac{z}{\omega}} d^{\frac{z}{\omega}} =$$

$$= \frac{\Gamma\left(n + 1 + \frac{z}{\omega}\right)}{\omega \Gamma(n+1)\, \Gamma\left(1 + \frac{z}{\omega}\right)} \left(\frac{d}{1+d}\right)^{\frac{z}{\omega}} \frac{1}{(1+d)^{n+1}} \tag{81}$$

for this limit. For the special value of the parameter d i. e. $d = 1/\varkappa\omega$ we may call the function (81) characteristic function corresponding to the orthogonal system of the generalized Laguerre and Kummer polynomials respectively or the generalized Pearson frequency curve of type III.

The interval, in which this function is defined, is infinite (o, ∞). For reasons which will be obvious below, we can denote the function (81) the enlarged Poisson function.

c) The Poisson function.

Putting

$$d = \frac{h}{n+1}$$

in the expressions (80) and (81) and computing the limiting value for $n \to \infty$, we obtain

$$\lim \varphi_\lambda(n, m, z) = h^{-\lambda}\left(1 - \lambda + \frac{z}{\omega}, \lambda\right)$$

$$\lim \Phi_0(n, m, z) = \frac{h^{\frac{z}{\omega}} e^{-h}}{\omega \Gamma\left(1 + \frac{z}{\omega}\right)}. \qquad (82)$$

Obviously the characteristic function $\Phi_0(z)$ is identical in the given case with the well-known Poisson frequency function for events very rarely ever occurring

$$\frac{h^x e^{-h}}{x!}.$$

The designation „enlarged Poisson function" which we have used above for the functions (81) and (83) is justified by this connection.

8. The limiting forms of the function Φ_λ in case that $s \to \infty$, $\omega \to 0$.

In part I the limiting form of the function $\Phi_\lambda(x)$ was deduced, if ω tends to zero and at the same time $s \to \infty$. Essentially we have obtained the Pearson frequency function of type I from the characteristic function $\Phi_0(x)$. Analogously the symmetric frequency function of Pearson of type II

$$\frac{\Gamma(2\overline{n + 1})}{2^{2n+1}\Gamma(n + 1)^2} (1 - x^2)^n \qquad (83)$$

corresponds to the simple specialisation of the parameters n, m quoted in section (6) under a) $n = m$. To function $\varphi_\lambda(n, m, x)$ corresponds the function

$$\frac{(2\overline{n + 1}, 2\lambda)}{(n + 1, \lambda)^2\, 2^{2\lambda}} (1 - x^2)^\lambda. \qquad (84)$$

By special choice

$$n = m = -\tfrac{1}{2}$$

we obtain the function

$$\frac{1}{\pi} : \sqrt{1 - x^2} \qquad (85)$$

from the characteristic function $\Phi_0(n, m, x)$ in the limiting case under consideration. From function $\varphi_\lambda(n, m, x)$ we obtain the function

$$\frac{(1, 2\lambda)}{2^{2\lambda} (\frac{1}{2}, \lambda)^2} (1 - x^2)^\lambda. \qquad (86)$$

Finally, if $n = m = 0$ the function φ_λ is reduced to the polynomial

$$\frac{(2, 2\lambda)}{2^{2\lambda}\,(1, \lambda)^2}\,(1 - x^2)^\lambda. \tag{87}$$

The characteristic function Φ_0 is in this case equal to $\frac{1}{2}$, as follows immediately from (83).

The limiting cases under consideration are especially interesting, if the degeneration of the function Φ_λ mentioned in section 7 under a) and b) is in question.

We approach first the investigation of the corresponding limiting value of the expression (75) and (77).

Let us suppose that

$$\omega = \frac{1}{\sqrt{s-1}}.$$

Then we can write the expression (77) in the form

$$\frac{(s-1)\sqrt{s-1}\,\Gamma(s-1)\,p^{p\overline{s-1}+x\sqrt{s-1}}\,q^{q\overline{s-1}-x\sqrt{s-1}}}{\Gamma(p\,\overline{s-1}+x\sqrt{s-1})\,\Gamma(q\,\overline{s-1}-x\sqrt{s-1})\,(\overline{s-1}\,p+x\sqrt{s-1})\,(\overline{s-1}\,q-x\sqrt{s-1})}. \tag{88}$$

We shall now determine the value $\Phi_0(n, m, x)$, if $s \to \infty$.

From the approximative relation valid for great n

$$\Gamma(an + \alpha\sqrt{n}) \backsim n^{an+\alpha\sqrt{n}}\,a^{an+\alpha\sqrt{n}}\,e^{-an}\sqrt{\frac{2\pi}{an}}\;e^{\frac{x^2}{2a}}$$

F. Eggenberger[31]) deduced the general relation

$$\frac{\Gamma(a_1 n + \alpha_1\sqrt{n})\,\Gamma(a_2 n + \alpha_2\sqrt{n})\ldots\Gamma(a_k n + \alpha_k\sqrt{n})}{\Gamma(b_1 n + \beta_1\sqrt{n})\,\Gamma(b_2 n + \beta_2\sqrt{n})\ldots\Gamma(b_l n + \beta_l\sqrt{n})} \backsim$$
$$\backsim \left(\frac{a_1^{\,a_1}\ldots a_k^{\,a_k}}{b_1^{\,b_1}\ldots b_l^{\,b_l}}\right)^n \left(\frac{a_1^{\,\alpha_1}\ldots a_k^{\,\alpha_k}}{b_1^{\,\beta_1}\ldots b_l^{\,\beta_l}}\right)^{\sqrt{n}} \left(\frac{2\pi}{n}\right)^{\frac{1}{2}(k-l)} \times \tag{89}$$
$$\times \sqrt{\frac{b_1\ldots b_l}{a_1\ldots a_k}}\; e^{\sum\limits_{\gamma=1}^{k}\frac{\alpha_\gamma^2}{2a_\gamma} - \sum\limits_{\gamma=1}^{l}\frac{\beta_\gamma^2}{2b_\gamma}}$$

valid for great n, if a_i, b_i are positive, and α_i, β_i arbitrary constants. By applying this relation to the expression (88) we obtain

$$\Phi_0(n, m, x) \backsim \frac{(s-1)^{3/2}\,p^{p\overline{s-1}+x\sqrt{s-1}}\,q^{q\overline{s-1}-x\sqrt{s-1}}}{(\overline{s-1}\,p + x\sqrt{s-1})\,(\overline{s-1}\,q - x\sqrt{s-1})} \times$$
$$\times \left(\frac{1}{p^p\,q^q}\right)^{s-1} \left(\frac{1}{p^x\,q^{-x}}\right)^{\sqrt{s-1}} \left(\frac{2\pi}{s-1}\right)^{-1/2} \sqrt{pq}\;e^{-\frac{x^2}{2pq}}$$

[31]) Cp. loc. cit.

whence

$$\lim \Phi_0(n, m, x) = \frac{1}{\sqrt{2\pi pq}} e^{-\frac{x^2}{2pq}}. \tag{90}$$

In this way we have obtained the normal Laplace-Gauss frequency function.

The limiting value of the function $\varphi_\lambda(x)$ is given by the expression

$$\lim \varphi_\lambda(n, m, x) = \lim \frac{\left(p - \frac{\lambda - 1}{s - 1} + \frac{x}{\sqrt{s-1}}, \lambda / \frac{1}{\sqrt{s-1}}\right)}{p^\lambda \left(1 - \frac{\lambda - 1}{s - 1}, \lambda / \frac{1}{\sqrt{s-1}}\right)} = 1. \tag{91}$$

We obtain a result somewhat more general, if we do not perform the limiting passage of the parameters n, m, s successively, but all at once.

For this purpose let us denote the quotient

$$\frac{s-1}{n+m+2} = d = \frac{s-1}{m+1} q = \frac{s-1}{n+1} p$$

and let us suppose, that this quotient remains finite, if s, n, m increase to infinity.

The corresponding limiting form of the characteristic function

$$\Phi_0(n, m, x) =$$

$$= \frac{\sqrt{s-1}\, \Gamma\left(\overline{s-1} \frac{p}{d} + \overline{s-1}\, p + x\sqrt{s-1}\right) \Gamma\left(\overline{s-1}\left(q + \frac{q}{d}\right) - x\sqrt{s-1}\right)}{\Gamma\left(\overline{s-1} \frac{p}{d}\right) \Gamma\left(\overline{s-1} \frac{q}{d}\right) \Gamma(\overline{s-1}\, p + x\sqrt{s-1})\, \Gamma(\overline{s-1}\, q - x\sqrt{s-1})} \times$$

$$\times \frac{\Gamma\left(\frac{s-1}{d}\right)(s-1)\,\Gamma(s-1)}{\Gamma\left(\frac{s-1}{d} + s - 1\right) . (p\overline{s-1} + x\sqrt{s-1}) . (q\,\overline{s-1} - x\sqrt{s-1})} \tag{92}$$

was deduced by K. Eggenberger in the paper „Wahrscheinlichkeits-ansteckung" by application of relation (89) i. e.

$$\Phi_0(n, m, x) \sim \frac{(s-1)^{3/2} \left(\frac{2\pi}{s-1}\right)^{-1/2} \sqrt{\frac{pq}{d+1}}}{(p\,\overline{s-1} + x\sqrt{s-1})(q\,\overline{s-1} - x\sqrt{s-1})} \times$$

$$\times\, e^{\frac{x^2}{2}\left(\frac{d}{(1+d)pq} - \frac{1}{pq}\right)}$$

whence

$$\lim \Phi_0(n, m, x) = \frac{1}{\sqrt{2\pi p.q\,(1+d)}}\, e^{-\frac{x^2}{2pq(1+d)}}.$$

If $d = 0$, this value coincides with expression (90). The function $\varphi_\lambda(x)$ is reduced again to the constant equal to 1 in the limit.

Finally we shall investigate the limiting process under consideration for functions (80) and (81).

We put

$$d = \frac{1}{\varkappa\omega}$$

whence

$$\lim \varphi_\lambda(n, m, z) = \lim_{\omega\to 0} \frac{\left(1 - \lambda + \frac{z}{\omega}, \lambda\right) \omega^\lambda \varkappa^\lambda}{(n+1, \lambda)} = \frac{z^\lambda \varkappa^\lambda}{(n+1, \lambda)} \tag{93}$$

and

$$\lim \Phi_0(n, m, z) =$$

$$= \lim \frac{\Gamma\left(n + 1 + \frac{z}{\omega}\right)}{\omega\,\Gamma(n+1)\,\Gamma\left(1 + \frac{z}{\omega}\right)(1+\varkappa\omega)^{\frac{z}{\omega}}\left(1 + \frac{1}{\varkappa\omega}\right)^{n+1}} = \frac{z^n e^{-z\varkappa} \varkappa^{n+1}}{\Gamma(n+1)}. \tag{94}$$

Thus we obtained the characteristic function of the orthogonal system of Kummer and Laguerre polynomials respectively from (81).

From the characteristic function $\Phi_0(n, m, x)$, it would be possible to deduce further special cases the investigation of which is, however, outside the scope of this paper.

PART III.

Generalized Legendre polynomials. Legendre polynomials. Tchebyshef's polynomials. Polynomials corresponding to the binomial frequency function. Hermite polynomials. Generalized Kummer polynomials. Kummer-Laguerre polynomials. Charlier-Jordan polynomials.

To every special case of the characteristic function $\Phi_0(n, m, x)$ deduced in part II corresponds a certain system of orthogonal polynomials either of summation or of integration. For their deduction we can always use either the fundamental definition of Jacobi's generalized polynomials expressed by formula (6) or functional equation (17). It is sufficient to perform the necessary specialisation of the parameters or the corresponding limiting process. Also the fundamental properties of these special polynomials follow from the analogous properties of Jacobi's generalized polynomials. We shall, therefore, confine ourselves to a brief summary of the most important results valid for a certain system of polynomials and be more specific about the systems of summation which, on the whole, are less known, than about the systems of integration deduced from them. We shall point to some applications and the most important literature of the subject as well.

1. The generalized Legendre polynomials.

This simplest system of orthogonal polynomials—which we shall denote $\mathfrak{P}_\lambda(x)$ — corresponds to the characteristic function

$$\Phi_0(x) = \text{const.}$$

and to the finite interval of summation $(\pm \frac{s-1}{2}\omega)$. It fulfils the orthogonal relation

$$\sum_{-a}^{a} \mathfrak{P}_\lambda(x)\, \mathfrak{P}_\mu(x) = 0, \;\; \lambda \gtrless \mu.$$

From the fundamental relations valid for Jacobi's generalized polynomials follow by a simple specialisation of the parameters

$$n = 0, \; m = 0,$$

quite analogous relations for the polynomials $\mathfrak{P}_\lambda(x)$. So we obtain e. g. from the equation (6), (6') and (8):

$$\mathfrak{P}_\lambda(x) = \frac{(1,\lambda)}{2^\lambda} \mathop{\Delta^\lambda}_{\omega} F_\lambda\left(\frac{s-1}{2}\omega + x\right) F_\lambda\left(\frac{s-1}{2}\omega + \lambda\omega - x\right) =$$
$$= \frac{(-)^\lambda}{(1,\lambda)\,2^\lambda} \mathop{\Delta^\lambda}_{\omega}\left(x + \frac{s-1}{2}\omega, \lambda/\omega\right)\left(x - \frac{s-1}{2}\omega, \lambda/\omega\right)^{32)} = (-)^\lambda \mathfrak{P}_\lambda(-x) =$$
$$= \frac{1}{2^\lambda}\sum_{k=0}^{\lambda}(-)^k \frac{(1,\lambda)}{(1,\lambda-k)}\cdot\frac{(\lambda+1,k)}{(1,k)}\binom{\frac{s-1}{2}+\frac{x}{\omega}}{k}\frac{}{(\overline{s-\lambda}\,\omega, \overline{\lambda-k}/\omega)\omega^k} \tag{95}$$

In the summation interval $(0, \overline{s-1\omega})$, the polynomial $\mathfrak{P}_\lambda(x)$ is expressed by

$$\mathfrak{P}_\lambda(z) = \frac{(1,\lambda)(-\omega)^\lambda}{2^\lambda}\sum_{k=0}^{\lambda}\binom{\lambda+k}{k}\binom{\lambda-s}{\lambda-k}\binom{\frac{z}{\omega}}{k}.^{33)} \tag{95'}$$

Using the development

$$\binom{x+a}{k} = \sum_{i=0}^{k}\binom{x-x_0}{i}\binom{x_0+a}{k-i}$$

---

[32]) Essentially Tchebyshef's expression in the paper: „Sur l'interpolation par la méthode de moindres carrés." Oeuvres I.

[33]) Cp. e. g. Jordan, Statistique mathématique, p. 27.

we can write the value $\left(\frac{x}{\omega} + \frac{s+1}{2}, \lambda - i\right)$ in the form

$$\left(\frac{x}{\omega} + \frac{s+1}{2}, \lambda - i\right) = \overline{\lambda - i}\,! \sum_{k=0}^{\lambda-i} \binom{\frac{x}{\omega} - \frac{s+1}{2} - i}{\lambda - i - k} \binom{s+\lambda}{k}$$

and substitute this value into the formula (6′). We obtain

$$\mathfrak{P}_\lambda(x) = \frac{(-)^\lambda \lambda!\, \omega^\lambda}{2^\lambda} \sum_{i=0}^{\lambda} \sum_{k=0}^{\lambda-i} \binom{\lambda}{\lambda - i} \binom{\lambda - k}{i} \binom{s+\lambda}{k} \binom{\frac{x}{\omega} - \frac{s+1}{2}}{\lambda - k} =$$

$$= \frac{(-\omega)^\lambda \lambda!}{2^\lambda} \sum_{k=0}^{\lambda} \sum_{i=0}^{\lambda-k} \binom{\lambda}{\lambda - i} \binom{\lambda - k}{i} \binom{s+\lambda}{k} \binom{\frac{x}{\omega} - \frac{s+1}{2}}{\lambda - k} =$$

$$= \frac{(-\omega)^\lambda \lambda!}{2^\lambda} \sum_{k=0}^{\lambda} \binom{2\lambda - k}{\lambda - k} \binom{s+\lambda}{k} \binom{\frac{x}{\omega} - \frac{s+1}{2}}{\lambda - k} =$$

$$= \frac{(-\omega)^\lambda \lambda!}{2^\lambda} \sum_{k=0}^{\lambda} \binom{\lambda + k}{k} \binom{s+\lambda}{\lambda - k} \binom{\frac{x}{\omega} - \frac{s+1}{2}}{k}.$$

In the interval $(\omega, s\omega)$ this expression assumes the form

$$\mathfrak{P}_\lambda(\zeta) = \frac{(-\omega)^\lambda \lambda!}{2^\lambda} \sum_{k=0}^{\lambda} \binom{\lambda + k}{k} \binom{s+\lambda}{\lambda - k} \binom{\frac{\zeta}{\omega} - s - 1}{k}. \qquad (95'')$$

[34]

The functional equation (17) is considerably simplified, if the polynomials $\mathfrak{P}_\lambda(x)$ are under consideration:

$$(\lambda + 1)\, \mathfrak{P}_{\lambda+1}(x) + (2\lambda + 1)\, x \mathfrak{P}_\lambda(x) + \frac{s^2 - \lambda^2}{4} \lambda \omega^2\, \mathfrak{P}_{\lambda-1}(x) = 0. \qquad (96)$$

Also the hypergeometric difference equation (23) assumes a simple form in this special case:

$$\left(x + \frac{s+3}{2}\omega\right)\left(\frac{s-3}{2}\omega - x\right) \underset{\omega}{\Delta^2}\, \mathfrak{P}_\lambda(x) - (2x - \overline{\lambda^2 + \lambda - 2}\,\omega) \times$$

$$\times \underset{\omega}{\Delta} \mathfrak{P}_\lambda(x) + \lambda(\lambda + 1)\, \mathfrak{P}_\lambda(x) = 0. \qquad (97)$$

---

[34]) Cp. e. g. Esscher, On graduation according to the method ef least squares by means of certain polynomials.

For the sum $I_\lambda = \sum_{-a}^{a} \mathfrak{P}_\lambda^2(x)\omega$ we obtain from (14′) the expression

$$I_\lambda = \frac{(1,\lambda)\,(\lambda+1,\lambda)}{2^{2\lambda}} \binom{s+\lambda}{2\lambda+1} \omega^{2\lambda+1} = \frac{(s^2-\lambda^2)(s^2-\overline{\lambda-1}^2)\ldots(s^2-1^2)\,s}{2^{2\lambda}(2\lambda+1)}\,\omega^{2\lambda+1} \tag{98}$$

whence follows the value of the factor:

$$\varkappa_\lambda = \frac{2^\lambda}{\omega^\lambda} \sqrt{\frac{2\lambda+1}{(s^2-\lambda^2)\,(s^2-\overline{\lambda-1}^2)\ldots(s^2-1^2)}}$$

which reduces $\mathfrak{P}_\lambda(x)$ to the normal form $\mathring{\mathfrak{P}}_\lambda(x)$ satisfying the relation:

$$\sum_{-a}^{a} \mathring{\mathfrak{P}}_\lambda^2(x)\,\omega = s\omega.$$

Tchebyshef[35]) deduced these polynomials from the development of the sum:

$$\frac{1}{z-1} + \frac{1}{z-2} + \ldots + \frac{1}{z-s}$$

in a continuous fraction as the denominator of the successive approximate values of this fraction. With regard to (25) the continuous fraction under consideration which satisfies the definition (95) of the polynomials $\mathfrak{P}_\lambda(x)$ is given by the expression:

$$\frac{\frac{1}{2}}{|-x} - \frac{\frac{s^2-1}{8}\,\omega^2|}{|-\frac{3}{2}x} - \frac{2\,\frac{s^2-2^2}{4\,.\,3}\,\omega^2|}{|-\frac{5}{3}x} - \ldots \tag{99}$$

Also the generalized Legendre polynomials of the second kind are defined by this fraction — let us denote them $Q_\lambda(x)$ — as numerators of the successive approximate values of the fraction. Otherwise they could be deduced in regard to (32) from the function

$$Q_\lambda(x) = \frac{\omega}{2} \sum_{-a}^{a} \frac{\mathfrak{P}_\lambda(y)}{x-y}$$

by a process described in part I (8). They, of course, satisfy the functional equation (96), which defines them completely, if we know the initial members

$$Q_1(x) = \tfrac{1}{2}, \qquad Q_2(x) = -\tfrac{3}{4}\,x.$$

As to the use of polynomials in the Gauss summation I refer to my article: „Zobecněné polynomy Legendrovy, užití jich v numerické sumaci“ (The generalized Legendre polynomials and their application in numerical summation:[36])

---

[35]) Sur les fractions continues, 1855, Oeuvres I.
[36]) Čas. mat. fys., Praha, 1927.

Of all the known summation systems of orthogonal polynomials these as the simplest became most widely used in the approximation of functions numerically given. This application concerns essentially the following problem:

We have to express the function y_x given by the values $y_1, y_2, \ldots y_s$ approximately by the series

$$a_0 + a_1 \mathfrak{P}_1(x) + a_2 \mathfrak{P}_2(x) + \ldots + a_\lambda \mathfrak{P}_\lambda(x), \quad \lambda < s \tag{100}$$

of polynomials $\mathfrak{P}_\lambda(x)$, where the coefficients a_i are to be determined either by the method of least squares or by the method of moments. In the given case the coefficient a_i is expressed by value

$$a_i = \frac{\sum_{-a}^{a} \mathfrak{P}_i(x)\, y(x)\, \omega}{\sum_{-a}^{a} \mathfrak{P}_i^2(x)\, \omega} \tag{101}$$

whatever method quoted above we are using.

In applying the formula (100) we meet with the greatest difficulties when computing these coefficients, especially if the number of the given values s is large. Several authors have tried to find a transformation of the expression (101) in order to simplify the calculation, but only F. Esscher and K. Jordan have suggested the most convenient method.

F. Esscher, whose method we are quoting here, used very conveniently the normal form of the polynomials $\mathring{\mathfrak{P}}_\lambda(x)$ corresponding to the interval $(1, s)$:

$$\mathring{\mathfrak{P}}_\lambda(\zeta) = \frac{(-)^\lambda \lambda!}{2^\lambda} \varkappa_\lambda \sum_{k=0}^{\lambda} \binom{\lambda + k}{k} \binom{s + \lambda}{\lambda - k} \binom{\zeta - s - 1}{k}.$$

The coefficient

$$a_i = \frac{1}{s} \sum_{1}^{s} \mathring{\mathfrak{P}}_\lambda(\zeta)\, y_\zeta.$$

corresponds to this expression.

If we denote the sum

$$S_0(s) = y_1 + y_2 + \ldots + y_s$$
$$S_1(s) = S_0(1) + S_0(2) + \ldots + S_0(s) \text{ etc.}$$

the relation

$$S_\nu(s) = (-)^\nu \sum_{x=1}^{s} \binom{x - s - 1}{\nu} y_x \tag{102}$$

holds good generally for the sum $S_\nu(s)$.

The values S_ν, which are the binomial moments of the function y_x, increase numerically very rapidly with the increasing s and render the performance of the calculation difficult. For this reason F. Esscher introduced the average moments

$$\overline{S}_\nu(s) = S_\nu(s) : \binom{s+\nu}{\nu+1}$$

The factor $1 : \binom{s+\nu}{\nu+1}$ corresponds to the value of the sum (102), if $y_x = 1$.

If we apply the average binomial moments $\overline{S}_\nu(s)$ the coefficient a_ν assumes the form

$$(-)^\nu a_\nu = \beta_{\nu,s}(\overline{S}_0 + b_{\nu 1}\overline{S}_1 + \ldots + b_{\nu\nu}\overline{S}_\nu), \tag{101'}$$

where

$$\beta_{\nu,s} = \sqrt{(2\nu+1)\,\frac{(s+1)(s+2)\ldots(s+\nu)}{(s-1)(s-2)\ldots(s-\nu)}}$$

and

$$b_{\nu,\mu} = \frac{(-)^\mu\,\nu!\binom{\nu+\mu}{\mu}}{\overline{\nu-\mu}!\,\overline{\mu+1}!}$$

The tabulation of the values $1 : \binom{s+\nu}{\nu+1}$ and $\beta_{\nu,s}$ simplifies considerably the calculation of a_i. The numbers $b_{\nu\mu}$ are smaller integers, the calculation of which does not offer any special difficulties. Let us quote a few initial values of a_i in the transformation (101'):

$$\begin{aligned}
a_0 &= \overline{S}_0 \\
-a_1 &= \beta_1(\overline{S}_0 - \overline{S}_1) \\
a_2 &= \beta_2(\overline{S}_0 - 3\overline{S}_1 + 2\overline{S}_2) \\
-a_3 &= \beta_3(\overline{S}_0 - 6\overline{S}_1 + 10\overline{S}_2 - 5\overline{S}_3) \\
a_4 &= \beta_4(\overline{S}_0 - 10\overline{S}_1 + 30\overline{S}_2 - 35\overline{S}_3 + 14\overline{S}_4) \\
-a_5 &= \beta_5(\overline{S}_0 - 15\overline{S}_1 + 70\overline{S}_2 - 140\overline{S}_3 + 126\overline{S}_4 - 42\overline{S}_5)
\end{aligned}$$

Knowing the coefficients a_i we approach first the calculation of the square of the standard deviation from the formula (30):

$$\sigma_\lambda^2 = \frac{1}{s}\,\Sigma y_x^2 - a_0^2 - a_1^2 - \ldots - a_\lambda^2.$$

In this manner, we obtain a basis for the solution of the problem, how many members have to be kept in the series (100), in order to obtain a satisfactory result.

For a systematic calculation of all the values of y_x we can use e. g. the values of the differences

$$\Delta^\nu y(s+1) = \frac{s}{\binom{s+\nu}{\nu+1}} \sum_{i=\nu}^{\lambda} b_{i\nu}\, a_i\, \beta_i (-)^i.$$

By adding them up successively we obtain y_x for all x. The detailed proceeding is discussed in the paper of Esscher and Jordan quoted above.

As we have already mentioned in the introduction, the polynomials $\mathfrak{P}_\lambda(x)$ were deduced by Tchebyshef; he discussed them in a whole series of papers, of which we may quote in particular: Sur les fractions continues, 1855, Sur une nouvelle série, 1858, Sur l'interpolation par la méthode de moindres carrés, 1859, Sur l'interpolation, 1864, Sur l'interpolation des valeurs équidistantes, 1875 (Oeuvres I, 11, 18, 23, II, 12). J. P. Gram deduced a very general system of orthogonal polynomials, a particular case of which are the polynomials $\mathfrak{P}_\lambda(x)$, in a very original manner in the paper „On expansions in series folllowing from the method of least squares"[37]) 1879. Gram discussed the application of the polynomials $\mathfrak{P}_\lambda(x)$ in the theory of graduation in the paper „Ueber partielle Angleichung mittels Orthogonalfunktionen" (Mitteilungen der Vereinigung schweiz. Versicherungsmathematiker, 1915). There is a very good survey of Tchebyshef's and Gram's work in R. Radau's „Etudes sur les formules d'interpolation" 1891. Among more recent papers we have to refer to Jordan's study: „Sur une série de polynomes dont chaque somme partielle représente la meilleure approximation d'un degré donné suivant la méthode des moindres carrés." (Proceedings of the London Math. Society. Vol. 20, p. 1921.)

We may quote the following publications from the rest of the literature of the subject as far as we have not referred to them in their proper places:

V. Pareto, Tables pour faciliter l'application de la méthode de moindres carrés, Journal de statistique suisse, 1899.

K. Petr, O interpolaci, 1903 (Výroční zpráva II. čes. st. gymnasia v Brně).

H. Poincaré, Calcul des probabilités (Chap. XV. Théorie d'interpolation), Paris, 1912.

A. Quiquet, Sur une méthode d'interpolation exposée par H. Poincaré et sur une application possible aux fonctions de survie d'ordre n, Bull. trim. Nro 91, 1912.

F. Esscher, Ueber die Sterblichkeit in Schweden 1886—1914, Meddelanden från Lunds Astron. Observatorium, Ser. II, Nro 23, 1920.

R. A. Fisher, Studies in Group Variation, Journal of Agric. Science, Vol. XI, part. II, 1921.

K. Pearson, On a general Method of Determining the Successive Terms in a Skew Regression Line, Biometrika XIII, 1923.

R. Risser, A propos de la méthode d'ajustement de Tchebycheff, Bull. trim. Nro 125, 1926.

M. Fréchet, P. Perenoud, M. Mahrer, Sur l'ajustement des tables de mortalité à l'aide de la méthode de Tchebicheff, Bull. trim. Nro 126, 1926.

L. Isserlis, Note on Tchebyshef's Interpolation Formula, Biom. XIX, 1927.

P. Lorenz, Der Trend, Vierteljahrshefte zur Konjunktur-Forschung, Sonderheft 9, Berlin 1928.

A. Sipos, Praktische Anwendung der Trendberechnungsmethode von Jordan. Mitteilungen der ungar. Landeskommission für Wirtschaftsstatistik und Konjunkturforschung, 1930.

Sipos-Jordan, Report on the practical application of Jordan's method for trend measurement, submitted by the Hungarian national committee for economic statistics to the second European economic conference held at Berlin in 1930.

---

[37]) Published in an enlarged version in the „Journal f. d. reine u. angew. Mathematik" under the title: „Ueber die Entwickelung reeller Funktionen in Reihen mittels der Methode der kleinsten Quadrate."

2. The Legendre polynomials.

We obtain the fundamental properties of this simplest system of orthogonal polynomials of the characteristic function $\Phi_0(x)=1$ — let us denote them $P_\lambda(x)$ — either from the analogous properties of the polynomials $\mathfrak{P}_\lambda(x)$ by the limiting process $s \to \infty$, $\omega \to 0$, $s\omega =$ const., or by a simple specialisation of the parameters

$$n = 0, \ m = 0,$$

from the corresponding relations valid for Jacobi's polynomials. From the equation (47) and (95) respectively follows their definition in the interval (± 1)

$$P_\lambda(x) = \frac{1}{2^\lambda \lambda!} \frac{d^\lambda}{dx^\lambda}(1 - x^2)^\lambda = (-)^\lambda P_\lambda(-x) =$$

$$= \frac{1}{2^\lambda} \sum_{k=0}^{\lambda} (-)^k \frac{(1, \lambda)}{(1, \lambda - k)} \frac{(\lambda + 1, k)}{(1, k)} \binom{1 + x}{k} 2^{\lambda - k}$$

The functional equation (17) and (96) respectively is reduced to the simple form

$$(\lambda + 1) P_{\lambda+1}(x) + (2\lambda + 1) x P_\lambda(x) + \lambda P_{\lambda-1}(x) = 0,$$

the difference equation (97) changes into the differential equation

$$(1 - x^2) P''_\lambda(x) - 2xP'_\lambda(x) + \lambda(\lambda + 1) P_\lambda(x) = 0.$$

The factor $\varkappa_\lambda$, which transforms $P_\lambda(x)$ into a normal form $\mathring{P}_\lambda(x)$ can be deduced from the limiting value of the sum (98)

$$J_\lambda = \frac{2}{2\lambda + 1} = \int_{-1}^{1} P_\lambda^2(x) dx$$

whence

$$\varkappa_\lambda = \sqrt{\frac{2\lambda + 1}{2}}.$$

From the continuous fraction (99) we obtain

$$\frac{\frac{1}{2}}{|-x} - \frac{\frac{1}{2}|}{|-\frac{3}{2}x} - \ldots,$$

n the limit, which defines not only the polynomials $P_\lambda(x)$, but also the corresponding polynomials of the second kind $Q_\lambda(x)$. The solution of the well-known problem of Gauss's mechanical quadrature (numerical integration) is expressed just by both kinds of Legendre polynomials. The corresponding formula follows in the limit from (37″) in the form

$$\int_{-1}^{1} p(x)\, dx = -2 \sum_{k=1}^{\lambda} \frac{Q_\lambda(x_k)}{P'_\lambda(x_k)} p(x_k) + R_{2\lambda}.$$

In applying the Legendre polynomials to the approximate expression of arbitrary functions by a series of polynomials it is not necessary to confine oneself to a finite number of members of the series

$$y = a_0 P_0 + a_1 P_1 + a_2 P_2 + \dots \tag{103}$$

as with the generalized Legendre polynomials.

If the number of members of this series grows to infinity, the question arises, what are the conditions of the convergence of this series with regard to the function y and how to evaluate the rest of the series, if we keep only the finite number of members. The coefficients a_i are expressed by the value

$$a_i = \frac{2\lambda + 1}{2} \int_{-1}^{1} y(x)\, P_i(x)\, dx,$$

From the recent literature on the series (103)[38]) I mention especially M. H. Stone's paper: „Developments in Legendre polynomials"[39]), which is, on the whole, very accessible in its mathematics.

3. Tchebyshef's polynomials.

From the other systems of orthogonal polynomials corresponding to the symmetrical form of the characteristic function $\Phi_0(x)$ we may, moreover, quote the case in which

$$n = m = -\tfrac{1}{2}.$$

The corresponding summation system of the polynomials $\mathfrak{T}_\lambda(x)$ of the interval $\pm \frac{1}{2}(s-1)\,\omega$ is defined by the relation

$$\mathfrak{T}_\lambda(x) = \frac{(\frac{1}{2}, \lambda)}{2^\lambda} \frac{F_{2\lambda}(\overline{s+\lambda-1}\,\omega)}{\Phi_0(-\frac{1}{2}, -\frac{1}{2}, x)} \mathop{\Delta^\lambda}_{\omega} \Phi_\lambda(-\tfrac{1}{2}, -\tfrac{1}{2}, x) =$$

$$= \frac{(-)^\lambda}{2^\lambda} \sum_{k=0}^{\lambda} (-)^k \binom{\lambda}{k} \frac{(\lambda, \lambda-k)}{(\frac{1}{2}, \lambda-k)} (x + \frac{s-1}{2}\,\omega, -\overline{\lambda - k/\omega})(\overline{s-\lambda}\,\omega, k/w) =$$

$$= \left(\frac{\omega}{2}\right)^\lambda \frac{\Gamma(s+\lambda)}{\Gamma(s)} F(\lambda, -\lambda, \frac{x}{\omega} + \frac{s}{2};\ \tfrac{1}{2}, s). \tag{104}$$

We refer to this system of polynomials, the principal properties of which follow from the results deduced for Jacobi's generalized polynomials directly by an substitution quoted above, because of the special importance of the corresponding integral system of polynomials $T_\lambda(x)$. It was especially Tchebyshef who discussed these polynomials.

---

[38]) From the earlier literature especially H. E. Heine: Handbuch der Kugelfunktionen, Berlin 1878.

[39]) Ann. of Math., Second Ser., Vol. 27.

Using the relations (6) and (8) respectively, we can define the polynomials of Tchebyshef either by the expression

$$T_\lambda(x) = \frac{\sqrt{1-x^2}}{2^\lambda\,(\frac{1}{2},\lambda)}\frac{d^\lambda}{dx^\lambda}(1-x^2)^{\lambda-\frac{1}{2}} =$$
$$= \frac{(-)^\lambda}{2^\lambda}\sum_{k=0}^{\lambda}(-)^k\binom{\lambda}{k}\frac{(\lambda,\lambda-k)}{(\frac{1}{2},\lambda-k)}(1+x)^{\lambda-k}\,2^k, \tag{105}$$

or from (25) as the denominator of the successive approximate values of the continuous fraction

$$\frac{\frac{1}{2}}{|-x} - \frac{1|}{|-2x} - \frac{1|}{|-2x} - \cdots$$

They satisfy the recurrence relation

$$T_{\lambda+1}(x) + 2xT_\lambda(x) + T_{\lambda-1}(x) = 0 \tag{106}$$

and the differential equation

$$(1-x^2)\,T''_\lambda(x) - xT'_\lambda(x) + \lambda^2 T_\lambda(x) = 0. \tag{107}$$

We deduce an important property of the polynomials $T_\lambda(x)$ investigated by Tchebyshef[40]) while computing the integral

$$\sigma = \int_{-1}^{1} p(x)\,\Phi_0(x)\,dx = -2\sum_{k=1}^{\lambda}\frac{Q_\lambda(x_k)}{T'_\lambda(x_k)}\,p(x_k) + R_{2\lambda}$$

analogous to the formula (37″). We can prove namely, that the ratio

$$\frac{Q_\lambda(x)}{T'_\lambda(x)} = -\frac{1}{2\lambda} \tag{108}$$

is constant, so that

$$\sigma = \int_{-1}^{1} p(x)\,\Phi_0(x)\,dx = \frac{1}{\lambda}\sum_{k=1}^{\lambda} p(x_k) + R_{2\lambda}. \tag{109}$$

To prove this we use both the recurrence equation (106) valid also for the corresponding polynomials of the second kind $Q_\lambda(x)$, and the relation

$$T_\lambda(x) = \frac{x}{\lambda}\,T'_\lambda(x) + \frac{1}{\lambda-1}\,T'_{\lambda-1}(x), \tag{110}$$

the validity of which can be checked in an easy way. From the expression (105) follows

$$T'_\lambda(x) = \frac{(-)^\lambda}{2^\lambda}\sum_{k=0}^{\lambda}(-)^k\binom{\lambda}{k}\frac{(\lambda,\lambda-k)}{(\frac{1}{2},\lambda-k)}(\lambda-k)\,(1+x)^{\lambda-k-1}\,2^k$$

---

[40]) Sur les quadratures, Journal de Mat. pures et appl. 1874.

and the analogous expression for $T'_{\lambda-1}(x)$. Having substituted these values into the equation (110), which can be written also in the form

$$\lambda(\lambda-1)\,T_\lambda(x) = (1+x)\,(\lambda-1)\,T'_\lambda(x) - (\lambda-1)\,T'_\lambda(x) + \lambda\,T'_{\lambda-1}(x)$$

and having compared the coefficients of the member $(1+x)^k$ on both sides of the equation, we see, that these coefficients are identical.

Since the relation (108) is valid for $\lambda = 1\ 2$, as follows from a direct substitution of the corresponding values

$$\begin{aligned} T_1(x) &= -x & Q_1 &= \tfrac{1}{2} \\ T_2(x) &= 2x^2 - 1 & Q_2 &= -x \\ T_3(x) &= -4x^3 + 3x & Q_3 &= 2x^2 - \tfrac{1}{2}, \end{aligned}$$

it is sufficient for the proof of their general validity to show, that from the supposition of their correctness for the index λ follows necessarily also the validity for the index $\lambda+1$. By differentiating (106) we obtain

$$T'_{\lambda+1}(x) = -2\,T_\lambda(x) - 2x\,T'_\lambda(x) - T'_{\lambda-1}(x).$$

Using the formula (108) and (110) we can write this expression in the form

$$T'_{\lambda+1}(x) = -2\frac{x}{\lambda}\,T'_\lambda(x) - \frac{2}{\lambda-1}\,T'_{\lambda-1}(x) + 4x\lambda Q_\lambda(x) +$$
$$+ 2\overline{\lambda-1}\,Q_{\lambda-1}(x) = -(2\lambda+2)\,Q_{\lambda+1}(x),$$

whereby the validity of the relation (108) is proved.

Special importance must be ascribed to the polynomials $T_\lambda(x)$ in the expression of arbitrary functions by a series of polynomials by application of the method of the best approximation, which may be only mentioned in the course of this paper.

The literature on the polynomials $T_\lambda(x)$ is already very extensive. Of the publications confined to the investigation whether x is a real variable, we may quote at least the following:

Tchebychef, Sur les questions de minima qui se rattachent à la représentation approximative des fonctions (Oeuvres I), 1859.
Tchebychef, Sur les fonctions qui s'écartent peu de zéro (Oeuvres II).
P. Kirchenberger, Ueber Tchebychefsche Annäherungsmethoden, 1902.
De la Vallée Poussin, Leçons sur l'approximations des fonctions d'une variable réelle.
S. Bernstein, Leçons sur les propriétés extrémales des fonctions analytiques.

4. The generalized Hermite polynomials.

To the characteristic function (77)

$$\Phi_0(x) = \frac{\Gamma(s)\,p^{p\overline{s-1}+\frac{x}{\omega}}\,q^{\overline{s-1}\,q-\frac{x}{\omega}}}{\omega\Gamma\left(p\overline{s-1}+1+\frac{x}{\omega}\right)\Gamma\left(\overline{s-1}\,q+1-\frac{x}{\omega}\right)}$$

corresponds in the interval $(-p\,\overline{s-1}\,\omega,\ q\,\overline{s-1}\,\omega)$ the summation system of the polynomials $\mathfrak{H}_\lambda(x)$ defined by the relation

$$\mathfrak{H}_\lambda(x) = \frac{\omega^{2\lambda}}{2^\lambda} q^\lambda \frac{\Gamma\left(p\,\overline{s-1}+1+\frac{x}{\omega}\right)\Gamma\left(\overline{s-1}\,q+1-\frac{x}{\omega}\right)}{p^{\frac{x}{\omega}}\, q^{-\frac{x}{\omega}}} \times$$

$$\times \underset{\omega}{\Delta^\lambda} \frac{p^{\frac{x}{\omega}-\lambda}\, q^{-\frac{x}{\omega}}}{\Gamma\left(p\,\overline{s-1}-\lambda+1+\frac{x}{\omega}\right)\Gamma\left(\overline{s-1}\,q+1-\frac{x}{\omega}\right)} = \tag{111}$$

$$= \frac{(-)^\lambda}{2^\lambda} \sum_{k=0}^{\lambda} (-)^k \binom{\lambda}{k} \frac{1}{p^{\lambda-k}} (x+p\,\overline{s-1}\,\omega,\ -\overline{\lambda-k}/\omega)\ (\overline{s-\lambda}\,\omega,\ k/\omega).$$

The polynomials $\mathfrak{H}_\lambda(x)$ are also defined by the functional equation

$$2p\,\mathfrak{H}_{\lambda+1}(x) + (x+\lambda\,\overline{p-q}\,\omega)\,\mathfrak{H}_\lambda(x) + \tfrac{1}{2}q\lambda\,\overline{s-\lambda}\,\omega^2\,\mathfrak{H}_{\lambda-1}(x) = 0 \tag{112}$$

following from the relation (17) and by the initial members

$$\mathfrak{H}_0 = 1, \quad \mathfrak{H}_1 = -\frac{x}{2p}.$$

The hypergeometric difference equation (23) assumes in this case the form

$$p\omega\,(\overline{s-1}\,q\,\omega-\omega-x)\,\underset{\omega}{\Delta^2}\,\mathfrak{H}_\lambda(x) - (x+\omega\,\overline{1-\lambda})\,\underset{\omega}{\Delta}\,\mathfrak{H}_\lambda(x) + \lambda\,\mathfrak{H}_\lambda(x) = 0 \tag{113}$$

The sum (14′) assumes the value

$$I_\lambda = \frac{\lambda!\,(s-\lambda,\lambda)}{2^{2\lambda}}\,\omega^{2\lambda}\left(\frac{q}{p}\right)^\lambda. \tag{114}$$

By a continuous fraction

$$\frac{\frac{1}{4p}}{|\,\varphi(x,0)} - \frac{\beta(1)\,|}{|\,\varphi(x,1)} - \frac{\beta(2)\,|}{|\,\varphi(x,2)} - \ldots, \tag{115}$$

where

$$\beta(\lambda) = \frac{q}{p}\,\frac{\lambda\,(s-\lambda)}{4}\,w^2; \quad \varphi(x,\lambda) = -\frac{x}{2p} \cdot -\frac{p-q}{2p}\,\lambda\omega$$

also the corresponding polynomials of the second kind are defined besides the polynomials $\mathfrak{H}_\lambda(x)$.

We can use the polynomials $\mathfrak{H}_\lambda(x)$ in the approximation of functions, especially frequency functions, in a way indicated for the polynomials $\mathfrak{J}_\lambda(x)$ in the first part, section 6.

Approximately, the given frequency function $y(x)$ can be expressed by the series

$$y(x) = \frac{1}{\omega}\binom{\overline{s-1}}{p\,\overline{s-1} + \frac{x}{\omega}} p^{\frac{x}{\omega}} q^{\overline{s-1}-\frac{x}{\omega}} \left(\frac{p}{q}\right)^{p\overline{s-1}} \times$$

$$\times\{1 + a_2\mathfrak{H}_2(x) + a_3\ \mathfrak{H}_3(x) + \ldots\}, \tag{116}$$

if we determine the parameter p from the condition

$$h = \sum xy(x)\,\omega = \sum x\Phi_0(x)\,\omega = p(s-1)\,\omega,$$

$$p = \frac{h}{\overline{s-1}\,\omega}.$$

By applying the average binomial moments $\overline{B}_k$ of the function y, the coefficients a_λ are given by

$$a_\lambda = s\left(\frac{2p}{q\omega}\right)^\lambda \sum_{k=0}^{\lambda} (-)^k \frac{\overline{B}_k}{p^k\,\overline{\lambda-k}!\,\overline{k+1}!}. \tag{117}$$

As far as I know, the polynomials $\mathfrak{H}_\lambda(x)$ do not appear till now in the mathematical analysis. But the limiting form of these polynomials if $s \to \infty$, $\omega \to 0$ which is the integral system of orthogonal Hermite polynomials, is very extensively used in pure and applied mathematics.

5. Hermite polynomials.

We shall first investigate a more general form of these polynomials which corresponds to the characteristic function (90′)

$$\Phi_0(x) = \frac{1}{\sqrt{2\pi\,pq\,(1+d)}}\,e^{-\frac{x^2}{2pq\,(1+d)}}$$

in the interval $(\pm\infty)$. They are defined either, according to (6), by the relation

$$H_\lambda(d, x) = \frac{(1+d)^\lambda q^\lambda}{2^\lambda}\,e^{\frac{x^2}{2pq\,(1+d)}}\,\frac{d^\lambda}{dx^\lambda}\,e^{-\frac{x^2}{2pq\,(1+d)}}, \tag{118}$$

or, according to (17), by the recurrence formula

$$H_{\lambda+1}(d, x) + \frac{x}{2p}H_\lambda(d, x) + \frac{q}{4p}\,\lambda(1+d)\,H_{\lambda-1}(d, x) = 0 \tag{119}$$

and by initial members

$$H_0(d, x) = 1,\quad H_1(d, x) = -\frac{x}{2p}.$$

These polynomials and the corresponding polynomials of the second kind are also determined by the continuous fraction

$$\frac{\frac{1}{4p}}{\left| -\frac{x}{2p} \right.} - \frac{\beta(1)\,|}{\left| -\frac{x}{2p} \right.} - \frac{\beta(2)\,|}{\left| -\frac{x}{2p} \right.} - \ldots, \tag{120}$$

where

$$\beta(\lambda) = \frac{(1+d)\,q\,.\,\lambda}{4p}.$$

The sum (14') assumes in this case the value

$$I_\lambda = (1, \lambda)\left(\frac{\overline{1+d}\,q}{4p}\right)^\lambda, \tag{121}$$

the difference equation (23) changes into the differential equation

$$pq\,(1+d)\,H''_\lambda(d, x) - xH'_\lambda(d, x) + \lambda H_\lambda(d, x) = 0. \tag{122}$$

The quantities $H'_\lambda(d, x)$ and $H_{\lambda-1}(d, x)$ are connected by the simple relation

$$H'_\lambda(d, x) = -\frac{\lambda}{2p} H_{\lambda-1}(d, x). \tag{123}$$

By application of this relation we can now express the polynomials H_λ explicitly according to the powers of x. From the Maclaurin development of the function $H_\lambda(y + x)$ we obtain

$$H_\lambda(y + x) = \sum_{i=0}^{\lambda} H_{\lambda-i}(y) \binom{\lambda}{i} (-)^i \frac{x^i}{(2p)^i}$$

and therefore for $y = 0$

$$H_\lambda(x) = \sum_{i=0}^{\lambda} H_{\lambda-1}(0)\,(-)^i \binom{\lambda}{i} \left(\frac{x}{2p}\right)^i.$$

The values $H_{\lambda-i}(0)$ can be computed easily from the recurrence relation (119) in the form

$$H_{2i}(0) = (-)^i \left(\frac{q\overline{1+d}}{2p}\right)^i \frac{\overline{2i}\,!}{i\,!\,2^i}; \quad H_{2i-1}(0) = 0.$$

Having substituted these values, we obtain the required expression

$$H_\lambda(d, x) = \frac{(-)^\lambda}{(2p)^\lambda} \sum_{i=0}^{\frac{\lambda}{2}, \frac{\lambda-1}{2}} (-)^i \frac{\overline{2i}!\,(pq\,\overline{1+d})^i}{i!} \binom{\lambda}{2i} x^{\lambda-2i}. \tag{124}$$

The polynomials $H_\lambda(d, x)$ can be used to an approximate expression of empirical frequency functions by the series

$$\frac{1}{\sqrt{2\pi pq\,(1+d)}}\, e^{-\frac{(x-a)^2}{2pq(1+d)}} \left[1 + a_1 H_1(d, x-a) + \ldots + a_\lambda H_\lambda(d, x-a)\right]. \tag{125}$$

The value α follows from the condition

$$h = \int_{-\infty}^{\infty} y\, x\, dx = \int_{-\infty}^{\infty} \Phi_0(x - \alpha)\, x\, dx = \alpha.$$

The further parameter $pq\,(1 + d) = \beta$ will be determined by the method of moments from the condition

$$\sigma^2 - h^2 = \int_{-\infty}^{\infty} x^2 y\, dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\sqrt{2\beta}\, \xi + \alpha)^2\, e^{-\xi^2}\, d\xi = \beta.$$

In consequence of this choice of parameters α and β the coefficients a_1 and a_2 in the series (125) vanish so that

$$y(x) = \Phi_0(x - h)\,[1 + a_3 H_3\,(x - h) + \ldots]. \qquad (125')$$

The coefficient a_i in this series can be expressed as a linear function of moments μ_i

$$\frac{a_i}{p^i} = \frac{2^i\,(-)^i \sum\limits_{k=0}^{i} (-)^k \dfrac{(2k)!\,\beta^k}{k!} \dbinom{\lambda}{2k} \mu_{i-2k}}{(1,\, i)\,\beta^i}. \qquad (126)$$

By the above mentioned application of the method of moments can not be, of course, determined the special relation between p and d, which would characterize the given frequency function $y(x)$.

Putting

$$x = \xi \sqrt{2pq}, \quad d = 0 \qquad (127)$$

and multiplying by the factor

$$\left(\frac{2p}{q}\right)^{\frac{1}{2}\lambda}$$

we get from the polynomials $H_\lambda(d, x)$ the usual form of Hermite polynomials[41]) defined by the formula

$$H_\lambda(\xi) = e^{\xi^2} \frac{d^\lambda}{d\xi^\lambda} e^{-\xi^2}. \qquad (118')$$

From the series (125′) we obtain in this case the well-known Charlier's series A[42]) forming also the basis of Bruns's series.[43])

From the numerous papers about the Hermite polynomials I refer to the following:

Tchebychef, Sur le développement des fonctions à une seule variable, Oeuvres I,

---

[41]) See Ch. Hermite: Sur un nouveau développement en séries des fonctions, C. R. LVIII.

[42]) C. V. L. Charlier: Ueber das Fehlergesetz, Arkiv för Mat., vol. II. Stockholm 1905.

[43]) H. Bruns: Wahrscheinlichkeitsrechnung und Kollektivmasslehre, 1906.

T. N. Thiele, Theory of Observations, London 1903,
C. V. L. Charlier, Researches into the Theory of Probability, Medd. från Lunds Astron. Obserw., Lund, 1906,
Edgeworth, The law of error, Phil. Trans. XX, 1905,
Mises, Ueber die Grundbegriffe der Kollektivmasslehre (1912),
J. Henderson, On expansions in tetrachoric functions, Biom. XIV.,
Einar Hille, A class of reciprocal functions, Annals of Math., Second Ser., Vol. 27,
J. V. Uspensky, On the development of arbitrary functions in series of Hermite 's and Laguerre's polynomials, Ann. of Math., Second Ser., Vol. 28,
M. H. Stone, Developments in Hermite polynomials, Ann. of Mat., Second Ser., Vol. 29.
J. Korous, O polynomech Hermitových, Rozpravy II. tř. Čes. Akad., roč. XXXVII. č. 11.

6. The generalized Kummer polynomials.

To the characteristic function (81)

$$\Phi_0(z) = \frac{\Gamma\left(n+1+\frac{z}{\omega}\right)}{\omega\Gamma(n+1)\Gamma\left(1+\frac{z}{\omega}\right)}\left(\frac{d}{1+d}\right)^{\frac{z}{\omega}}\frac{1}{(1+d)^{n+1}} \tag{128}$$

corresponds in the interval $(0, \infty)$ the orthogonal system of polynomials $\mathfrak{L}_\lambda(z)$ defined by the relation

$$\mathfrak{L}_\lambda(z) = \lim_{\substack{s\to\infty \\ m\to\infty}} \frac{2^\lambda}{(m+1, \lambda)} \cdot \mathfrak{J}_\lambda(n, m, z) =$$

$$= \frac{\omega^{2\lambda}(1+d)^{\lambda-n-1}}{\Phi_0(z)} \mathop{\Delta^\lambda}_{\omega} \frac{\Gamma\left(n+1+\frac{z}{\omega}\right)}{\omega\Gamma(n+\lambda+1)\,\Gamma\left(1-\lambda+\frac{z}{\omega}\right)}\left(\frac{d}{1+d}\right)^{\frac{z}{\omega}} \tag{129}$$

$$= \sum_{k=0}^{\lambda} (-)^k \frac{(1, \lambda)}{(1, \lambda-k)} \frac{1}{(n+1, k)} \binom{\frac{z}{\omega}}{k} d^{\lambda-k}\,\omega^\lambda.$$

The expression by a hypergeometric series of the third order (10′) assumes the following form in this case

$$\mathfrak{L}_\lambda(z) = (\omega d)^\lambda \left[1 + \frac{-\lambda}{n+1}\frac{z}{d\omega} + \frac{(-\lambda)(-\lambda+1)}{(n+1)(n+2)}\frac{z(z-\omega)}{(d\omega)^2} + \ldots\right]. \tag{130}$$

The polynomials $\mathfrak{L}_\lambda(z)$ satisfy the recurrence relation

$$(n+\lambda+1)\,\mathfrak{L}_{\lambda+1}(z) + [z - (\lambda + d\,\overline{n+2\lambda+1})\,\omega]\,\mathfrak{L}_\lambda(z) + \\ + d\lambda\,(1+d)\,\omega^2\,\mathfrak{L}_{\lambda-1}(z) = 0 \tag{131}$$

following from the equation (17′). The initial members are

$$\mathfrak{L}_0(z) = 1, \quad \mathfrak{L}_1(z) = -\frac{z}{n+1} + d\omega.$$

From the other formulas valid for the polynomials $\mathfrak{L}_\lambda(z)$ we may quote the following:

$$(z+\overline{n+1}\,\omega)\,d\omega\,\underset{\omega}{\Delta^2}\mathfrak{L}_\lambda(z) - [z-\overline{n+1}\,d\omega+\omega-\lambda\omega]\,\underset{\omega}{\Delta}\,\mathfrak{L}_\lambda(z)+\lambda\mathfrak{L}_\lambda(z)=0$$

$$I_\lambda = \frac{(1,\lambda)\,(1+d)^\lambda\,d^\lambda\,\omega^{2\lambda}}{(n+1,\lambda)}. \tag{132}$$

In applying the polynomials $\mathfrak{L}_\lambda(z)$ to the approximate expression of a frequency function $y(z)$ which satisfies the relations

$$\sum_{-a}^{\infty} y(z)\,\omega = 1; \qquad \sum_{-a}^{\infty} zy(z)\,\omega = 0$$

we shall transfer the origin by the value a chosen so that the following relation be valid:

$$\sum_{-a}^{\infty} z\Phi_0(z+a)\,\omega = 0 = d(n+1)\,\omega - a$$

$$a = d(n+1)\,\omega.$$

The parameters n, d will be now determined by the method of moments from the conditions

$$\mu_2 = \sum_{-a}^{\infty} z(z-\omega)\,y(z) = \sum_{-a}^{\infty} z(z-\omega)\,\Phi_0(z+a)$$

$$\mu_3 = \sum_{-a}^{\infty} z(z-\omega)\,(z-2\omega)\,y(z) = \sum_{-a}^{\infty} z(z-\omega)\,(z-2\omega)\,\Phi_0(z+a).$$

From the value of binomial moments of the function $\Phi_0(n, m, z)$ contained in the formula (19′) we get in the limit the binomial moments of the function $\Phi_0(z)$ in the form

$$B_k = \binom{n+k}{k} d^k. \tag{133}$$

Using these values we can compute in an easy way

$$\mu_2 = \omega^2 d(d+1)\,(n+1)$$
$$\mu_3 = 2\omega^3 d(d+1)\,(d-1)\,(n+1)$$

or

$$(d-1)\,\omega = \frac{\mu_3}{2\mu_2}; \quad n+1 = \frac{4\mu_2^{\,3}}{(2\mu_2\omega+\mu_3)\,(4\mu_2\omega+\mu_3)}. \tag{134}$$

In consequence of this choice of parameters a, d, n the coefficients

a_1, a_2, a_3, in the corresponding expression of the function $y(z)$ vanish so that

$$y(z) = \Phi_0(z+\alpha)\,[1 + a_4\mathfrak{L}_4(z+\alpha) + \ldots + a_\lambda\mathfrak{L}_\lambda(z+\alpha)]. \qquad (135)$$

From the polynomials $\mathfrak{L}_\lambda(z)$ we can deduce two important systems of orthogonal polynomials i. e. either the Kummer polynomials and Laguerre polynomials respectively or the Charlier-Jordan polynomials.

7. The Kummer polynomials.

If $d = \dfrac{1}{\varkappa\omega}$ and if ω converges to zero, the characteristic function (128) assumes the form (98)

$$\frac{z^n\, e^{-z\varkappa}\, \varkappa^{n+1}}{\Gamma(n+1)}$$

to which, in the interval $(0, \infty)$, correspond the polynomials $L_\lambda(z)$ defined by the formula

$$L_\lambda(z) = \frac{z^{-n}\, e^{z\varkappa}}{\varkappa^\lambda}\,\frac{d^\lambda}{dz^\lambda}\,\frac{z^{n+\lambda}\, e^{-z\varkappa}}{(n+1, \lambda)} = \sum_{k=0}^{\lambda} (-)^k \binom{\lambda}{k} \frac{z^k}{(n+1, k)\, \varkappa^{\lambda-k}}. \qquad (136)$$

By use of a hypergeometric series (10′) we obtain

$$L_\lambda(z) = \frac{1}{\varkappa^\lambda}\left[1 - \frac{\lambda}{n+1}\,\frac{z\varkappa}{1!} + \frac{\lambda(\lambda-1)}{(n+1)(n+2)}\,\frac{(z\varkappa)^2}{2!} - \ldots\right]. \qquad (137)$$

This is the well-known confluent hypergeometric series, which was first studied by Kummer in the paper: Ueber die hypergeometrische Reihe.[44])

The elementary properties of the polynomials $L_\lambda(z)$ are obvious from the formulas following directly from the analogous formulas valid for the polynomials $\mathfrak{L}_\lambda(z)$

$$(n+\lambda+1)\, L_{\lambda+1}(z) + \left(z - \frac{n+2\lambda+1}{\varkappa}\right) L_\lambda(z) + \frac{\lambda}{\varkappa^2}\, L_{\lambda-1}(z) = 0$$

$$\frac{z}{\varkappa}\, L''_\lambda(z) - \left(z - \frac{n+1}{\varkappa}\right) L'_\lambda(z) + \lambda L_\lambda(z) = 0 \qquad (138)$$

$$I_\lambda = \frac{(1, \lambda)}{(n+1, \lambda)\, \varkappa^{2\lambda}}.$$

The polynomials of Sonin, Laguerre, Abel, Milne, and Vaney do not differ materially from the polynomials $L_\lambda(z)$; we can deduce also the Hermite polynomials from them by a convenient substitution of the variable z. A more detailed account can be found in P. Humbert's „Monographie des polynomes de Kummer.“[45])

[44]) Journal f. Math. 15 (1836).
[45]) Nouvelles Ann. de Math. 1922.

In the expression of the frequency function $y(z)$ by a series of polynomials $L_\lambda(z)$, we shall use the same proceeding as for the polynomials $\mathfrak{L}_\lambda(z)$ and determine first

$$\alpha = \frac{n+1}{\varkappa},$$

further

$$\mu_2 = \frac{n+1}{\varkappa^2}; \qquad \mu_3 = \frac{2(n+1)}{\varkappa^3},$$

so that

$$\varkappa = \frac{2\mu_2}{\mu_3}, \qquad n+1 = \frac{4\mu_2^3}{\mu_3^2}.$$

The corresponding series is given by the expression

$$y(z) = \Phi_0(z+\alpha)\,[1 + a_4 L_4(z+\alpha) + \ldots + a_\lambda L_\lambda(z+\alpha)]. \qquad (139)$$

This development was deduced by V. Romanovský in his above mentioned paper: „Generalisation of some Types of the Frequency Curves of Professor Pearson."[46])

From other papers about the special cases of polynomials $L_\lambda(\varkappa, n, z)$, the Kummer polynomials $L_\lambda(1, n, z)$ and the Laguerre polynomials $L_\lambda(1, 0, z)$ we shall refer to the following ones:

G. Szegö, Beiträge zur Theorie der Laguerreschen Polynome, Mat. Zeitschrift, Bd 25.
J. V. Uspensky, On the development of arbitrary function in series of Hermite's and Laguerre's polynomials. Ann. Math. Sec. ser., Vol. 28.
J. Korous, O řadách Laguerrových polynomů, Rozpravy II. tř. České Akad., roč. XXXVII, č. 40.
W. Rotach, Reihenentwicklungen einer willkürlichen Funktion nach Hermite'schen und Laguerre'schen Polynomen (Geneva, 1925).

8. The Charlier-Jordan polynomials.

This summation system of orthogonal polynomials — let us denote it $\mathfrak{G}_\lambda(z)$ — corresponds to the characteristic function (82)

$$\frac{h^{\frac{z}{\omega}}\, e^{-h}}{\omega\Gamma\left(1+\frac{z}{\omega}\right)}$$

and to the interval $(0, \infty)$.

Their definition is given by

$$\mathfrak{G}_\lambda(z) = \lim \frac{2^\lambda(n+1,\lambda)}{(m+1,\lambda)}\,\mathfrak{J}_\lambda(n, m, z) = \frac{\Gamma\left(1+\frac{z}{\omega}\right)}{h^{\frac{z}{\omega}}}\,\omega^{2\lambda}\,\underset{\omega}{\Delta^\lambda}\,\frac{h^{\frac{z}{\omega}}}{\Gamma\left(1-\lambda+\frac{z}{\omega}\right)} =$$

---

[46]) Biometrika XVI, 1924.

$$=(-)^{\lambda}\,\lambda!\sum_{k=0}^{\lambda}\frac{(-)^k}{k!}\binom{\frac{z}{\omega}}{\lambda-k}h^k\,\omega^{\lambda-k}, \tag{140}$$

or by the recurrence relation

$$\mathfrak{G}_{\lambda+1}(z)-(z-\lambda+\overline{h}\,\omega)\,\mathfrak{G}_\lambda(z)+\lambda h\omega^2\mathfrak{G}_{\lambda-1}(z)=0 \tag{141}$$

and by the polynomials

$$\mathfrak{G}_0=1,\ \ \mathfrak{G}_1=-z+h\omega.$$

By the continuous fraction

$$\frac{\frac{1}{2}\omega}{|-z+h\,\omega}-\frac{h\omega^2\,|}{|-z+\overline{h+1}\,\omega}-\frac{2h\omega^2\,|}{|-z+\overline{h+2}\,\omega}-\ldots \tag{142}$$

are defined also simultaneously the corresponding polynomials of the second kind. The sum (14′) assumes in this case the form

$$I_\lambda=(1,\lambda)\,h^\lambda\,\omega^{2\lambda}, \tag{143}$$

the difference equation (23′) is reduced to the form

$$h\omega^2\underset{\omega}{\Delta^2}\mathfrak{G}_\lambda(z)+(z-\overline{h+\lambda-1}\,\omega)\,\underset{\omega}{\Delta}\mathfrak{G}_\lambda(z)+\lambda\mathfrak{G}_\lambda(z)=0. \tag{144}$$

Between $\underset{\omega}{\Delta}\,\mathfrak{G}_\lambda(z)$ and $\mathfrak{G}_{\lambda-1}(z)$ the simple relation

$$\underset{\omega}{\Delta}\mathfrak{G}_\lambda(z)=-\lambda\mathfrak{G}_{\lambda-1}(z). \tag{145}$$

holds good.

The series

$$y(z)=\frac{h^{\frac{z}{\omega}}\,e^{-h}}{\Gamma\left(1+\frac{z}{\omega}\right)}\left[1+a_1\mathfrak{G}_1(z)+\ldots+a_\lambda\mathfrak{G}_\lambda(z)\right] \tag{146}$$

assumed special importance in mathematical statistics. It was investigated by C. V. L. Charlier in the paper „Ueber die zweite Form des Fehlergesetzes". This is the well-known Charlier series B. Also K. Jordan discussed it in detail in the paper „Sur la probabilité des épreuves répétées".

If we determine the parameter h from the condition

$$\mu_1=\sum_0^\infty zy(z)=\sum_0^\infty z\Phi_0(z)=h,$$

it is obviously $a_1=0$. For the computation of single members $\Phi_0\mathfrak{G}_\lambda(z)$ of the series (146) for equidistant values of the argument z, we can use with advantage the tables of the function

$$\frac{h^z e^{-h}}{z!}$$

contained e. g. in Pearson's publication: „Tables for Statisticians and Biometricians.“

The convergence of the expansion (146) was investigated by H. Pollaczek-Geiringer in the paper „Ueber die Poissonsche Verteilung und die Entwickelung willkürlicher Verteilungen“ and in very recent times by J. V. Uspensky: „Ch. Jordan's series for probability.“

Besides the papers quoted above we may refer to the following publications on the polynomials $\mathfrak{G}_\lambda(z)$:

Ch. Jordan, On Poisson's and Lexis' problem of probability of repeated trials. Phil. Magaz. (7) 3.

Pollaczek-Geiringer, Die Charlier'sche Entwickelung willkürlicher Verteilungen, Skand. Aktuarietidskrift 1928.

L. Truksa, Poznámka k polynomům Charlier-Jordanovým, Čas. mat. a fys., Praha 1928.

Note sur le calcul de la table d'activité.

Par *Vladimir Šupík.*

Dans le premier numéro du tome IIème de ce journal M. Schönbaum, professeur à l'Université Charles de Prague a publié un article sur le calcul numérique de la table d'activité que l'on a pris pour le fondament des valeurs mathématiques de l'assurance sociale des ouvriers en Tchécoslovaquie. M. Schönbaum a traité ce problème par deux méthodes diverses dont les résultats coïncident suffisamment. De l'initiative de M. Schönbaum, l'auteur de cet article a entrepris le contrôle de ces résultats par l'application des méthodes sommatoires de l'integration numérique qui n'a pas encore été appliquée dans ce cas.

I.

Soit à intégrer la fonction $y = f(x)$ entre les limites $x_0 - \frac{1}{2}h$ et $x_n + \frac{1}{2}h$ resp. x_0 et x_n si celle-ci est donnée pour les valeurs aequidistantes de l'argument $x_k = x_{k-1} + h$ $(k = -1, 0, 1, \ldots, n + 2)$. Dans la littérature du calcul numérique on déduit d'une manière qui s'appuie sur les formules de Bessel et de Stirling des formules suivantes:

$$\frac{1}{h}\int_{x_0-\frac{1}{2}h}^{x_n+\frac{1}{2}h} y\,dx = (n + \tfrac{1}{2}, -1) + \tfrac{1}{24}(n + \tfrac{1}{2}, 1) - \tfrac{17}{5760}(n + \tfrac{1}{2}, 3), \quad (1)$$