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SOME INFINITE SUMS IDENTITIES

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Abstract. We find the sum of series of the form

$$\sum_{i=1}^{\infty} \frac{f(i)}{i^r}$$

for some special functions f. The above series is a generalization of the Riemann zeta function. In particular, we take f as some values of Hurwitz zeta functions, harmonic numbers, and combination of both. These generalize some of the results given in Mező's paper (2013). We use multiple zeta theory to prove all results. The series sums we have obtained are in terms of Bernoulli numbers and powers of π.

Keywords: multiple zeta values; multiple Hurwitz zeta values

MSC 2010: 11M32, 11M36

1. Introduction

For $s > 1$, the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The values of $\zeta(s)$ at integers have attracted many mathematicians. The values of $\zeta(s)$ at odd positive integers are quite mysterious. The only proper result in this direction is about the irrationality of $\zeta(3)$. However, at even integers the values of $\zeta(s)$ are well known. The values are in terms of powers of π and Bernoulli numbers. This shows that $\zeta(2i)$ is transcendental for any positive integer i. The nth Bernoulli number B_n is a rational number defined by the power series expansion

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{\mathrm{e}^t - 1}.$$

The Hurwitz zeta function which is a generalization of the Riemann zeta function is defined as

$$\zeta(s;a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}\quad (s>1;\ a\in\mathbb{R}\setminus\mathbb{Z}_-),$$

where $\mathbb{R}$ and $\mathbb{Z}_-$ are the sets of real numbers and non-positive integers, respectively. The nth generalized harmonic number of order r is defined as

$$H_{n,r}=\sum_{i=1}^{n}\frac{1}{i^r}.$$

More generally, if we consider series of the form

$$\sum_{i=1}^{\infty}\frac{f(i)}{i^r}, \tag{1.1}$$

it is interesting to find the sum of the above series for some special functions f. The above series is a generalization of the Riemann zeta function. Mező [1] has considered series of the form (1.1) where f are some particular values of Hurwitz zeta function and generalized harmonic numbers. In [1], the sums of various series were computed by applying the Weierstrass product theorem. The aim of this paper is to generalize some of the identities which have been proved in [1] only in particular cases. We use the theory of multiple zeta values to prove our results. The series sums we have obtained are in terms of Bernoulli numbers and powers of π.

2. Series arising from multiple zeta values

For $s_1>1$ and $s_i\geqslant 1$ for $2\leqslant i\leqslant r$, the multiple zeta values are defined as

$$\zeta(s_1,s_2,\ldots,s_r)=\sum_{n_1>n_2>\ldots>n_r\geqslant 1}\frac{1}{{n_1}^{s_1}\ldots {n_r}^{s_r}}.$$

We have the following identity which generalizes identities 4 and 7 of [1].

Proposition 2.1. *For any positive integer k,*

$$\sum_{i=1}^{\infty}\frac{\zeta(2k;i)}{i^{2k}}=\Big\{\Big(\frac{B_{2k}}{2(2k)!}\Big)^2-\frac{B_{4k}}{2(4k)!}\Big\}\frac{(2\pi)^{4k}}{2}. \tag{2.1}$$

P r o o f. To prove the above identity, we need the following identity for multiple zeta values:

$$\zeta(s_1)\zeta(s_2)=\zeta(s_1,s_2)+\zeta(s_2,s_1)+\zeta(s_1+s_2). \tag{2.2}$$

Proof of (2.2) is not difficult. The left hand side product is a double sum

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{1}{{n_1}^{s_1}}\frac{1}{{n_2}^{s_2}},$$

which can be written as

$$\sum_{n_1,n_2=1;\; n_1>n_2}^{\infty}\frac{1}{{n_1}^{s_1}}\frac{1}{{n_2}^{s_2}}+\sum_{n_1,n_2=1;\; n_2>n_1}^{\infty}\frac{1}{{n_1}^{s_1}}\frac{1}{{n_2}^{s_2}}+\sum_{n_1,n_2=1;\; n_1=n_2}^{\infty}\frac{1}{{n_1}^{s_1}}\frac{1}{{n_2}^{s_2}}.$$

This proves (2.2). Now, if $s_1 = s_2 = 2k$, then from (2.2) we get

$$\zeta(2k,2k)=\frac{\zeta(2k)^2-\zeta(4k)}{2}. \tag{2.3}$$

Next, we recall the famous theorem due to Euler on the values of Riemann zeta function at even numbers. We have

$$2\zeta(2k)=(-1)^{k-1}(2\pi)^{2k}\frac{B_{2k}}{(2k)!}. \tag{2.4}$$

Now

$$\zeta(2k,2k)=\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=i+1}^{\infty}\frac{1}{j^{2k}}=\sum_{i=1}^{\infty}\frac{1}{i^{2k}}(\zeta(2k)-H_{i,2k})=\zeta(2k)^2-\sum_{i=1}^{\infty}\frac{H_{i,2k}}{i^{2k}}. \tag{2.5}$$

Arranging appropriately, we see that

$$\sum_{i=1}^{\infty}\frac{H_{i,2k}}{i^{2k}}=\sum_{i=1}^{\infty}\frac{\zeta(2k;i)}{i^{2k}}.$$

Thus from the above identity, (2.3) and (2.5), we deduce

$$\sum_{i=1}^{\infty}\frac{\zeta(2k;i)}{i^{2k}}=\zeta(2k)^2-\frac{\zeta(2k)^2-\zeta(4k)}{2}=\frac{\zeta(2k)^2+\zeta(4k)}{2}.$$

Applying (2.4) to the above equality, we finally get

$$\sum_{i=1}^{\infty}\frac{\zeta(2k;i)}{i^{2k}}=\left\{\left(\frac{B_{2k}}{2(2k)!}\right)^2-\frac{B_{4k}}{2(4k)!}\right\}\frac{(2\pi)^{4k}}{2}.$$

This proves Proposition 2.1. □

Our second result stated below generalizes identities 1 and 8 of [1].

Proposition 2.2. *For any positive integer k we have*

$$\sum_{i=1}^{\infty}\frac{H_{i,2k}}{i^{2k}}\zeta(2k;i)=\Big\{(-1)^{3k-1}\frac{B_{6k}}{(6k)!}+(-1)^{3k-3}\frac{(B_{2k})^3}{8(2k)!^3}+(-1)^{3k-2}\frac{3B_{2k}B_{4k}}{4(2k)!(4k)!}\Big\}\frac{(2\pi)^{6k}}{6}. \tag{2.6}$$

Proof. As in (2.2), it is not difficult to see that

$$\zeta(s_1,s_2,s_3)+\zeta(s_1,s_2+s_3)+\zeta(s_1,s_3,s_2)+\zeta(s_1+s_3,s_2)+\zeta(s_3,s_1,s_2)=\zeta(s_1,s_2)\zeta(s_3). \tag{2.7}$$

Substituting $s_1=s_2=s_3=2k$ in the above equation and using (2.2), we obtain

$$\zeta(2k,2k,2k)=\frac{\zeta(2k,2k)\zeta(2k)-\zeta(2k)\zeta(4k)+\zeta(6k)}{3}. \tag{2.8}$$

Now

$$\begin{aligned}\zeta(2k,2k,2k)&=\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=i+1}^{\infty}\frac{1}{j^{2k}}\sum_{t=j+1}^{\infty}\frac{1}{t^{2k}}=\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=i+1}^{\infty}\frac{1}{j^{2k}}(\zeta(2k)-H_{j,2k})\\&=\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\bigg(\sum_{j=i+1}^{\infty}\frac{\zeta(2k)}{j^{2k}}-\sum_{j=i+1}^{\infty}\frac{H_{j,2k}}{j^{2k}}\bigg)\\&=\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\bigg[\zeta(2k)(\zeta(2k)-H_{i,2k})-\bigg(\sum_{j=1}^{\infty}\frac{H_{j,2k}}{j^{2k}}-\sum_{j=1}^{i}\frac{H_{j,2k}}{j^{2k}}\bigg)\bigg].\end{aligned}$$

Thus we have

$$\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=1}^{i}\frac{H_{j,2k}}{j^{2k}}=\zeta(2k,2k,2k)-\sum_{i=1}^{\infty}\frac{1}{i^{2k}}(\zeta(2k)^2-\zeta(2k)H_{i,2k})+\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=1}^{\infty}\frac{H_{j,2k}}{j^{2k}}. \tag{2.9}$$

Also we have

$$\sum_{i=1}^{\infty}\frac{1}{i^{2k}}(\zeta(2k)^2-\zeta(2k)H_{i,2k})=\zeta(2k)\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=i+1}^{\infty}\frac{1}{j^{2k}}=\zeta(2k)\zeta(2k,2k),$$

and

$$\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=1}^{\infty}\frac{H_{j,2k}}{j^{2k}}=\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\{\zeta(2k)^2-\zeta(2k,2k)\}=\zeta(2k)^3-\zeta(2k)\zeta(2k,2k).$$

From (2.9), we also have

$$\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=1}^{i}\frac{H_{j,2k}}{j^{2k}} = \zeta(2k,2k,2k)-\zeta(2k)\zeta(2k,2k)+\zeta(2k)^3-\zeta(2k)\zeta(2k,2k)$$
$$= \zeta(2k,2k,2k)-2\zeta(2k)\zeta(2k,2k)+\zeta(2k)^3.$$

Using (2.3) and (2.8) in the above identity, we get

$$\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=1}^{i}\frac{H_{j,2k}}{j^{2k}} = \frac{1}{6}\{2\zeta(6k)+\zeta(2k)^3+3\zeta(2k)\zeta(4k)\}. \tag{2.10}$$

Using (2.4) in the above equation, we obtain

$$\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=1}^{i}\frac{H_{j,2k}}{j^{2k}} = \frac{(2\pi)^{6k}}{6}\Big[(-1)^{3k-1}\frac{B_{6k}}{(6k)!}+\frac{1}{8}(-1)^{3k-3}\frac{B_{2k}^3}{((2k)!)^3} + \frac{3}{4}(-1)^{3k-2}\frac{B_{2k}B_{4k}}{(2k)!(4k)!}\Big]. \tag{2.11}$$

Arranging appropriately, we see that

$$\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=1}^{i}\frac{H_{j,2k}}{j^{2k}} = \sum_{i=1}^{\infty}\frac{H_{i,2k}}{i^{2k}}\zeta(2k;i).$$

Now using (2.11) and the above identity, we finally obtain

$$\sum_{i=1}^{\infty}\frac{H_{i,2k}}{i^{2k}}\zeta(2k;i) = \frac{(2\pi)^{6k}}{6}\Big[(-1)^{3k-1}\frac{B_{6k}}{(6k)!}+\frac{1}{8}(-1)^{3k-3}\frac{B_{2k}^3}{((2k)!)^3} + \frac{3}{4}(-1)^{3k-2}\frac{B_{2k}B_{4k}}{(2k)!(4k)!}\Big].$$

□

3. Series arising from multiple Hurwitz zeta values

For $s_1 > 1$, $s_1 + s_2 > 1$, …, $s_1 + s_2 + \ldots + s_r > r$, and $\alpha_1, \alpha_2, \ldots, \alpha_r \in \mathbb{R} \setminus \mathbb{Z}$, the multiple Hurwitz zeta function is defined by

$$\zeta(s_1, s_2, \ldots, s_r;\ \alpha_1, \alpha_2, \ldots, \alpha_r) = \sum_{n_1 > n_2 \ldots > n_r \geqslant 1} \frac{1}{(n_1 + \alpha_1)^{s_1} \ldots (n_r + \alpha_r)^{s_r}}.$$

For more details on multiple Hurwitz zeta function, see [2]. For $n \in \mathbb{N}$ we define

$$H'_{2n-1,r} = \sum_{i=1}^{n} \frac{1}{(2i-1)^r}.$$

Our next result stated below generalizes identity 2 of [1].

Proposition 3.1. *For any positive integer k, we have*

$$\sum_{i=1}^{\infty} \frac{\zeta(2k; i - \frac{1}{2})}{(2i-1)^{2k}} = 2^{6k-2}\pi^{4k}\Big[\frac{1}{2}\Big(1 - \frac{1}{2^{2k}}\Big)^2 \frac{B_{2k}^2}{((2k)!)^2} - \Big(1 - \frac{1}{2^{4k}}\Big)\frac{B_{4k}}{(4k)!}\Big].$$

P r o o f. To prove the above identity, we need the following identity for multiple Hurwitz zeta values:

$$\begin{aligned}\zeta(s_1;\alpha)\zeta(s_2;\alpha) = {} & \zeta(s_1, s_2; \alpha, \alpha) + \zeta(s_2, s_1; \alpha, \alpha) + \zeta(s_1 + s_2; \alpha) \\ & + \frac{1}{\alpha^{s_1}}\zeta(s_2;\alpha) + \frac{1}{\alpha^{s_2}}\zeta(s_1;\alpha) - \frac{2}{\alpha^{s_1+s_2}}.\end{aligned} \tag{3.1}$$

Proof of (3.1) is similar to the proof of (2.2), hence we omit it. Substituting $s_1 = s_2 = 2k$, $\alpha = -1/2$ in (3.1), we obtain

$$2\zeta\Big(2k, 2k; -\frac{1}{2}, -\frac{1}{2}\Big) = \zeta\Big(2k; -\frac{1}{2}\Big)^2 - \zeta\Big(4k; -\frac{1}{2}\Big) - 2^{2k+1}\zeta\Big(2k; -\frac{1}{2}\Big) + 2^{4k+1}. \tag{3.2}$$

Now

$$\begin{aligned}\zeta\Big(2k, 2k; -\frac{1}{2}, -\frac{1}{2}\Big) &= \sum_{i=1}^{\infty} \frac{1}{(i - \frac{1}{2})^{2k}} \sum_{j=i+1}^{\infty} \frac{1}{(j - \frac{1}{2})^{2k}} \\ &= 2^{4k} \sum_{i=1}^{\infty} \frac{1}{(2i-1)^{2k}} \sum_{j=i+1}^{\infty} \frac{1}{(2j-1)^{2k}}\end{aligned}$$

$$= 2^{4k}\sum_{i=1}^{\infty}\frac{1}{(2i-1)^{2k}}\Big\{\Big(1-\frac{1}{2^{2k}}\Big)\zeta(2k)-H'_{2i-1,2k}\Big\}$$

$$= 2^{4k}\Big\{\Big(1-\frac{1}{2^{2k}}\Big)\zeta(2k)\Big\}^2-2^{4k}\sum_{i=1}^{\infty}\frac{H'_{2i-1,2k}}{(2i-1)^{2k}}.$$

Thus from the above identity, we obtain

$$2^{4k}\sum_{i=1}^{\infty}\frac{H'_{2i-1,2k}}{(2i-1)^{2k}} = 2^{4k}\Big\{\Big(1-\frac{1}{2^{2k}}\Big)\zeta(2k)\Big\}^2-\zeta\Big(2k,2k;-\frac{1}{2},-\frac{1}{2}\Big). \tag{3.3}$$

Arranging appropriately, we see that

$$2^{2k}\sum_{i=1}^{\infty}\frac{H'_{2i-1,2k}}{(2i-1)^{2k}} = \sum_{i=1}^{\infty}\frac{\zeta(2k;i-\frac{1}{2})}{(2i-1)^{2k}}. \tag{3.4}$$

Using (3.4) in (3.3), we get

$$2^{2k}\sum_{i=1}^{\infty}\frac{\zeta(2k;i-\frac{1}{2})}{(2i-1)^{2k}} = 2^{4k}\Big\{\Big(1-\frac{1}{2^{2k}}\Big)\zeta(2k)\Big\}^2-\zeta\Big(2k,2k;-\frac{1}{2},-\frac{1}{2}\Big).$$

Now using (3.2), the identity

$$2^{-2k}\zeta\Big(2k;-\frac{1}{2}\Big)=\zeta(2k)-\frac{\zeta(2k)}{2^{2k}}+1,$$

and (2.4) in the above equality, we finally obtain

$$\sum_{i=1}^{\infty}\frac{\zeta(2k;i-\frac{1}{2})}{(2i-1)^{2k}} = 2^{6k-2}(\pi)^{4k}\Big[\frac{1}{2}\Big(1-\frac{1}{2^{2k}}\Big)^2\frac{B_{2k}^2}{((2k)!)^2}-\Big(1-\frac{1}{2^{4k}}\Big)\frac{B_{2k}}{(4k)!}\Big].$$

□

Our next result stated below generalizes identity 3 of [1].

Proposition 3.2. *For any positive integer k, we have*

$$\sum_{n=1}^{\infty}\frac{H'_{2n-1,2k}}{(2n-1)^{2k}}\zeta\Big(2k;n-\frac{1}{2}\Big) \tag{3.5}$$
$$= 2^{8k}\pi^{6k}\Big[\frac{(-1)^k}{8}\Big(1-\frac{1}{2^{2k}}\Big)\Big(1-\frac{1}{2^{4k}}\Big)\frac{B_{2k}B_{4k}}{(2k)!(4k)!}$$
$$+\frac{(-1)^{3k-3}}{48}\Big(1-\frac{1}{2^{2k}}\Big)^3\frac{B_{2k}^3}{(2k)!^3}+\frac{(-1)^{3k-1}}{6}\Big(1-\frac{1}{2^{6k}}\Big)\frac{B_{6k}}{(6k)!}\Big].$$

P r o o f. To prove the above identity, we need the following identity for multiple Hurwitz zeta values, proof of which is similar to the proof of (3.1).

$$\begin{aligned}\zeta(s_1,s_2;\alpha,\alpha)\zeta(s_3;\alpha) &= \frac{1}{\alpha^{s_3}}\zeta(s_1,s_2;\alpha,\alpha)+\zeta(s_1,s_2,s_3;\alpha,\alpha,\alpha)\\&\quad+\zeta(s_1,s_3,s_2;\alpha,\alpha,\alpha)+\zeta(s_3,s_1,s_2;\alpha,\alpha,\alpha)\\&\quad+\zeta(s_1+s_3,s_2;\alpha,\alpha)+\zeta(s_1,s_2+s_3;\alpha,\alpha).\end{aligned}$$

Substituting $s_1=s_2=s_3=2k$ and $\alpha=-1/2$ in the above equality and using (3.1), we obtain

$$\begin{aligned}\zeta\Big(2k,2k,2k;-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\Big) &= \frac{2^{6k}}{3}\Big[\frac{\zeta(2k)^3}{2}\Big(1-\frac{1}{2^{2k}}\Big)^3\\&\quad-\frac{3}{2}\zeta(2k)\zeta(4k)\Big(1-\frac{1}{2^{2k}}\Big)\Big(1-\frac{1}{2^{4k}}\Big)+\Big(1-\frac{1}{2^{6k}}\Big)\zeta(6k)\Big].\end{aligned}\tag{3.6}$$

Now

$$\begin{aligned}&\zeta\Big(2k,2k,2k;-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\Big)\\&=\sum_{i=1}^{\infty}\frac{1}{(i-\frac{1}{2})^{2k}}\sum_{j=i+1}^{\infty}\frac{1}{(j-\frac{1}{2})^{2k}}\sum_{t=j+1}^{\infty}\frac{1}{(t-\frac{1}{2})^{2k}}\\&=2^{6k}\Big[\Big(1-\frac{1}{2^{2k}}\Big)^3\zeta(2k)^3-\Big(1-\frac{1}{2^{2k}}\Big)\zeta(2k)\sum_{i=1}^{\infty}\frac{H'_{2i-1,2k}}{(2i-1)^{2k}}\\&\quad-\sum_{i=1}^{\infty}\frac{1}{(2i-1)^{2k}}\sum_{j=1}^{\infty}\frac{H'_{2j-1,2k}}{(2j-1)^{2k}}\Big]+2^{6k}\sum_{i=1}^{\infty}\frac{1}{(2i-1)^{2k}}\sum_{j=1}^{i}\frac{H'_{2j-1,2k}}{(2j-1)^{2k}}.\end{aligned}\tag{3.7}$$

Substituting the values of $\sum\limits_{j=1}^{\infty}H'_{2j-1,2k}/(2j-1)^{2k}$ and $\zeta(2k,2k,2k;-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$ from (3.3) and (3.6), respectively, in the above identity, we deduce that

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{1}{(2i-1)^{2k}}\sum_{j=1}^{i}\frac{H'_{2j-1,2k}}{(2j-1)^{2k}} &= \frac{1}{2}\zeta(2k)\zeta(4k)\Big(1-\frac{1}{2^{2k}}\Big)\Big(1-\frac{1}{2^{4k}}\Big)\\&\quad+\frac{1}{3}\Big[\frac{1}{2}\Big(1-\frac{1}{2^{2k}}\Big)^3\zeta(2k)^3+\zeta(6k)\Big(1-\frac{1}{2^{6k}}\Big)\Big].\end{aligned}\tag{3.8}$$

Also arranging appropriately, we see that

$$\sum_{i=1}^{\infty}\frac{1}{(2i-1)^{2k}}\sum_{j=1}^{i}\frac{H'_{2j-1,2k}}{(2j-1)^{2k}}=2^{-2k}\sum_{i=1}^{\infty}\frac{H'_{2i-1,2k}}{(2i-1)^{2k}}\zeta\Big(2k;i-\frac{1}{2}\Big).$$

Now using the above identity and (2.4) in (3.8), we finally obtain

$$
\begin{aligned}
(3.9) \qquad & \sum_{n=1}^{\infty} \frac{H'_{2n-1,2k}}{(2n-1)^{2k}} \zeta\Big(2k; n-\frac{1}{2}\Big) \\
&= 2^{8k}\pi^{6k}\Big[\frac{(-1)^k}{8}\Big(1-\frac{1}{2^{2k}}\Big)\Big(1-\frac{1}{2^{4k}}\Big)\frac{B_{2k}B_{4k}}{(2k)!(4k)!} \\
&\quad + \frac{(-1)^{3k-3}}{48}\Big(1-\frac{1}{2^{2k}}\Big)^3\frac{B_{2k}^3}{(2k)!^3} + \frac{(-1)^{3k-1}}{6}\Big(1-\frac{1}{2^{6k}}\Big)\frac{B_{6k}}{(6k)!}\Big].
\end{aligned}
$$

□

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