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Kybernetika, Vol. 51 (2015), No. 4, 639–654

Persistent URL: http://dml.cz/dmlcz/144472

K Y B E R N E T I K A — V O L U M E *51* (2 0 1 5) , N U M B E R 4 , P A G E S 6 3 9 – 6 5 4

DISTRIBUTED CONSENSUS CONTROL FOR DISCRETE-TIME LINEAR MULTI-AGENT SYSTEMS WITH REDUCED-ORDER OBSERVER

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In this paper, we investigate multi-agent consensus problem with discrete-time linear dynamics under directed interaction topology. By assumption that all agents can only access the measured outputs of its neighbor agents and itself, a kind of distributed reduced-order observer-based protocols are proposed to solve the consensus problem. A multi-step algorithm is provided to construct the gain matrices involved in the protocols. By using of graph theory, modified discrete-time algebraic Riccati equation and Lyapunov method, the proposed protocols can be proved to solve the discrete-time consensus problem. Furthermore, the proposed protocol is generalized to solve the model-reference consensus problem. Finally, a simulation example is given to illustrate the effectiveness of our obtained results.

Keywords: multi-agent system, discrete-time system, distributed control, consensus, observer

Classification: 93A14, 93C10

1. INTRODUCTION

In recent years, more and more scholars have paid their attention on the coordination control for its useful applications, including consensus, formation, flocking, and coverage [22, 28]. Among them, consensus control is well-accepted as one of the most important and fundamental issues in the fields of automata theory and coordination control of multi-agent systems, which aims to find the control law that enables a group of agents to reach an agreement on some quantities [16, 21]. Numerous interesting results for multi-agent consensus problems have been obtained (see [1, 22] and the references therein).

Discrete dynamical systems can be used to model and analyze many real-world problems. In real situation, the interaction among agents may only occur at discrete sampling instant, due to the extensive application of digital sensors and controllers. Therefore, it is significant to study the discrete-time consensus algorithms in real application. The distributed state-feedback protocols were proposed by [10, 23] to solve discrete-time consensus problem with general linear dynamics. The consensus problems via sampled-data control were reported in [6, 17]. A unified framework was established by [20] to deal with the consensus for the discrete-time delayed multi-agent system.

DOI: 10.14736/kyb-2015-4-0639

However it is often difficult or even unavailable to obtain the full state information in many circumstances. To achieve control objective, state-observers are often adopted in the protocols to estimate these unmeasurable state variables. In the past decade, the state estimation problems have been intensively studied for complex networks with various kinds of incomplete measurements. In [4, 8, 11, 12, 13, 14], the observer-based consensus problems with first- or seconder-order dynamics have been investigated. The adaptive observer-based consensus protocols for seconder-order multi-agent systems were proposed by [15, 27]. For multi-agent systems with general linear dynamics, a unified framework was provided by [18] for the observer-based consensus protocols. In [29], the authors proposed three different architectures of controllers and observers to solve the leader-following multi-agent consensus problems. The approach of [29] were developed to solve the discrete-time consensus problem by [9]. The distributed reduced-order observer-based protocols were proposed by [19] for both continuous-time and discrete-time multi-agent systems. In [7], another reduced-order observer design approach is proposed to solve the consensus problem with continuous-time general linear dynamics under directed switching topology. In [26], the discrete-time leader-following consensus problem was investigated via the observer-based protocols under switching topologies. The full-order observer and reduced-order observer-based consensus protocols were provided in [24] to solve discrete-time leader-following consensus problem. In [5], the authors investigated H_∞ consensus problem for discrete-time multi-agent systems with general linear dynamics via dynamic output feedback method.

Motivated by the above works, we investigate the multi-agent consensus problem with discrete-time general linear dynamics under directed interaction topology. Both the leaderless and leader-following cases are discussed. By assumption that the output information can be accessible, a reduced-order state-observer is adopted for each following agent. A multi-step algorithm is provided to construct the protocol's gain matrices. Following that, the distributed consensus protocols are constructed, which is the main contribution of this paper. This paper is a extension of the preliminary conference paper [25], in which only a special case was addressed. Based on theory of matrix and Riccati equation, a multi-step algorithm is provided to construct the gain matrices involved in the protocols. Then, a sufficient condition for consensus is established. Certainly, as a special case, the consensus condition for the well-known second-order multi-agent system can be obtained directly. In comparison with the existed references, our proposed design approach has several advantages.

The rest of the paper is organized as follows. Section 2 gives preliminaries and the problem formulation. In section 3, a kind of distributed reduced-order observer-based protocols are provided to solve the leaderless consensus problem, which is the main result of this paper. Following that, the proposed approach is generalized to solve the model-reference consensus problem in section 4. Section 5 provides a simulation example to illustrate our obtained result, and finally, the concluding remarks are given in section 6.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1. Notations and graph theory

The notations of this paper are standard. Denote $R^{m\times n}$ and $C^{m\times n}$ as the set of $m \times n$ real matrices and complex matrices. For a complex number s, $\bar{s}$ represents its conjugate.

Let I be the n-order unit matrix . $\mathbf{1}_n \in R^n$ is the column vector with all elements equal to one. For matrix A, A^T and A^H denote respectively its transpose matrix and conjugate transpose matrix. A matrix is said to be Schur-stable if all its eigenvalues are inside unit circle. For a symmetric matrix, $P > 0$ means that P is a positive definite matrix. $\otimes$ represents Kronecker product, which satisfies $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

The interaction relationships among N agents are described by a directed weighted diagraph $\mathcal{G} = (\nu, \varepsilon, W)$, where $\nu = \{v_1,\ v_2,\ \ldots,\ v_N\}$ is the set of vertices and $\varepsilon \subseteq \nu \times \nu$ is the set of edges. The index set of neighbors for vertex i is denoted by $\mathcal{N}_i = \{j \in \nu | (v_i,\ v_j) \in \varepsilon\}$. $W = [w_{ij}]_{N \times N}$ represents weighted adjacency matrix associated with digraph $\mathcal{G}$, where $w_{ij} > 0$ if $(i,j) \in \varepsilon$ and $w_{ij} = 0$ otherwise. The degree matrix $D = diag\{d_1,\ d_2,\ \ldots,\ d_N\}$ of digraph $\mathcal{G}$ is a diagonal matrix with diagonal elements $d_i = \sum_{j=1}^{N} w_{ij}$. Correspondingly, the Laplacian matrix related with graph $\mathcal{G}$ is defined as $L = D - W$. In the sequel, let $\Gamma = I - D^{-1}W$, which will play a key role in consensus analysis. Since 0 is an eigenvalue of Laplacian matrix L, 0 is an eigenvalue of Γ. Let λ_i, $i = 1, 2, \ldots, N$ be ith eigenvalue of Γ with $\lambda_1 = 0$ too.

Definition 2.1. A covering circle $\bar{C}(c_0, r_0)$ related to matrix Γ is a closed circle in the complex plane centered at $c_0 + 0j$, $c_0 \in R$ and $\lambda(\Gamma) \in \bar{C}(c_0, r_0)$ for all $\lambda(\Gamma) \neq 0$.

Remark 2.2. The concept covering circle can be referred to [9]. Obviously, $D^{-1}W$ is a row-stochastic matrix. According to the result of [19], we know that 1 is an eigenvalue of $D^{-1}W$ with $\mathbf{1}_N$ and a nonnegative vector $r^T \in R^{1 \times N}$, respectively, as the corresponding right and left eigenvectors, and all other eigenvalues of $D^{-1}W$ are in the open unit disk. Furthermore, 1 is a simple eigenvalue of $D^{-1}W$ if and only if $\mathcal{G}$ contains a directed spanning tree. Since the interaction topology $\mathcal{G}$ contains a directed spanning tree, 0 is one simple eigenvalue of matrix Γ, and all its other eigenvalues lie within the disk of radius 1 centered at the point $1 + 0j$. Furthermore, we know that there must exist a covering circle with $\frac{r_0}{c_0} < 1$ for matrix Γ.

2.2. Problem formulation and preliminary results

Consider the multi-agent system consisting of N identical agents, whose dynamics is modeled by the following discrete-time linear system

$$\begin{aligned} x_i(k+1) &= Ax_i(k) + Bu_i(k), \\ y_i(k) &= Cx_i(k), \end{aligned} \tag{1}$$

where $x_i(k) \in R^n$, $u_i(k) \in R^p$ and $y_i(k) \in R^q$ are, respectively, the state variable, control input and measured output of agent i. A, B and C are constant matrices with appropriate dimensions. It is always assumed that $(A,\ B,\ C)$ is stabilizable and detectable. For simplicity, C is assumed to have full row rank, that is, $rank(C) = q$.

The discrete-time multi-agent system is said to achieve consensus, if the state variables of all agents satisfy $\lim_{k\to\infty}(x_i(k) - x_j(k)) = 0$, $i, j = 1, 2, \ldots, N$ for any initial state. We say that the protocol $u_i(k)$ can solve the consensus problem, if the closed-loop system achieves consensus.

For agent i, assume that $\{i\} \bigcup \mathcal{N}_i \triangleq \{j_1, j_2, \ldots, j_l\}$. Then, a state feedback

$$u_i = k_i(x_{j_1}, \ldots, x_{j_l}) \tag{2}$$

is said to be a protocol with topology $\mathcal{G}$, that is, the control law u_i is distributed. Since only the output information can be available, the state feedback (2) can not be applied directly in our problem. To do this, the local observers are adopted by the agents to estimate state information. Because part of state information is contained in the output information, the state information may be reconstructed via the reduced-order observers. Here, we are interested to the reduced-order observer. The main aim of this paper is to construct a distributed reduced-order observer-based consensus protocol $u_i(k)$ to solve the consensus problem.

Now, we introduce some preliminary results which will be used later. Consider the modified discrete-time algebraic Ricatti equations(MDARE)

$$APA^T - P - \delta APB(I + B^T PB)^{-1}B^T PA^T + Q = 0 \tag{3}$$

where Q is any given positive definite matrix. Since Q is positive definite, $(A, Q^{1/2})$ must be detectable. The solvability of the modified discrete-time algebraic Riccati equation is addressed by the following lemma.

Lemma 2.3. (Franceschetti et al. [3]) If $(A, Q^{\frac{1}{2}})$ is detectable, $(A,\ B)$ is stablizable, then there exists a $\delta_c \in [0, 1)$ such that the modified discrete time algebraic Riccati equation (3) has a unique positive-definite solution P for any $\delta_c < \delta \leq 1$. Furthermore, $P = \lim_{k\to\infty} P_k$ for any initial condition $P_0 \geq 0$, where P_k satisfies

$$P_{k+1} = A^T P_k A - \delta A^T P_k B(I + B^T P_k B)^{-1}B^T P_k A + Q. \tag{4}$$

Remark 2.4. The MDARE (3) is reduced respectively to the well-known discrete-time Riccati equation (DARE) and Stain equation as $\delta = 1$ and $\delta = 0$. The Stain equation has a unique positive-definite solution if A is Schur-stable. It is well-known that the discrete-time Riccati equation has a unique positive-definite solution, while $(A,\ B)$ is stablizable. If matrix A has some eigenvalues with magnitude larger than 1, it is easy to see that $0 < \delta_c < 1$. Moreover, if the matrix A has no eigenvalues with magnitude larger than 1 and $(A,\ B)$ is stablizable, MDARE (3) has a unique positive-definite solution P for any δ satisfying $0 < \delta \leq 1$. More details for issue δ_c can be referred to [3].

3. CONSENSUS PROTOCOL WITH REDUCED-ORDER STATE-OBSERVER

In this section, we propose the following reduced-order observer-based consensus protocol for agent i, which is composed of a reduced-order state observer and a neighbor-based feedback control law.

- A reduced-order observer for agent i is

$$\begin{aligned} z_i(k+1) &= (TA - GCA)M_2 z_i(k) + [(TA - GCA)M_2 G \\ &\quad + (TA - GCA)M_1]y_i(k) + (T - GC)Bu_i(k) \\ \bar{y}_i(k) &= z_i(k) + Gy_i(k) \end{aligned} \tag{5}$$

  where $z_i(k)$ is the protocol state, $\bar{y}_i(k)$ is the restructured variable of $Tx(k)$, $G \in R^{(n-q)\times q}$, $T \in R^{(n-q)\times n}$, $M_1 \in R^{n\times q}$ and $M_2 \in R^{n\times(n-q)}$ are given gain matrices to be designed.

- Distributed feedback control law for agent i is

$$\begin{aligned} u_i(k) = & - K_1(M_1 y_i(k) + M_2 \bar{y}_i(k)) \\ & - \frac{c}{d_i} K[M_1 \sum_{j\in\mathcal{N}_i} w_{ij}(y_i(k) - y_j(k)) - M_2 \sum_{j\in\mathcal{N}_i} w_{ij}(\bar{y}_i(k) - \bar{y}_j(k))] \end{aligned} \tag{6}$$

where K_1 and K are the designed gain matrices, $c > 0$ is the coupling parameter.

The following algorithm is provided to design the gain matrices G, T, M_1, M_2, K_1 and K used in the protocols (5) and (6).

Algorithm 3.1. For $(A,\ B,\ C)$, suppose that (A, B) is stabilizable, $(A,\ C)$ is detectable, C is full row rank. Then, matrices G, T, M_1, M_2, K_1 and K are constructed as follows:

(1) Choose T such that $\begin{bmatrix} C \\ T \end{bmatrix}$ is nonsingular. Then, M_1 and M_2 are obtained by computing $[M_1\ M_2] = \begin{bmatrix} C \\ T \end{bmatrix}^{-1}$.

(2) Choose K_1 such that $\tilde{A} = A - BK_1$ is Schur-stable.

(3) For a given positive definite matrix Q and small enough positive constant δ, solve the following modified discrete-time algebraic Riccati equation

$$\tilde{A}^T P \tilde{A} - P - \delta \tilde{A}^T PB(I + B^T PB)^{-1} B^T P \tilde{A} + Q = 0 \tag{7}$$

to get the unique positive definite solution P. Then, the gain matrix K can be designed as $K = (I + B^T PB)^{-1} B^T P \tilde{A}$.

(4) Choose G such that $TAM_2 - GCAM_2$ is Schur-stable.

Remark 3.2. Since C has full row rank, the matrix T can be chosen easily such that $[C^T,\ T^T]^T$ is nonsingular. Because $(A,\ B)$ is stabilizable, it is easy to see that $(\tilde{A},\ B)$ is stabilizable too. Note that $\tilde{A}$ is Shur-stable. According to Remark 2.4, MDARE (7) has a unique positive definite solution P with any $0 < \delta \leq 1$. The pair $(A,\ C)$ is detectable if and only if $Rank \begin{bmatrix} sI - A \\ C \end{bmatrix} = n$ for any $s \in C:\ \ Re(s) \geq 0$ [2]. On the other hand, we have

$$\begin{bmatrix} sI - CAM_1 & -CAM_2 \\ -TAM_1 & sI - TAM_2 \\ I & 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} C \\ T \end{bmatrix} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sI - A \\ C \end{bmatrix} \begin{bmatrix} M_1 & M_2 \end{bmatrix}.$$

While $(A,\ C)$ is detectable, we have $Rank \begin{bmatrix} sI - TAM_2 \\ -CAM_2 \end{bmatrix} = n-q$, that is, $(TAM_2,\ CAM_2)$ is detectable. For the detectable pair $(TAM_2,\ CAM_2)$, we can use the pole assignment algorithm to construct G such that $TAM_2 - GCAM_2$ is Schur-stable. Another well-known approach to construc G is based on the discrete-time algebraic Riccati equation. Let $\bar{A} = TAM_2$ and $\bar{C} = CAM_2$. For $\bar{Q} > 0$, solve the following DARE

$$\bar{A}\bar{P}\bar{A}^T - \bar{P} - \bar{A}\bar{P}\bar{C}^T(I + \bar{C}\bar{P}\bar{C}^T)^{-1}\bar{C}\bar{P}\bar{A}^T + \bar{Q} = 0 \tag{8}$$

to get the unique positive definite solution $\bar{P}$. Then, the gain matrix G can be designed as $G = \bar{A}\bar{P}\bar{C}^T(I + \bar{C}\bar{P}\bar{C}^T)^{-1}$.

Now, we present our main result as follows.

Theorem 3.3. For multi-agent system (1) whose interaction topology $\mathcal{G}$ contains a directed spanning tree, if matrix Γ has a covering circle $\bar{C}(c_0, r_0)$ with $0 < \frac{r_0}{c_0} < 1$, then the discrete-time consensus problem can be solved via the protocols (5) and (6). Furthermore, the gain matrices G, T, M_1, M_2, K_1 and K can be constructed by Algorithm 3.1 with δ satisfying

$$\frac{r_0}{c_0} \leq \sqrt{1-\delta}, \tag{9}$$

and the coupling strength c being chosen by

$$c = \frac{1}{c_0}. \tag{10}$$

Proof. Denote $e_i(k) = Tx_i(k) - \bar{y}_i(k)$ and $e(k) = [e_1^T(k), \ldots, e_n^T(k)]^T$. Then, it is easy to obtain

$$\begin{aligned} e_i(k+1) &= Tx_i(k+1) - (z_i(k+1) + Gy_i(k+1)) \\ &= TAx_i(k) + TBu_i(k) - (TA - GCA)M_2 z_i(k) - (T - GC)Bu_i(k) \\ &\quad + [(TA - GCA)M_2 G + (TA - GCA)M_1]y_i(k) - GCAx_i(k) - GCBu_i(k) \\ &= (TAM_2 - GCAM_2)e_i(k). \end{aligned} \tag{11}$$

Then, we can get the following error dynamics of observer system

$$e(k+1) = [I \otimes (TAM_2 - GCAM_2)]e(k). \tag{12}$$

Let $x(k) = [x_1^T(k), x_2^T(k), \ldots, x_n^T(k)]^T$. By the definition of M_1 and M_2, we have $M_1 C + M_2 T = I$. By (1) and (6), we can get

$$\begin{aligned} x_i(k+1) &= Ax_i(k) + Bu_i(k) \\ &= Ax_1(k) - BK_1(M_1 y_i(k) + M_2 \bar{y}_i(k)) \\ &\quad - \frac{c}{d_i} BK[M_1 \sum_{j \in \mathcal{N}_i} w_{ij}(y_i(k) - y_j(k)) + M_2 \sum_{j \in \mathcal{N}_i} w_{ij}(\bar{y}_i(k) - \bar{y}_j(k)] \\ &= (A - BK_1)x_i(k) - \frac{c}{d_i} BK \sum_{j \in \mathcal{N}_i} w_{ij}(x_i(k) - x_j(k)) \\ &\quad + BK_1 M_2 e_i(k) + \frac{c}{d_i} BKM_2 \sum_{j \in \mathcal{N}_i} w_{ij}(e_i(k) - e_j(k)), \end{aligned} \tag{13}$$

equivalently,

$$x(k+1) = [I \otimes \tilde{A} - \Gamma \otimes (cBK)]x(k) + [I \otimes BK_1 M_2 + \Gamma \otimes (cBKM_2)]e(k). \tag{14}$$

By (12) and (14), we get the following equivalent dynamics for the closed loop system

$$\begin{bmatrix} x(k+1) \\ e(k+1) \end{bmatrix} = \begin{bmatrix} I \otimes \tilde{A} - \Gamma \otimes (cBK) & I \otimes BK_1 M_2 + \Gamma \otimes (cBKM_2) \\ 0 & I \otimes (TAM_2 - GCAM_2) \end{bmatrix} \begin{bmatrix} x(k) \\ e(k) \end{bmatrix}. \tag{15}$$

By noticing that $\Gamma\mathbf{1} = 0$, there must exist a Schur transform orthogonal matrix with form $U = [\frac{1}{\sqrt{N}}\mathbf{1}, U_1]$ such that

$$U^T\Gamma U = \begin{pmatrix} 0 & \alpha \\ 0 & \Delta \end{pmatrix} \triangleq \bar{\Gamma},$$

where $\Delta \in C^{(N-1)\times(N-1)}$ is upper triangular matrix whose diagonal entries are the nonzero eigenvalues of Γ. Let $\bar{x}(k) = (U^T \otimes I_n)x(k)$ and $\bar{e}(k) = (U^T \otimes I_{n-q})e(k)$. Then, system (13) can be expressed as the following equivalent system

$$\begin{bmatrix} \bar{x}(k+1) \\ \bar{e}(k+1) \end{bmatrix} = \begin{bmatrix} I \otimes \tilde{A} - \bar{\Gamma} \otimes (cBK) & I \otimes BK_1M_2 + \bar{\Gamma} \otimes (cBKM_2) \\ 0 & I \otimes (TAM_2 - GCAM_2) \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\ \bar{e}(k) \end{bmatrix} \tag{16}$$

which can be divided into two subsystems

$$\begin{bmatrix} \bar{x}^0(k+1) \\ \bar{e}^0(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{A} & BK_1M_2 \\ 0 & TAM_2 - GCAM_2 \end{bmatrix} \begin{bmatrix} \bar{x}^0(k) \\ \bar{e}^0(k) \end{bmatrix} - \alpha \otimes (cBK)\bar{x}^1(k) + \alpha \otimes (cBKM_2)\bar{e}^1(k) \tag{17}$$

and

$$\begin{bmatrix} \bar{x}^1(k+1) \\ \bar{e}^1(k+1) \end{bmatrix} = \begin{bmatrix} I \otimes \tilde{A} - \Delta \otimes (cBK) & I \otimes BK_1M_2 + \Delta \otimes (cBKM_2) \\ 0 & I \otimes (TAM_2 - GCAM_2) \end{bmatrix} \begin{bmatrix} \bar{x}^1(k) \\ \bar{e}^1(k) \end{bmatrix} \tag{18}$$

where $\bar{x} = [\bar{x}^{0T},\ \bar{x}^{1T}]^T$ and $\bar{e} = [\bar{e}^{0T},\ \bar{e}^{1T}]^T$ with $\bar{x}^0$ and $\bar{e}^0$ being their first n and $n-p$ components respectively.

It is easy to see that system (18) is Schur-stable if all matrices $\tilde{A} - c\lambda_i BK$, $i = 2, 3, \ldots, N$ and $TAM_2 - GCAM_2$ are Schur-stable. Now, we prove subsystem (18) is Schur-stable. Under the condition of this theorem, we have $|c\lambda_i - 1| \le \sqrt{1-\delta}$ $(i = 2, 3, \ldots, N)$. For any s satisfying $|s-1| \le \sqrt{1-\delta}$, we have

$$\begin{aligned}
&(\tilde{A} - sBK)^H P(\tilde{A} - sBK) - P \\
&= \tilde{A}^T P\tilde{A} - (s+\bar{s})\tilde{A}^T PB(I + B^T PB)^{-1}B^T P\tilde{A} + ss^* K^T B^T PBK - P \\
&= \tilde{A}^T P\tilde{A} - P - (s+\bar{s} - s\bar{s})\tilde{A}^T PB(I + B^T PB)^{-1}B^T P\tilde{A} - |s|^2 K^T K \\
&= \tilde{A}^T P\tilde{A} - P - (1 - |s-1|^2)\tilde{A}^T PB(I + B^T PB)^{-1}B^T P\tilde{A} - |s|^2 K^T K \\
&\le \tilde{A}^T P\tilde{A} - P - \delta\tilde{A}^T PB(I + B^T PB)^{-1}B^T P\tilde{A} \\
&\le -Q < 0,
\end{aligned} \tag{19}$$

which implies that matrix $\tilde{A} - sBK$ is Schur-stable. Thus, system (18) is Schur-stable under the condition of Theorem 3.3, that is, $\lim_{k\to\infty} \bar{x}^1(k) = 0$ and $\lim_{k\to\infty} \bar{e}^1(k) = 0$. Then, we have

$$\begin{aligned}
x(k) &= (U \otimes I_n)\bar{x}(k) = [\frac{1}{\sqrt{N}}\mathbf{1}, \otimes I_n, U_1 \otimes I_n] \begin{bmatrix} \bar{x}^0(k) \\ \bar{x}^1(k) \end{bmatrix} \\
&\to [\frac{1}{\sqrt{N}}\mathbf{1} \otimes I_n, U_1 \otimes I_n] \begin{bmatrix} \bar{x}^0(k) \\ 0 \end{bmatrix} = \mathbf{1} \otimes [\frac{1}{\sqrt{N}}\bar{x}^0(k)],\ \text{as } k \to \infty,
\end{aligned}$$

which means that the multi-agent system achieves consensus. □

Now, we probe the special case that $K_1 = 0$. In this case, the consensus protocol for agent i contains a observer with form (5) and a feedback control law (6) with degenerated form

$$u_i(k) = -\frac{c}{d_i}K[M_1 \sum_{j\in\mathcal{N}_i} w_{ij}(y_i(k) - y_j(k)) - M_2 \sum_{j\in\mathcal{N}_i} w_{ij}(\bar{y}_i(k) - \bar{y}_j(k))]. \tag{20}$$

Corollary 3.4. For multi-agent system (1) whose interaction topology $\mathcal{G}$ contains a directed spanning tree, if matrix Γ has a covering circle $\bar{C}(c_0, r_0)$ satisfying

$$0 < \frac{r_0}{c_0} < \sqrt{1-\delta_c}, \tag{21}$$

then the discrete-time consensus problem can be solved by the protocols (5) and (20). Moreover, gain matrices G, T, M_1, M_2 and K involved in the protocols are constructed by Step 1, 3, 4 of Algorithm 3.1 with $K_1 = 0$ and δ satisfying

$$\frac{r_0}{c_0} \le \sqrt{1-\delta} < \sqrt{1-\delta_c}, \tag{22}$$

and the coupling strength c being chosen as

$$c = \frac{1}{c_0}.$$

Proof. According to the proof of Theorem 3.3, it is easy to see that the multi-agent system can achieve consensus via the protocols (5) and (20) if all matrices $A - c\lambda_i BK$, $i = 2, 3, \ldots, N$ and $TAM_2 - GCAM_2$ are Schur-stable. From (22), we have $1 > \delta > \delta_c$, which means that

$$A^TPA - P - \delta A^TPB(I + B^TPB)^{-1}B^TPA + Q = 0 \tag{23}$$

has a unique positive definite solution of P. Then, $K = (I + B^TPB)^{-1}B^TPA$. Under the given condition, we can also prove that $A - c\lambda_i BK$, $i = 2, 3, \ldots, N$ are Schur-stable similar as (19). The other detailed proof is omitted here. □

Remark 3.5. Essentially, protocols (20) is only based on the relative state errors of a agent with its neighbors. Most existed references investigated the consensus protocols with this formula. In continuous-time case, the consensus protocols with this formula can be successful to solve the consensus problem. Unfortunately, protocols (20) may not be successful to solve the discrete-time consensus problem while the system matrices A is unstable. As for protocols (6), while the parameter δ involved in MDARE (7) is selected small enough, the discrete-time consensus problem can be solve successfully. In comparison with protocols (6), protocols (20) can be generalized to leader-following consensus problem easily, which will be discussed in the next section.

The design approach to construct the discrete-time reduced-order observer-based protocols has been proposed by [19]. In comparison with the design approaches provided in [19], our proposed design approach has advantages at least in two respects: (1) According to Theorem 8.M6 in [2], to construct the consensus protocols successfully by the approaches proposed in [19], $(A,\ C)$ need to be observable. By the design approach of this paper, it only requires that $(A,\ C)$ is detectable. (2) It is easy to get a T such that $[C^T,\ T^T]^T$ is nonsingular in this paper. But in [19], it needs to solve a Sylvester equation to get T to make $[C^T,\ T^T]^T$ nonsingular. Unfortunately, the obtained T can not guarantee the $[C^T,\ T^T]^T$ must be nonsingular. It needs to solve the Sylvester equation again until $[C^T,\ T^T]^T$ is nonsingular. Additionally, it requires that the system matrix A has no eigenvalues with magnitude larger than 1 in [19], but it need not this limitation in our approach.

4. CONSENSUS WITH RESPECT TO A REFERENCE STATE

In many real situation, we expect that all the agent's states coverage to a common reference state, which is called as model-reference consensus problem. The dynamics of following agents is still expressed by (1), and the dynamics of reference state x_r is

$$\begin{aligned} x_r(k+1) &= Ax_r(k) + Bu_r(k), \\ y_r(k) &= Cx_r(k). \end{aligned} \tag{24}$$

The reference state is assumed only available by a subset of the following agents. But the input $u_r(k)$ is regarded as a common policy, which is assumed to be known by all the following agents. In many references such as [9, 16, 24, 29], u_r is assumed to be $u_r = 0$.

The proposed reduced-order observer-based distributed consensus protocol for agent i concludes an observer with same form (5) and a feedback control law with form

$$\begin{aligned} u_i(k) = u_r(k) - \frac{c}{d_i+g_i}K[M_1 \sum_{j\in\mathcal{N}_i} w_{ij}(y_i(k) - y_j(k)) + g_i(y_i(k) - Cx_r(k)) \\ + M_2 \sum_{j\in\mathcal{N}_i} w_{ij}(\bar{y}_i(k) - \bar{y}_j(k)) + g_i(\bar{y}_i(k) - Tx_r(k)] \end{aligned} \tag{25}$$

where g_i is the weighted constant. $g_i > 0$ if agent i is connected to the leader, otherwise $g_i = 0$. Let $G_d = diag\{g_1, g_2, \ldots, g_N\}$.

With the reduced-order observer-based protocol, the model-reference consensus problem is said to be solved if the state of any agent satisfies that $\lim_{t\to\infty}(x_i(t) - x_r(t)) = 0$, $i, = 1, 2, \ldots, N$ for any initial state.

Let $e_i(k) = Tx_i(k) - \bar{y}_i(k)$ and $e(k) = [e_1^T(k), \ldots, e_n^T(k)]^T$. Similar as (11), we also have

$$e(k+1) = [I \otimes (TAM_2 - GCAM_2)]e(k). \tag{26}$$

Let $\xi_i = x_i - x_r$ and $\xi = [\xi_1^T, \xi_2^T, \ldots, \xi_N^T]^T$. By (1), (5), (24) and (25), we have

$$\begin{aligned} \xi_i(k+1) &= A\xi_i(k) + B(u_i(k) - u_r(k)) \\ &= A\xi_i(k) - \frac{c}{d_i+g_i}BK[M_1 \sum_{j\in\mathcal{N}_i} w_{ij}(y_i(k) - y_j(k)) + g_i(y_i(k) - y_r(k)) \\ &\quad + M_2 \sum_{j\in\mathcal{N}_i} w_{ij}(\bar{y}_i(k) - \bar{y}_j(k)) + g_i(\bar{y}_i(k) - Tx_r)(k)] \end{aligned}$$

$$= A\xi_i(k) - \frac{c}{d_i + g_i} BK[\sum_{j\in\mathcal{N}_i} w_{ij}(\xi_i(k) - \xi_j(k)) + g_i\xi_i(k)$$
$$+M_2[\sum_{j\in\mathcal{N}_i} w_{ij}(e_i(k) - e_j(k)) + g_i(t)e_i(k)] \tag{27}$$

equivalently,

$$\xi(k+1) = [I \otimes A - H \otimes (cBK)]\xi + [H \otimes (cBKM_2)]e(k) \tag{28}$$

where $H = (D + G_d)^{-1}(D + G_d - W)$. The interaction topology $\bar{\mathcal{G}}$ contains graph $\mathcal{G}$, vertex v_r and edges from other vertices to vertex v_r. To achieve consensus, it is necessary that the interaction topology $\bar{\mathcal{G}}$ contains a directed spanning tree with root v_r. More discussion of $\bar{\mathcal{G}}$ and its covering circle can be referred to [24]

By (26) and (28), we get the error dynamics for the closed loop system as

$$\begin{bmatrix} \xi(k+1) \\ e(k+1) \end{bmatrix} = \begin{bmatrix} I \otimes A - H \otimes (cBK) & H \otimes (cBKM_2) \\ 0 & I \otimes (TAM_2 - GCAM_2) \end{bmatrix} \begin{bmatrix} \xi(k) \\ e(k) \end{bmatrix}. \tag{29}$$

The multi-agent system achieves consensus if the error dynamical system (29) is stable. Similarly as Corollary 3.4, the following theorem can be obtained easily, and the proof is omitted.

Theorem 4.1. For multi-agent system (1) whose interaction topology $\bar{\mathcal{G}}$ contains a directed spanning tree with root v_r, if matrix H has a covering circle $\bar{C}(c_0, r_0)$ satisfying

$$0 < \frac{r_0}{c_0} < \sqrt{1 - \delta_c}, \tag{30}$$

then the discrete-time the model-reference consensus can be solved by the protocols (5) and (25). Moreover, the gain matrices G, T, M_1, M_2 and K are constructed by Step 1, 3, 4 of Algorithm 3.1 with $k_1 = 0$ and δ satisfying

$$\frac{r_0}{c_0} \le \sqrt{1 - \delta} < \sqrt{1 - \delta_c}, \tag{31}$$

and the coupling strength c can be chosen as

$$c = \frac{1}{c_0}.$$

5. SIMULATION EXAMPLE

In this section, we provide an example to illustrate our result. Consider a multi-agent system consisting of $N = 4$ following agents. The dynamical systems for agents and leader are modeled by (1) and (24) respectively, whose system matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 1.5 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, \ C = [0, 5, 1].$$

The Laplacian matrix L for the interaction graph $\mathcal{G}$ is given by

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Let $\delta_i(t) = x_i - \frac{1}{N}\sum_{j=1}^{N} x_j$ and $\delta_i(t) = x_i - x_0$ be the state consensus error for agent i in leaderless case and leader-following case respectively. In the simulation example, we always take $T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.Then, one has $M_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -5 & 0 \end{bmatrix}$.

Case 1: We use the proposed protocols (5) and (20) to solve the consensus problem, which constructed by Algorithm 3.1. Take $K_1 = 0$. There exists covering circle $\bar{C}(c_0, r_0)$ related to matrix Γ with $c_0 = 1.25$ and $r_0 = 0.25$. Solve the Riccati equation (7) with $\delta = 1 - \frac{r_0^2}{c_0^2}$ and $Q = I$ to get a positive definite solution

$$P = \begin{bmatrix} 1.2703 & -0.0146 & 0.0955 \\ -0.0146 & 2.2670 & -0.0127 \\ 0.09557 & -0.0127 & 2.2519 \end{bmatrix}.$$

Then $K = [-0.2295,\ 0.0094,\ 0.4008]$. Take $G = \begin{bmatrix} 0.1536 \\ -0.0307 \end{bmatrix}$ and $c = \frac{1}{c_0}$. Therefore, the protocols (5) and (20) are constructed successfully. The three components of error trajectories are depicted in Figure 1, which shows that the multi-agent system can reach consensus under the protocols (5) and (20).

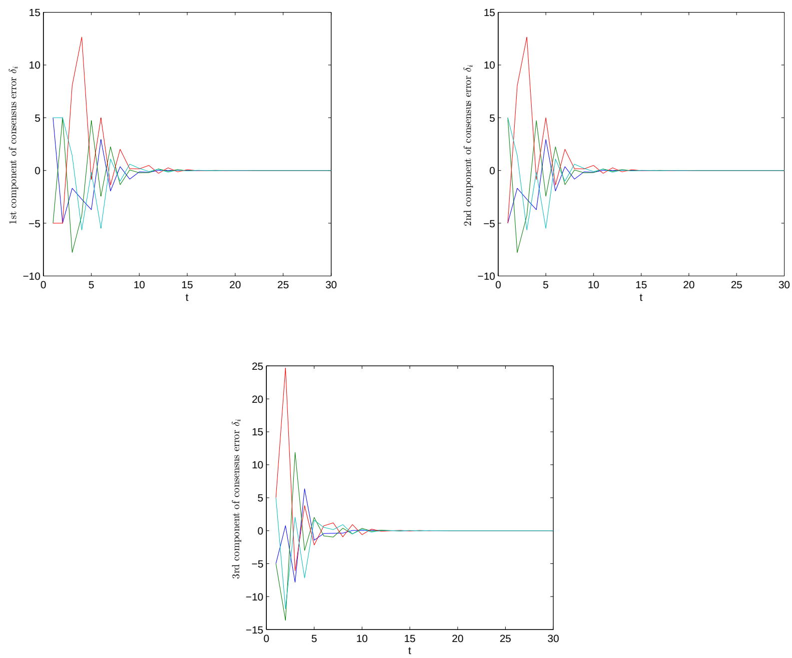

Fig. 1. Trajectories of three error components by the protocols (5) and (20).

Case 2: Take another interaction graph $\mathcal{G}$ whose Laplacian matrix is given by

$$L = \begin{bmatrix} 9 & -2 & -1 & -6 \\ -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix}.$$

There exists a covering circle $\bar{C}(c_0, r_0)$ related to matrix Γ with $c_0 = 0.9793$ and $r_0 = 0.7725$, which has the smallest $\frac{r_0}{c_0}$. By taking $\delta = 1 - \frac{r_0^2}{c_0^2}$ and $K_1 = 0$, the Riccati equation (7) can not be solvable. Therefore, protocols (5) and (20) can not be constructed successfully in this case.

Case 3: But, under the same interaction topology as Case 3, take $K_1 = [-0.2281,\ 0.0255,\ 0.4007]$ such that $\tilde{A} = A - BK_1$ is Schur-stable. Then, the Riccati equation (7) with $\delta = 1 - \frac{r_0^2}{c_0^2}$ and $Q = I$ has a positive definite solution

$$P = \begin{bmatrix} 1.1237 & -0.0269 & 0.3057 \\ -0.0269 & 2.0798 & -0.0594 \\ 0.30574 & -0.0594 & 1.7712 \end{bmatrix}.$$

Then $K = [-0.0064,\ 0.0140,\ 0.0124]$. $c = \frac{1}{c_0}$. The other parameters are chosen as Case 1. Therefore, protocols (5) and (6) is constructed successfully. The three components of error trajectories are depicted in Figure 2, which shows that the multi-agent system can reach consensus under the protocols (5) and (6).

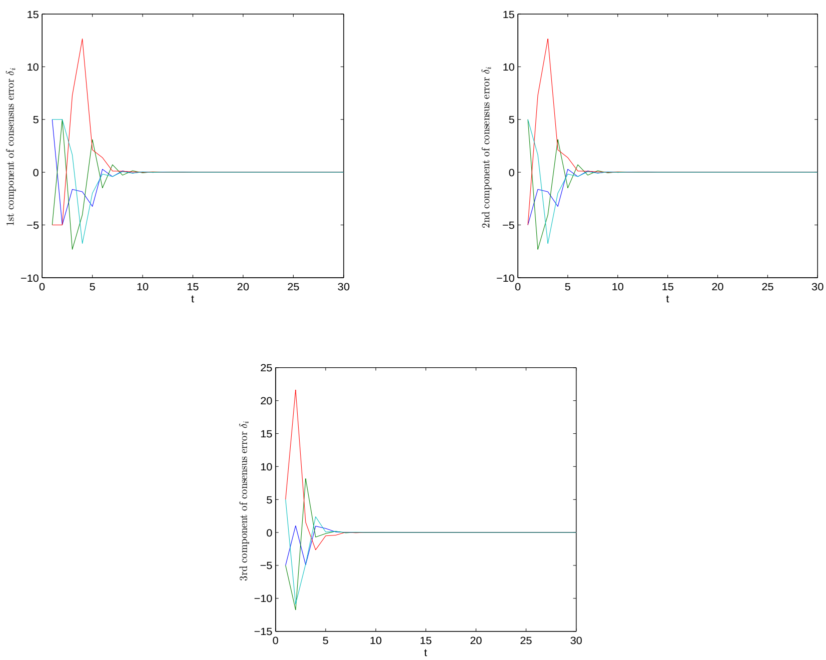

Fig. 2. Trajectories of three error components by the protocols (5) and (6) .

Case 4: For reference state tracking case, matrices L and G_d related with the interaction graph $\bar{\mathcal{G}}$ are given by

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}, \quad G_d = diag\{0,\ 4,\ 0,\ 5\}.$$

There exists a covering circle $\bar{C}(c_0, r_0)$ related to matrix H with $c_0 = 0.8474$ and $r_0 = 0.5322$. Solve the Riccati equation (7) with $\delta = 1 - \frac{r_0^2}{c_0^2}$ and $Q = I$ to get a positive definite solution

$$P = \begin{bmatrix} 11.9805 & 6.2883 & -13.6724 \\ 6.2883 & 9.5390 & -3.4460 \\ -13.6724 & -3.4460 & 26.8235 \end{bmatrix}.$$

Then $K = [-0.2520,\ -0.1174,\ 0.3677]$. The other parameters are chosen as Case 1. By using protocols (5) and (25), the three components of error trajectories are depicted in Figure 3, which shows that the multi-agent system can reach consensus

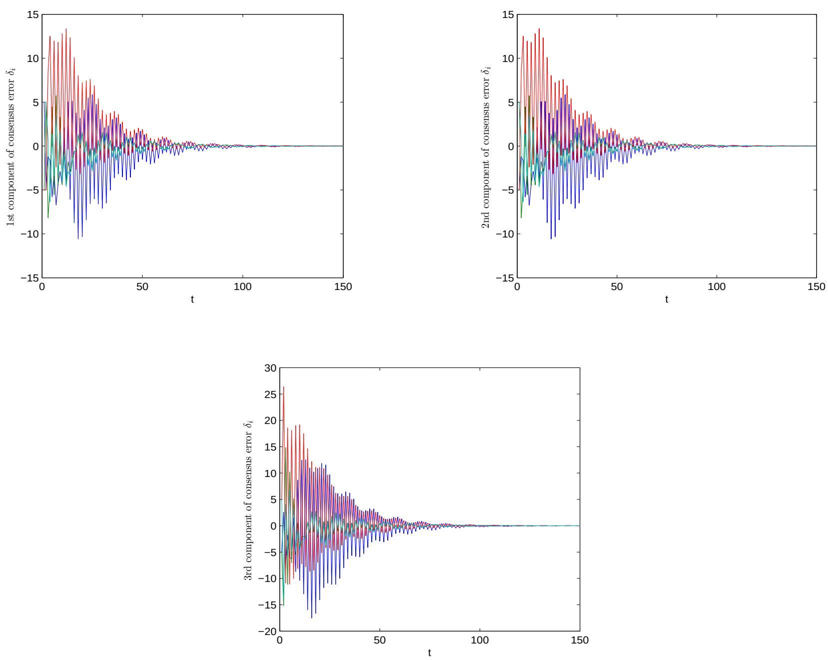

Fig. 3. Trajectories of three error components by the protocols (5) and (25) .

6. CONCLUSION

This paper investigated the multi-agent consensus problems with discrete-time general linear dynamics under directed interaction topology. The proposed distributed reduced-order observer-based consensus protocols were constructed by solving a modified discrete-time Riccati equation. With help of graph theory, matrix theory and Lyapunov method, a sufficient consensus condition has been established. More generalized cases, such as adaptive gain design, non-linear dynamics and time delays, will be investigated in our future works.

ACKNOWLEDGMENT

This work was supported by Zhejiang Provincial Natural Science Foundation of China under Grant Nos. LY15F030009, LY13F030005 and the National Natural Science Foundation of China under grant Nos. 61074123, 61174063.

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