Kybernetika

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Kybernetika, Vol. 51 (2015), No. 4, 667–677

Persistent URL: `http://dml.cz/dmlcz/144474`

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SEMI-T-OPERATORS ON A FINITE TOTALLY ORDERED SET

Yong Su and Hua-wen Liu

Recently, Drygaś generalized nullnorms and t-operators and introduced semi-t-operators by eliminating commutativity from the axiom of t-operators. This paper is devoted to the study of the discrete counterpart of semi-t-operators on a finite totally ordered set. A characterization of semi-t-operators on a finite totally ordered set is given. Moreover, The relations among nullnorms, t-operators, semi-t-operators and pseudo-t-operators (i. e., commutative semi-t-operators) on a finite totally ordered set are shown.

Keywords: fuzzy connectives, finite chain, t-operator, semi-t-operator, pseudo-t-operator

Classification: 03E72

1. INTRODUCTION

Nullnorms and t-operators were introduced in [3, 10], respectively, which are also generalizations of the notions of t-norms and t-conorms. It is pointed out that nullnorms and t-operators are equivalent since they have the same block structures in $[0,1]^2$ (see [11] for detail). The t-operators have the same properties that t-norms and t-conorms but without boundary conditions, i. e., the existence of any identity is not required and this condition is substituted simply by continuity on boundary. It is reasoned in [17] that exactly these properties are adequated for operators that have to be used in fuzzy preorders. Recently, Drygaś generalized nullnorms and t-operators and introduced semi-t-operators [4] by eliminating commutativity from the axiom of t-operators.

Qualitative information is often interpreted to take values in a finite scale like:

$$\mathcal{L} = \{\textbf{Extremely Good}, \textbf{Very Good}, \textbf{Good}, \textbf{Fair}, \textbf{Bad}, \textbf{Very Bad}, \textbf{Extremely Bad}\}.$$

In these cases, a finite totally ordered set $\mathcal{L}$ is usually considered and several researchers have developed an extensive study of aggregation functions on $\mathcal{L}$, usually called **discrete aggregation functions**. Dealing with discrete operators, the smoothness condition is usually considered as the discrete counterpart of continuity. Thus, many classes of discrete aggregation functions with some smoothness condition have been studied and characterized. For instance, smooth discrete t-norms and t-conorms were characterized in [12, 13], see also [1], uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ and nullnorms in [9], nullnorms

DOI: 10.14736/kyb-2015-4-0667

without the commutative property in [5], idempotent discrete uninorms in [2], uninorms with smooth underlying operators in [8, 15] and weighted means in [7].

In the framework of $[0,1]$, t-operators are continuous on boundary and not continuous on $[0,1]$. "t-operators" on a finite totally ordered set, as the discrete counterpart of t-operators, shall be smooth on the boundary and not be smooth on the global domain. In this paper, we introduce the definition of semi-t-operators on a finite totally ordered set $\mathcal{L}$ by eliminating commutativity and another property smoothness on the boundary trying to retain the idea of border continuity. This paper is organized as follows. Section 2, we recall some operators on a finite totally ordered set $\mathcal{L}$. In section 3, the main results are present. We introduce the notion of semi-t-operators, which is a generalization of t-operators and give out a characterization of this kind of these operators.

2. PRELIMINARIES

We assume that the reader is familiar with the classical results concerning basic fuzzy connectives. To make this work self-contained, we recall some concepts and results used in the rest of the paper.

Definition 2.1. (Drygaś [4], Klemenet et al. [6]) A semi-t-norm T is an increasing, associative operation $T : [0,1]^2 \to [0,1]$ with neutral element 1.

A semi-t-conorm S is an increasing, associative operation $S : [0,1]^2 \to [0,1]$ with neutral element 0.

A t-norm T is a commutative semi-t-norm.

A t-conorm S is a commutative semi-t-conorm.

Definition 2.2. (Calvo et al. [3]) Operation $V : [0,1]^2 \to [0,1]$ is called nullnorm if it is commutative, associative, increasing, has a zero element $k \in [0,1]$ and that satisfies

$$V(0,x) = x \quad \text{for all } x \leq k, \tag{1}$$

$$V(1,x) = x \quad \text{for all } x \geq k. \tag{2}$$

By definition, the case $k = 0$ leads back to t-norms, while the case $k = 1$ leads back to t-conorms (cf. [6]). The next theorem shows that it is built up from a t-norm, a t-conorm and the zero element.

Theorem 2.3. (Calvo et al. [3]) Let $k \in (0,1)$. A binary operation V is a nullnorm with zero element k if and only if there exist a t-norm T and a t-conorm S such that

$$F(x,y) = \begin{cases} kS\left(\frac{x}{k}, \frac{y}{k}\right) & \text{if } x,y \in [0,k], \\ k + (1-k)T\left(\frac{x-k}{1-k}, \frac{y-k}{1-k}\right) & \text{if } x,y \in [k,1], \\ k & \text{otherwise.} \end{cases} \tag{3}$$

Definition 2.4. (Mas et al. [9]) Operation $F : [0,1]^2 \to [0,1]$ is called t-operator if it is commutative, associative, increasing and such that

$$F(0,0) = 0, F(1,1) = 1, \text{ and} \tag{4}$$

$$F_0 \text{ and } F_1 \text{ are continuous, where } F_0(x) = F(0,x), F_1(x) = F(1,x). \tag{5}$$

It has been pointed out that nullnorms and t-operators are equivalent ([11]).

Definition 2.5. (Drygaś [4]) Operation $F : [0,1]^2 \to [0,1]$ is called semi-t-operator if it is associative, increasing, fulfils (4) and such that the functions F_0, F_1, F^0, F^1 are continuous, where $F_0(x) = F(0,x)$, $F_1(x) = F(1,x)$, $F^0(x) = F(x,0)$ and $F^1(x) = F(x,1)$.

Denote $\mathcal{F}_{a,b}$ the family of all semi-t-operators, such that $F(0,1) = a$ and $F(1,0) = b$.

Theorem 2.6. (Drygaś [4]) Let $F : [0,1]^2 \to [0,1]$, $F(0,1) = a$ and $F(1,0) = b$. Operation $F \in \mathcal{F}_{a,b}$ if and only if there exist semi-t-norm T_F and semi-t-conorm S_F such that

$$F(x,y) = \begin{cases} aS_F\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } x,y \in [0,a], \\ b + (1-b)T_F\left(\frac{x-b}{1-b}, \frac{y-b}{1-b}\right) & \text{if } x,y \in [b,1], \\ a & \text{if } x \le a \le y, \\ b & \text{if } y \le b \le x, \\ x & \text{if } a \le x \le b, \end{cases} \tag{6}$$

when $a \le b$ and

$$F(x,y) = \begin{cases} bS_F\left(\frac{x}{b}, \frac{y}{b}\right) & \text{if } x,y \in [0,b], \\ a + (1-a)T_F\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } x,y \in [a,1], \\ a & \text{if } x \le a \le y, \\ b & \text{if } y \le b \le x, \\ y & \text{if } b \le y \le a, \end{cases} \tag{7}$$

when $b \le a$.

Let $\mathcal{L}$ be the set given by,

$$\mathcal{L} = \{x_0 \prec x_1 \prec \cdots \prec x_n \prec x_{n+1}\}.$$

We use the notation $0 := x_0$ and $1 := x_{n+1}$ in the sequel. For any $x_i, x_j \in \mathcal{L}$, $x_i \preceq x_j$ let us define

$$\langle x_i, x_j \rangle = \{x_k \in \mathcal{L} \mid x_i \preceq x_k \preceq x_j\}$$

which can be considered as the "closed interval" of points in $\mathcal{L}$ between x_i and x_j. Note that x_i and x_j are always in $\langle x_i, x_j \rangle$.

In the sequel, unless otherwise stated, $\mathcal{L}$ always represents any given finite totally ordered set with maximal element 1 and minimal element 0.

Definition 2.7. (Fodor [5]) Let F be a binary operator on $\mathcal{L}$. We say that F is smooth in the first place if

$$F(x_i, x_j) = x_k \text{ and } F(x_{i-1}, x_j) = x_l, \quad \text{imply } k-1 \le l \le k. \tag{8}$$

For any $x_i, x_j, x_{i-1}, x_k, x_l \in \mathcal{L}$.

Definition 2.8. (Fodor [5]) Let F be a binary operator on $\mathcal{L}$. We say that F is smooth in the second place if

$$F(x_i, x_j) = x_k \text{ and } F(x_i, x_{j-1}) = x_m, \quad \text{imply } k-1 \leq m \leq k. \tag{9}$$

For any $x_i, x_j, x_{j-1}, x_k, x_m \in \mathcal{L}$.

A binary operator F on $\mathcal{L}$ is said to be smooth if it is smooth in each place (see [5] for detail). In [9], a smooth binary operator is also said to verify the 1-smoothness condition.

Definition 2.9. (Torrens [16], Mas et al. [9]) Let $(\mathcal{L}, \preceq)$ be a finite totally ordered set with minimum and maximum which will be represented by 0 and 1, respectively. A triplet (T, S, N) defines a "directed algebra structure" on $\mathcal{L}$, or $(\mathcal{L}, \preceq, T, S, N)$ is said to be a "directed algebra" if it verifies:

(a) T and S are binary operations on $\mathcal{L}$ associative, commutative, and such that $T(1,1) = 1$ and $S(0,0) = 0$;

(b) $N : \mathcal{L} \to \mathcal{L}$ is an involution such that $N(T(x,y)) = S(N(x), N(y))$;

(c) $x \preceq y$ iff there exists $z \in \mathcal{L}$ such that $x = T(y,z)$
iff there exists $w \in \mathcal{L}$ such that $y = S(x,w)$.

In that case the operator T will be called an AND operator and the operator S an OR operator of the directed algebra on $\mathcal{L}$. We will also say that N is a negation on $\mathcal{L}$ and that S and T are N-dual operators.

Remark 2.10. (i) In [16], it is shown that T, S, N preserve the main properties of t-norms, t-conorms, and strong negations, respectively. Namely, N must be a decreasing bijection, such that $N(0) = 1$ and $N(1) = 0$. T must be nondecreasing, must have 1 as neutral and 0 as absorbing elements, and for all $i, j = 0, 1, ..., n+1$ it verifies $T(x_i, x_j) \preceq x_i, x_j$. Dual properties are also valid for S.

(ii) Mas et al. [9] showed that if $(\mathcal{L}, \preceq, T, S, N)$ is a directed algebra, then T and S verify 1-smoothness.

A semi-t-norm (semi-t-conorm) on $\mathcal{L}$ is a non-commutative t-norm (t-conorm) on $\mathcal{L}$.

Definition 2.11. (Mas et al. [9]) A binary operator $F : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ verifying 1-smoothness is said to be a t-operator on $\mathcal{L}$ if it is associative, commutative, and such that $F(0,0) = 0, F(1,1) = 1$. (Recall that 1-smoothness directly implies that F is nondecreasing in each place.)

Theorem 2.12. (Mas et al. [9]) $F : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is a t-operator on $\mathcal{L}$ if and only if there exists an OR operator S of directed algebra on $\mathcal{L}_{0,x_k} = \{0 = x_0, x_1, ..., x_k\}$ and an AND operator T of directed algebra on $\mathcal{L}_{x_k,1} = \{x_k, x_{k+1}, ..., 1\}$, where $x_k = F(0,1)$ such that

$$F(x,y) = \begin{cases} S(x,y) & \text{if } x, y \in \mathcal{L}_{0,x_k}, \\ T(x,y) & \text{if } x, y \in \mathcal{L}_{x_k,1}, \\ x_k & \text{otherwise.} \end{cases} \tag{10}$$

Fodor [5] studied smooth associative operations on $\mathcal{L}$ and give a characterization as follows:

Theorem 2.13. (Fodor [5]) Suppose a function M with properties $M(0,0) = 0$ and $M(1,1) = 1$. Let $\lambda := M(1,0)$, $\mu := M(0,1)$ and suppose $\lambda \preceq \mu$. If M is associative, increasing, and smooth, then M is of the following form:

$$M(x,y) = \begin{cases} S_{\langle 0,\lambda\rangle}(x,y) & \text{if } x,y \in \langle 0,\lambda\rangle, \\ \lambda & \text{if } x \in \langle \lambda,1\rangle,\ y \in \langle 0,\lambda\rangle, \\ y & \text{if } x \in \mathcal{L},\ y \in \langle \lambda,\mu\rangle, \\ \mu & \text{if } x \in \langle 0,\mu\rangle,\ y \in \langle \mu,1\rangle, \\ T_{\langle \mu,1\rangle}(x,y) & \text{if } x,y \in \langle \mu,1\rangle, \end{cases} \tag{11}$$

where $T_{\langle \mu,1\rangle}$ is a smooth t-norm on $\langle \mu,1\rangle$ and $S_{\langle 0,\lambda\rangle}$ is a smooth t-conorm on $\langle 0,\lambda\rangle$.

Conversely, if $\lambda,\mu \in \mathcal{L}$ are such that $\lambda \preceq \mu$, $T_{\langle \mu,1\rangle}$ is a smooth t-norm on $\langle \mu,1\rangle$, $S_{\langle 0,\lambda\rangle}$ is a smooth t-conorm on $\langle 0,\lambda\rangle$ and M is defined by formula (11), then M is associative, increasing and smooth with $M(0,0) = 0$, $M(1,1) = 1$, $M(1,0) = \lambda$, $M(0,1) = \mu$.

Theorem 2.14. (Fodor [5]) Suppose a function M with properties $M(0,0) = 0$ and $M(1,1) = 1$. Let $\lambda := M(1,0)$, $\mu := M(0,1)$ and suppose $\mu \preceq \lambda$. If M is associative, increasing, and smooth, then M is of the following form:

$$F(x,y) = \begin{cases} S_{\langle 0,\mu\rangle}(x,y) & \text{if } x,y \in \langle 0,\mu\rangle, \\ \mu & \text{if } x \in \langle 0,\mu\rangle,\ y \in \langle \mu,1\rangle, \\ x & \text{if } x \in \langle \mu,\lambda\rangle,\ y \in \mathcal{L}, \\ \lambda & \text{if } x \in \langle \lambda,1\rangle,\ y \in \langle 0,\lambda\rangle, \\ T_{\langle \lambda,1\rangle}(x,y) & \text{if } x,y \in \langle \lambda,1\rangle, \end{cases} \tag{12}$$

where $T_{\langle \lambda,1\rangle}$ is a smooth t-norm on $\langle \lambda,1\rangle$ and $S_{\langle 0,\mu\rangle}$ is a smooth t-conorm on $\langle 0,\mu\rangle$.

Conversely, if $\lambda,\mu \in \mathcal{L}$ are such that $\mu \preceq \lambda$, $T_{\langle \lambda,1\rangle}$ is a smooth t-norm on $\langle \lambda,1\rangle$, $S_{\langle 0,\mu\rangle}$ is a smooth t-conorm on $\langle 0,\mu\rangle$ and M is defined by formula (12), then M is associative, increasing and smooth with $M(0,0) = 0$, $M(1,1) = 1$, $M(1,0) = \lambda$, $M(0,1) = \mu$.

Riera and Torrens introduced nullnorms on $\mathcal{L}$ and give a characterisation of this kind of operators.

Definition 2.15. (Riera and Torrens [14]) A nullnorm on $\mathcal{L}$ is a two-place function $G : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ which is associative, increasing in each place, commutative, and such that there exists some element $k \in \mathcal{L}$, called absorbing element, such that $G(k,x) = k$ for all $x \in \mathcal{L}$, and satisfies

$$\begin{aligned} G(0,x) &= x \quad \text{for all } x \preceq k, \\ G(1,x) &= x \quad \text{for all } x \succeq k. \end{aligned}$$

Theorem 2.16. (Riera and Torrens [14]) A binary operation $G : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is a nullnorm if and only if there exist $k \in \mathcal{L}$, a t-conorm S on $[0,k]$ and a t-norm T on $[k,n]$ such that for all $x,y \in \mathcal{L}$, G is given by

$$G(x,y) = \begin{cases} S(x,y) & \text{if } x,y \in \langle 0,k\rangle, \\ T(x,y) & \text{if } x,y \in \langle k,1\rangle, \\ k & \text{otherwise.} \end{cases} \tag{13}$$

Moreover, G is smooth if and only if T and S are smooth.

Remark 2.17. By Theorems 2.12 and 2.16, we know that:

(i) On $\mathcal{L}$, nullnorms and t-operators are not equivalent.

(ii) On $\mathcal{L}$, t-operators are smooth nullnorms.

3. SEMI-T-OPERATORS ON $\mathcal{L}$

Obviously, t-operators F on $\mathcal{L}$ verify 1-smoothness on $\mathcal{L}$, however, counterparts of t-operators on $[0,1]$, do not need to satisfy continuous condition on $[0,1]$. Note that 1-smoothness is a global notion. As a counterpart of t-operators on $[0,1]$, t-operators on $\mathcal{L}$ shall be smooth on the boundary. Below, we give first the notion of smoothness on segments.

Definition 3.1. Let F be a binary operator on $\mathcal{L}$ and $x_i, x_j \in \mathcal{L}$.

(i) We say that $F(\cdot, x_j)$ is smooth if

$$F(x_s, x_j) = x_u \text{ and } F(x_{s-1}, x_j) = x_v, \quad \text{imply } u-1 \le v \le u \tag{14}$$

for any $x_s, x_{s-1} \in \mathcal{L}$.

(ii) We say that $F(x_i, \cdot)$ is smooth if

$$F(x_i, x_k) = x_l \text{ and } F(x_i, x_{k-1}) = x_m, \quad \text{imply } l-1 \le m \le l \tag{15}$$

for any $x_k, x_{k-1} \in \mathcal{L}$.

Now we establish equivalent formulations of (14) and (15) in the following lemma.

Lemma 3.2. Let F be a binary operator on $\mathcal{L}$ and $x_i, x_j \in \mathcal{L}$. Then:

(i) $F(x_i, \cdot)$ is smooth if and only if the following property holds:

$$F(x_i, x_s) \preceq x_l \preceq F(x_i, x_t) \Leftrightarrow \exists x_m \in \mathcal{L},\ s.t., x_s \preceq x_m \preceq x_t \text{ and } x_l = F(x_i, x_m)$$

for any $x_s, x_t \in \mathcal{L}$.

(ii) $F(\cdot, x_j)$ is smooth if and only if the following property holds:

$$F(x_u, x_j) \preceq x_l \preceq F(x_v, x_j) \Leftrightarrow \exists x_k \in \mathcal{L},\ s.t., x_u \preceq x_k \preceq x_v \text{ and } x_l = F(x_k, x_j)$$

for any $x_u, x_v \in \mathcal{L}$.

Proof. We refer to the proof of Theorem 2 in [5]. □

The commutativity property is not desired for these aggregation operators in a lot of cases, for instance when not all the inputs to be aggregated have the same reliability. In this way, we want to study a new family of operators on $\mathcal{L}$: those like t-operators by deleting the commutative property and substituting the global smoothness (or 1-smoothness) by the smoothness on the boundary.

Definition 3.3. A binary operator $F : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is said to be a semi-t-operator on $\mathcal{L}$ if it is associative, nondecreasing in each place, and such that $F(0,0) = 0, F(1,1) = 1$ and $F(0,\cdot)$, $F(\cdot,0)$, $F(1,\cdot)$ $F(\cdot,1)$ are smooth.

Now we are able to give a characterization of semi-t-operators on $\mathcal{L}$ similar to the one for $[0,1]$. Denote $\lambda := F(1,0)$ and $\mu := F(0,1)$. Since $\lambda, \mu \in \mathcal{L}$, there are two cases: either $\lambda \preceq \mu$ or $\mu \preceq \lambda$. We consider only the first case in detail, the second one can be studied in a similar way.

Theorem 3.4. Suppose F is a semi-t-operator on $\mathcal{L}$. Let $\lambda = F(1,0)$, $\mu = F(0,1)$ and $\lambda \preceq \mu$. Then there exist a semi-t-norm $T_{\langle\mu,1\rangle}$ on $\langle\mu,1\rangle$ and a semi-t-conorm $S_{\langle 0,\lambda\rangle}$ on $\langle 0,\lambda\rangle$ such that F is of the following form:

$$F(x,y) = \begin{cases} S_{\langle 0,\lambda\rangle}(x,y) & \text{if } x,y \in \langle 0,\lambda\rangle, \\ \lambda & \text{if } x \in \langle\lambda,1\rangle,\ y \in \langle 0,\lambda\rangle, \\ y & \text{if } x \in \mathcal{L},\ y \in \langle\lambda,\mu\rangle, \\ \mu & \text{if } x \in \langle 0,\mu\rangle,\ y \in \langle\mu,1\rangle, \\ T_{\langle\mu,1\rangle}(x,y) & \text{if } x,y \in \langle\mu,1\rangle. \end{cases} \tag{16}$$

Conversely, if $\lambda, \mu \in \mathcal{L}$ are such that $\lambda \preceq \mu$, $T_{\langle\mu,1\rangle}$ is a semi-t-norm on $\langle\mu,1\rangle$, $S_{\langle 0,\lambda\rangle}$ is a semi-t-conorm on $\langle 0,\lambda\rangle$ and F is defined by formula (16), then F is a semi-t-operator.

Proof. Suppose F is a semi-t-operator on $\mathcal{L}$. Let $\lambda = F(1,0)$, $\mu = F(0,1)$ and $\lambda \preceq \mu$.

• The first step is to show that $F(\lambda,0) = F(1,\lambda) = \lambda$ and $F(0,\mu) = F(\mu,1) = \mu$.

By $F(1,0) = \lambda$ and $F(0,0) = 0$, we have $F(\lambda,0) = F(F(1,0),0) = F(1,F(0,0)) = F(1,0) = \lambda$. In a similar way, we can show that $F(1,\lambda) = \lambda$ and $F(a,\mu) = F(\mu,b) = \mu$.

• Secondly, we will prove that $F(x,y) = \lambda$ for all $(x,y) \in \langle\lambda,1\rangle \times \langle 0,\lambda\rangle$ and $F(x,y) = \mu$ for all $(x,y) \in \langle 0,\mu\rangle \times \langle\mu,1\rangle$.

For any $(x,y) \in \langle\lambda,1\rangle \times \langle 0,\lambda\rangle$, by monotonicity of F, $\lambda = F(0,\lambda) \preceq F(x,y) \preceq F(1,\lambda) = \lambda$, i.e., $F(x,y) = \lambda$. In an analogous way, we can show that $F(x,y) = \mu$ for all $(x,y) \in \langle 0,\mu\rangle \times \langle\mu,1\rangle$.

• Thirdly, we will prove that F is a semi-t-conorm on $\langle 0,\lambda\rangle$.

Let us take any $x \in \langle 0,\lambda\rangle$. From $F(0,0) = 0$, $F(\lambda,0) = \lambda$, Lemma 3.2 and Definition 3.3, there exists $x^{'} \in \langle 0,\lambda\rangle$ such that $x = F(x^{'},0)$. Thus, $F(x,0) = F(F(x^{'},0),0) = F(x^{'},F(0,0)) = F(x^{'},0) = x$ for all $x \in \langle 0,\lambda\rangle$. Let us take any $y \in \langle 0,\mu\rangle$. It follows from $F(0,0) = 0$, $F(0,\mu) = \mu$, Lemma 3.2 and Definition 3.3 that there exists $y^{'} \in \langle 0,\mu\rangle$ such that $y = F(0,y^{'})$. Thus, $F(0,y) = F(0,F(0,y^{'})) = F(0,y^{'}) = y$ for all $y \in \langle 0,\mu\rangle$. From Definition 3.3, we know that F is a semi-t-conorm on $\langle 0,\lambda\rangle$.

• The fourth step is to show that $F(x,y) = y$ for all $(x,y) \in \mathcal{L} \times \langle\lambda,\mu\rangle$.

Let us take any $y \in \langle\lambda,\mu\rangle$. It follows from $F(1,\lambda) = \lambda$, $F(1,1) = 1$ Lemma 3.2 and Definition 3.3 that there exists $y^{'} \in \langle\lambda,1\rangle$ such that $y = F(1,y^{'})$. Thus, $F(1,y) = F(1,F(1,y^{'})) = F(1,y^{'}) = y$ for all $y \in \langle\lambda,1\rangle$. So, for any $(x,y) \in \mathcal{L} \times \langle\lambda,\mu\rangle$, $y = F(0,y) \preceq F(x,y) \preceq F(1,y) = y$, i.e, $F(x,y) = y$.

• Finally, we will prove that F is a semi-t-norm on $\langle\mu,1\rangle$.

From the fourth step and Definition 3.3, we only need to prove $F(x,1) = x$ for all $x \in \langle\mu,1\rangle$. Let us take any $x \in \langle\mu,1\rangle$. It follows from $F(\mu,1) = 1$, $F(1,1) = 1$, Lemma 3.2 and Definition 3.3 that there exists $x^{'} \in \langle \mu, 1\rangle$ such that $x = F(x^{'}, 1)$. Thus, $F(x, 1) = F(F(x^{'}, 1), 1) = F(x^{'}, 1) = x$ for all $x \in \langle \mu, 1\rangle$.

The second part can be justified by easy computations. □

We have above considered semi-t-operators when $\lambda \preceq \mu$, however, when $\mu \preceq \lambda$, the results of semi-t-operators would be listed in what follows since their proofs are similar.

Theorem 3.5. Suppose F is a semi-t-operator on $\mathcal{L}$. Let $\lambda = F(1, 0)$, $\mu = F(0, 1)$ and $\mu \preceq \lambda$. Then there exist a semi-t-norm $T_{\langle\lambda,1\rangle}$ on $\langle \lambda, 1\rangle$ and a semi-t-conorm $S_{\langle 0,\mu\rangle}$ on $\langle 0, \mu\rangle$ such that F is of the following form:

$$F(x,y) = \begin{cases} S_{\langle 0,\mu\rangle}(x,y) & \text{if } x, y \in \langle 0,\mu\rangle, \\ \mu & \text{if } x \in \langle 0,\mu\rangle,\ y \in \langle \mu, 1\rangle, \\ x & \text{if } x \in \langle \mu,\lambda\rangle,\ y \in \mathcal{L}, \\ \lambda & \text{if } x \in \langle \lambda,1\rangle,\ y \in \langle 0,\lambda\rangle, \\ T_{\langle\lambda,1\rangle}(x,y) & \text{if } x, y \in \langle \lambda, 1\rangle. \end{cases} \tag{17}$$

Conversely, if $\lambda, \mu \in \mathcal{L}$ are such that $\mu \preceq \lambda$, $T_{\langle\lambda,1\rangle}$ is a semi-t-norm on $\langle \lambda, 1\rangle$, $S_{\langle 0,\mu\rangle}$ is a semi-t-conorm on $\langle 0, \mu\rangle$ and F is defined by formula (17), then F is a semi-t-operator.

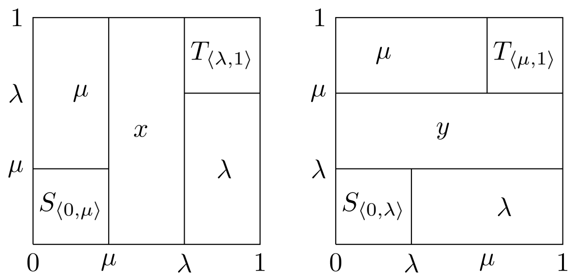

Fig. 1. Structure of F from Theorems 3.4 and 3.5 (left (17), right (16))

Remark 3.6. (i) By comparison, the different point between our work and Fodor's work [5] and the work of Mas et al. [9] is that we do not demand that the binary operator is smooth on $\mathcal{L}$.

(ii) Without the smoothness, we cannot get the commutativity of $S_{\langle 0,\lambda\rangle}$, $T_{\langle\mu,1\rangle}$ in Theorem 3.4 (or $S_{\langle 0,\mu\rangle}$, $T_{\langle\lambda,1\rangle}$ in Theorem 3.5). Example 3.7 demonstrates these.

Example 3.7. Let $\mathcal{L}_8 = \{0 \prec x_1 \prec \cdots \prec x_6 \prec 1\}$. Definite a binary operator F on $\mathcal{L}$ as follows:

1	x_5	x_5	x_5	x_5	x_5	x_5	x_6	1
x_6	x_5	x_5	x_5	x_5	x_5	x_5	x_6	x_6
x_5	x_5	x_5	x_5	x_5	x_5	x_5	x_5	x_5
x_4	x_4	x_4	x_4	x_4	x_4	x_4	x_4	x_4
x_3	x_3	x_3	x_3	x_3	x_4	x_4	x_4	x_4
x_2	x_2	x_3	x_3	x_3	x_4	x_4	x_4	x_4
x_1	x_1	x_1	x_2	x_3	x_4	x_4	x_4	x_4
0	0	x_1	x_2	x_3	x_4	x_4	x_4	x_4
F	0	x_1	x_2	x_3	x_4	x_5	x_6	1

Routine calculation shows that F is a semi-t-operator on $\mathcal{L}_8$. However, $F(x_1, x_1) = x_1$ and $F(x_1, x_2) = x_3$, we know that F is not smooth on $\mathcal{L}_8$. Moreover, $F(x_1, x_2) = x_3 \neq x_2 = F(x_2, x_1)$, i.e., F is non-commutative.

Example 3.8. Let $\mathcal{L}_8 = \{0 \prec x_1 \prec \cdots \prec x_6 \prec 1\}$. Definite a binary operator F on $\mathcal{L}$ as follows:

1	x_4	x_4	x_4	x_4	x_4	x_5	x_6	1
x_6	x_4	x_4	x_4	x_4	x_4	x_5	x_6	x_6
x_5	x_4	x_4	x_4	x_4	x_4	x_5	x_5	x_5
x_4	x_4	x_4	x_4	x_4	x_4	x_4	x_4	x_4
x_3	x_3	x_3	x_3	x_3	x_4	x_4	x_4	x_4
x_2	x_2	x_3	x_3	x_3	x_4	x_4	x_4	x_4
x_1	x_1	x_1	x_2	x_3	x_4	x_4	x_4	x_4
0	0	x_1	x_2	x_3	x_4	x_4	x_4	x_4
F	0	x_1	x_2	x_3	x_4	x_5	x_6	1

Routine calculation shows that F is a semi-t-operator with $F(0, 1) = F(1, 0) = x_4$ on $\mathcal{L}_8$. However, $F(x_1, x_1) = x_1$ and $F(x_1, x_2) = x_3$, we know that F is not smooth on $\mathcal{L}_8$. Moreover, $F(x_1, x_2) = x_3 \neq x_2 = F(x_2, x_1)$, i.e., F is non-commutative.

Definition 3.9. A semi-t-operator is called pseudo-t-operator if it is commutative.

Corollary 3.10. Suppose F is a pseudo-t-operators on $\mathcal{L}$. Let $\lambda = F(1, 0)$, $\mu = F(0, 1)$. Then $k := \lambda = \mu$ and there exist a t-norm $T_{\langle k,1\rangle}$ on $\langle k, 1\rangle$ and a t-conorm $S_{\langle 0,k\rangle}$ on $\langle 0, k\rangle$ such that F is of the following form:

$$F(x, y) = \begin{cases} S_{\langle 0,k\rangle}(x, y) & \text{if } x, y \in \langle 0, k\rangle, \\ T_{\langle k,1\rangle}(x, y) & \text{if } x, y \in \langle k, 1\rangle, \\ k & \text{otherwise.} \end{cases} \tag{18}$$

Conversely, if $k \in \mathcal{L}$, $T_{\langle k,1\rangle}$ is a t-norm on $\langle k, 1\rangle$, $S_{\langle 0,k\rangle}$ is a t-conorm on $\langle 0, k\rangle$ and F is defined by formula (18), then F is a pseudo-t-operator.

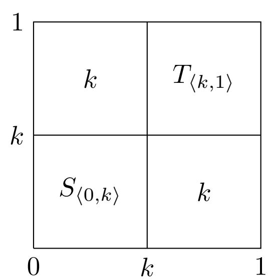

Fig. 2. Structure of F from Corollary 3.8 (Operator (18))

Remark 3.11. (i) From Theorem 2.16 and Corollary 3.10, we can see that nullnorms and pseudo-t-operators are equivalent on $\mathcal{L}$.

(ii) Denote $\mathcal{NULL}$, $\mathcal{STOPER}$, $\mathcal{PTOPER}$ and $\mathcal{TOPER}$ the class of all nullnorms, the class of semi-t-operators, the class of pseudo-t-operators and the class of t-operators on $\mathcal{L}$, respectively. Then

$$\mathcal{TOPER} \subsetneqq \mathcal{NULL} \equiv \mathcal{PTOPER} \subsetneqq \mathcal{STOPER}.$$

4. CONCLUSIONS

In this paper, we have introduced semi-t-operators on a finite totally ordered set $\mathcal{L}$, which is a generalization of t-operators. A characterization of semi-t-operators on a finite totally ordered set is given. The relations among nullnorms, t-operators, pseudo-t-operators and semi-t-operators on a finite totally ordered set have been shown.

ACKNOWLEDGEMENT

This work is supported by the National Natural Science Foundation of China (No.61174099) and the Research Found for the Doctoral Program of Higher Education of China (No.20120131110001).

(Received October 24, 2014)

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