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BAIRE CLASSES OF COMPLEX L_1-PREDUALS

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Abstract. Let X be a complex L_1-predual, non-separable in general. We investigate extendability of complex-valued bounded homogeneous Baire-α functions on the set $\operatorname{ext} B_{X^*}$ of the extreme points of the dual unit ball B_{X^*} to the whole unit ball B_{X^*}. As a corollary we show that, given $\alpha \in [1, \omega_1)$, the intrinsic α-th Baire class of X can be identified with the space of bounded homogeneous Baire-α functions on the set $\operatorname{ext} B_{X^*}$ when $\operatorname{ext} B_{X^*}$ satisfies certain topological assumptions. The paper is intended to be a complex counterpart to the same authors' paper: Baire classes of non-separable L_1-preduals (2015). As such it generalizes former work of Lindenstrauss and Wulbert (1969), Jellett (1985), and ourselves (2014), (2015).

Keywords: complex L_1-predual; extreme point; Baire function

MSC 2010: 46B25, 26A21

1. Introduction

A complex (or real) Banach space X is called an L_1*-predual* (or a *Lindenstrauss space*) if its dual X^* is isometric to a complex (or real) space $L^1(X, \mathcal{S}, \mu)$ for a measure space $(X, \mathcal{S}, \mu)$. Complex L_1-preduals were studied, e.g., in [4], [6], [11], [18], [20] or recently in [17]. Our contribution to the subject of L_1-preduals can be found in [13], [14] and [15].

After intensive studies of real L_1-preduals, the investigation of its complex version came more into focus. In [3], Effros provided a "simplex-like" characterization of complex L_1-preduals, which allowed to involve many real case techniques also in the complex case.

The present paper is intended to be a complex counterpart to the paper [13]. As such it generalizes some results of Lindenstrauss and Wulbert in [12], Jellett in [7]

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and of ours in [13], [14] as well. Although the very general strategy of the proofs is the same as in the paper [13], the complex setting demands introducing new; more intricate notions. It is not obvious whether the complex analogues of the real notions could work considerably well. The main goal of the paper is to show that the answer is affirmative. Nevertheless, the complex case requires more delicate approach and elaborated arguments. At some points we also needed to prove stronger results than in the real case (e.g., Lemma 2.13).

All topological spaces involved in the paper are considered to be Hausdorff. Let $\mathbb{F}$ represent the field of either real or complex numbers.

For a topological space K, let $\mathcal{C}(K,\mathbb{F})$ be the space of all continuous functions on K with values in $\mathbb{F}$, $\mathcal{B}(K,\mathbb{F})$ be the space of all Borel functions on K with values in $\mathbb{F}$ and $\mathcal{B}^b(K,\mathbb{F})$ be the space of all bounded Borel functions on K with values in $\mathbb{F}$. If K is compact, we write $\mathcal{M}(K,\mathbb{F})$ for the space of $\mathbb{F}$-valued Radon measures on K and $\mathcal{M}^1(K)$ for the set of all Radon probability measures on K. (By a Radon positive measure on a compact space K we mean a complete inner regular Borel measure. An $\mathbb{F}$-valued Radon measure μ on K is an $\mathbb{F}$-valued measure such that its total variation $|\mu|$ is a Radon positive measure.) For a point $x \in K$, ε_x stands for the Dirac measure at x. A set $B \subset K$ is universally measurable if B is measurable with respect to any Radon measure on K. If $B \subset K$ is a universally measurable subset of K, we write $\mathcal{M}(B,\mathbb{F})$ for the subset of $\mathcal{M}(K,\mathbb{F})$ containing measures μ satisfying $|\mu|(K \setminus B) = 0$. Similarly, $\mathcal{M}^1(B)$ stands for the probability measures carried by B. For a universally measurable set $B \subset K$, a bounded Borel function f on B and $\mu \in \mathcal{M}(K,\mathbb{F})$, we write $\mu(f)$ for the integral $\int_K \tilde{f}\,\mathrm{d}\mu$, where $\tilde{f} = f$ on B and 0 on $K \setminus B$.

Let K be a topological space and $\mathcal{H}$ be a subset of $\mathcal{C}(K,\mathbb{F})$. We set $\mathcal{B}^0(\mathcal{H}) = \mathcal{H}$ and, for $\alpha \in (0,\omega_1)$, let $\mathcal{B}^\alpha(\mathcal{H})$ consist of all pointwise limits of elements from $\bigcup\limits_{\beta<\alpha} \mathcal{B}^\beta(\mathcal{H})$. Further, we denote by $\mathcal{B}^{\alpha,b}(\mathcal{H})$ the set of all bounded elements from $\mathcal{B}^\alpha(\mathcal{H})$. The symbol $\mathcal{B}^{\alpha,bb}(\mathcal{H})$ denotes the inductive families created by means of pointwise limits of bounded sequences of lower classes, where $\mathcal{B}^{0,bb}(\mathcal{H}) = \mathcal{H}$.

If we start the inductive procedure from the space of all continuous functions, we write simply $\mathcal{B}^\alpha(K,\mathbb{F})$ and $\mathcal{B}^{\alpha,b}(K,\mathbb{F})$ for the spaces of Baire-α functions. Then we have $\mathcal{B}^{\alpha,b}(K,\mathbb{F}) = \mathcal{B}^{\alpha,bb}(K,\mathbb{F})$. Let us remind that for a metrizable space K the identity $\mathcal{B}^b(K,\mathbb{F}) = \bigcup\limits_{\alpha<\omega_1} \mathcal{B}^{\alpha,b}(K,\mathbb{F})$ holds. Having started with the space $\mathcal{A}(K,\mathbb{F})$ of all continuous affine functions on a compact convex set K in a locally convex space, we obtain spaces $\mathcal{A}^\alpha(K,\mathbb{F})$, $\mathcal{A}^{\alpha,b}(K,\mathbb{F})$ and $\mathcal{A}^{\alpha,bb}(K,\mathbb{F})$. As a consequence of the uniform boundedness principle we get $\mathcal{A}^{\alpha,bb}(K,\mathbb{F}) = \mathcal{A}^{\alpha,b}(K,\mathbb{F}) = \mathcal{A}^\alpha(K,\mathbb{F})$ (see e.g. [16], Lemma 5.36) and the elements of this set we call *functions of affine class α*.

If X is a (either real or complex) Banach space and B_{X^*} is its dual unit ball endowed with the weak* topology, X is isometrically embedded in $\mathcal{C}(B_{X^*}, \mathbb{F})$ via the canonical embedding. We recall the definitions of Baire classes of X^{**} from [2]. For $\alpha \in [0, \omega_1)$, we call $\mathcal{B}^\alpha(X, \mathbb{F})$ the *intrinsic α-Baire class of* X^{**}. Following [2], page 1044, we denote the intrinsic α-th Baire class by X^{**}_α. Let us remark that our definition differs from the one in [2]. While in our case elements of X^{**}_α are restrictions of uniquely determined elements from X^{**} to the closed unit ball B_{X^*}, the functions considered in [2] are precisely these extensions.

Still considering X to be a subspace of $\mathcal{C}(B_{X^*}, \mathbb{F})$, the *$\alpha$-th Baire class* of X^{**} is defined as the set of those elements $x^{**} \in X^{**}$ whose restriction to B_{X^*} is a Baire-α function and which satisfy the barycentric formula, i.e.,

$$x^{**}\left(\int_{B_{X^*}} \mathrm{id}\, \mathrm{d}\mu\right) = \int_{B_{X^*}} x^{**}\, \mathrm{d}\mu$$

for every probability measure $\mu \in \mathcal{M}^1(B_{X^*})$. Where no confusion can arise, we do not distinguish between $X^{**}_{\mathcal{B}_\alpha}$ and $X^{**}_{\mathcal{B}_\alpha}|_{B_{X^*}}$.

Obviously, $X^{**}_\alpha \subset X^{**}_{\mathcal{B}_\alpha}$, but the converse need not hold by [22], Theorem on page 184. We refer the reader for a detailed exposition on Baire classes of Banach spaces to [2], pages 1043–1048.

We have proven in [13], Theorems 2.14, 2.15: *Let X be a real L_1-predual.*

(a) *If* $\operatorname{ext} B_{X^*}$ *is Lindelöf and $\alpha \in [0, \omega_1)$, then for every odd function $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$ there exists a function h on B_{X^*} extending f such that*

$\triangleright$ *$h \in X^{**}_{\alpha+1}$ if $\alpha \in [0, \omega_0)$,*

$\triangleright$ *$h \in X^{**}_\alpha$ if $\alpha \in [\omega_0, \omega_1)$.*

(b) *If* $\operatorname{ext} B_{X^*}$ *is a Lindelöf H-set and $\alpha \in [1, \omega_1)$, then for every odd function $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$ there exists a function $h \in X^{**}_\alpha$ extending f.*

The first goal of this paper is to extend the validity of the previous assertions to the complex setting. This is accomplished by Theorems 2.1 and 2.2.

The second goal of our paper is to extend [13], Corollary 2.16, which states: *Let X be a real L_1-predual such that* $\operatorname{ext} B_{X^*}$ *is a Lindelöf H-set. Then for any $\alpha \in [1, \omega_1)$, the space X^{**}_α is isometric to the space of all real bounded odd Baire-α functions on* $\operatorname{ext} B_{X^*}$.

Corollary 2.3 carries the result to the context of complex L_1-preduals. It is also a generalization of [12], Theorem 1, by Lindenstrauss and Wulbert.

It is worth pointing out that for a separable Banach space X, the set $\operatorname{ext} B_{X^*}$ of extreme points in B_{X^*} is an F_σ set if and only if it is a Lindelöf H-set. In the non-separable case only one implication remains valid in general: $\operatorname{ext} B_{X^*}$ is a Lindelöf H-set provided it is of type F_σ. For a detailed argument consult, e.g. [15], page 4.

2. Results

Before we attain the results promised in the introduction we are obliged to provide definitions of H-sets and homogeneous functions.

A set A of a topological space K is called an *H-set* (or a *resolvable set*) if for any nonempty $B \subset K$ (equivalently, for any nonempty closed $B \subset K$) there exists a relatively open $U \subset B$ such that either $U \subset A$ or $U \cap A = \emptyset$. It is easy to see that the family of all H-sets is an algebra (see, e.g. [8], §12, VI).

Let $\mathbb{T}$ stand for the unit circle endowed with the unit Haar measure $\mathrm{d}\alpha$. The following notions are due to Effros (see [3]). A set $B \subset B_{X^*}$ is called *homogeneous* if $\alpha B = B$ for each $\alpha \in \mathbb{T}$. An example of a homogeneous set is $\operatorname{ext} B_{X^*}$. A function f on a homogeneous set $B \subset B_{X^*}$ is called *homogeneous* (see, e.g. [3], page 53, and [9], page 240) if

$$f(\alpha x^*) = \alpha f(x^*), \quad (\alpha, x^*) \in \mathbb{T} \times B.$$

The main aim of this section is to infer the following results.

Theorem 2.1. *Let X be a complex L_1-predual with $\operatorname{ext} B_{X^*}$ being Lindelöf and $\alpha \in [0, \omega_1)$. Then for every homogeneous function $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*}, \mathbb{C})$ there exists a function h on B_{X^*} extending f such that*

- ▷ $h \in X^{**}_{\alpha+1}$ *if* $\alpha \in [0, \omega_0)$,
- ▷ $h \in X^{**}_{\alpha}$ *if* $\alpha \in [\omega_0, \omega_1)$.

Theorem 2.2. *Let X be a complex L_1-predual such that $\operatorname{ext} B_{X^*}$ is a Lindelöf H-set. Let $\alpha \in [1, \omega_1)$. Then for every homogeneous function $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*}, \mathbb{C})$ there exists a function $h \in X^{**}_{\alpha}$ extending f.*

As a consequence of the preceding theorem we obtain:

Corollary 2.3. *Let X be a complex L_1-predual such that $\operatorname{ext} B_{X^*}$ is a Lindelöf H-set. Let $\alpha \in [1, \omega_1)$. Then the space X^{**}_{α} is isometric to the space of all bounded homogeneous Baire-α functions on $\operatorname{ext} B_{X^*}$.*

To meet our goals we have to supply the reader with several further notions which are necessary within our proofs.

Let K be a compact convex set in a locally convex topological vector space. For a point $x \in K$, we can assign the set $\mathcal{M}^1_x(K)$ consisting of all probability measures on K satisfying $\int_K \mathrm{id}\, \mathrm{d}\mu = x$ (equivalently, $\mu(h) = h(x)$ for any continuous affine function h on K). Given a measure $\mu \in \mathcal{M}^1(K)$, we write $r(\mu)$ for the unique point $x \in K$ satisfying $x = \int_K \mathrm{id}\, \mathrm{d}\mu$ (see [1], Proposition I.2.1, or [9], Chapter 7, §20).

A function $f\colon K \to \mathbb{F}$ is *strongly affine* if f is μ-measurable for each $\mu \in \mathcal{M}^1(K)$ and $f(x) = \mu(f)$ for any $x \in K$ and $\mu \in \mathcal{M}^1_x(K)$.

The usual dilation order $\prec$ on $\mathcal{M}^1(K)$ is defined as $\mu \prec \nu$ if $\mu(f) \leqslant \nu(f)$ for any convex continuous function f on K. We write $\mathcal{M}^1_{\max}(K)$ for the set of all probability measures on K which are maximal with respect to $\prec$. A measure $\mu \in \mathcal{M}(K, \mathbb{F})$ is *boundary* if either $\mu = 0$ or the probability measure $|\mu|/\|\mu\|$ is in $\mathcal{M}^1_{\max}$. The symbol $\mathcal{M}^{\mathrm{bnd}}(K, \mathbb{F})$ denotes the space of all boundary measures on K.

By the Choquet representation theorem, for any $x \in K$ there exists $\mu \in \mathcal{M}^1_x(K) \cap \mathcal{M}^1_{\max}(K)$ (see [9], Corollary on page 192). The set K is called *simplex* if this measure is uniquely determined for each $x \in K$ (see [9], §20, Theorem 3). If K is metrizable, maximal measures are carried by the G_δ set $\operatorname{ext} K$ of extreme points of K (see [9], §20, Theorem 5). If K is a simplex, the space $\mathcal{A}(K, \mathbb{F})$ is an example of an L_1-predual (see [9], §23, Theorem 6).

We recall that a topological space X is *K-analytic* if it is an image of a Polish space under an upper semicontinuous compact-valued map (see [19], Section 2.1). Let us just recall that the family of K-analytic sets contains compact sets and is stable with respect to countable unions and countable intersections.

If K is a topological space, a *zero set* in K is an inverse image of a closed set in $\mathbb{R}$ under a continuous function $f\colon K \to \mathbb{R}$. The complement of a zero set is a *cozero set*. A countable union of closed sets is called an *F_σ set*, the complement of an F_σ set is a *G_δ set*. If K is normal, it follows from Tietze's theorem that a closed set is a zero set if and only if it is also a G_δ set. We recall that *Borel sets* are elements of the σ-algebra generated by the family of all open subsets of K and *Baire sets* are elements of the σ-algebra generated by the family of all cozero sets in K.

We say that a function $f\colon K \to \mathbb{F}$ from a topological space K is a *Baire function* if it is measurable with respect to the σ-algebra of Baire sets (i.e., $f^{-1}(U)$ is a Baire set for every open set $U \subset \mathbb{F}$). It is well known that any Baire function belongs to some $\mathcal{B}^\alpha(K, \mathbb{F})$ for a suitable ordinal $\alpha \in [0, \omega_1)$.

The subsequent notion of the mapping hom means another Effros' contribution to our paper (see [3]).

Definition 2.4. Let X be a complex Banach space. If f is a Borel function defined on a homogeneous set $B \subset B_{X^*}$, we set

$$(\hom f)(x^*) = \int_{\mathbb{T}} \alpha^{-1} f(\alpha x^*) \,\mathrm{d}\alpha, \quad x^* \in B.$$

The basic properties of the mapping hom are summarized by the following lemma.

Lemma 2.5. *Let $B \subset B_{X^*}$ be a homogeneous set and $f \in \mathcal{B}^b(B, \mathbb{C})$.*

(a) *The function* $\hom f$ *is homogeneous on B.*

(b) *The function f is homogeneous if and only if* $\hom f = f$.

(c) *If f is continuous on B, then* $\hom f$ *is continuous on B.*

(d) *If $f \in \mathcal{B}^{\alpha,b}(B, \mathbb{C})$, then* $\hom f \in \mathcal{B}^{\alpha,b}(B, \mathbb{C})$.

Proof. (a) The homogeneity of $\hom f$ can be observed by taking into account the following equations valid for any $x^* \in B$ and $\beta \in \mathbb{T}$:

$$(\hom f)(\beta x^*) = \int_{\mathbb{T}} \alpha^{-1} f(\alpha\beta x^*)\,\mathrm{d}\alpha = \beta \int_{\mathbb{T}} (\alpha\beta)^{-1} f(\alpha\beta x^*)\,\mathrm{d}\alpha = \beta(\hom f)(x^*).$$

(b) If $\hom f = f$, then f is homogeneous by (a).

If f is homogeneous,

$$(\hom f)(x^*) = \int_{\mathbb{T}} \alpha^{-1} f(\alpha x^*)\,\mathrm{d}\alpha = \int_{\mathbb{T}} f(x^*)\,\mathrm{d}\alpha = f(x^*)$$

for any $x^* \in B$.

(c) Let

$$g(\alpha, x^*) = \alpha^{-1} f(\alpha x^*), \quad (\alpha, x^*) \in \mathbb{T} \times B.$$

Then g is a continuous function on $\mathbb{T} \times B$. We want to show that the mapping

$$\varphi\colon\ x^* \mapsto \int_{\mathbb{T}} g(\alpha, x^*)\,\mathrm{d}\alpha, \quad x^* \in B,$$

is continuous on B. To this end, let $x^* \in B$ and $\varepsilon > 0$ be given.

For each $\alpha \in \mathbb{T}$ we find an open neighborhood U_α of α and V_α of x^* such that

$$|g(\alpha, x^*) - g(\beta, y^*)| < \varepsilon, \quad (\beta, y^*) \in U_\alpha \times V_\alpha.$$

By the compactness of $\mathbb{T} \times \{x^*\}$ there exist finitely many $\alpha_1, \ldots, \alpha_n \in \mathbb{T}$ such that

$$\mathbb{T} \times \{x^*\} \subset \bigcup_{i=1}^{n} (U_{\alpha_i} \times V_{\alpha_i}).$$

We set $V = \bigcap\limits_{i=1}^{n} V_{\alpha_i}$. For any $\alpha \in \mathbb{T}$ we have α_k such that $\alpha \in U_{\alpha_k}$ and then, for any $y^* \in V$,

$$|g(\alpha, x^*) - g(\alpha, y^*)| < |g(\alpha, x^*) - g(\alpha_k, x^*)| + |g(\alpha_k, x^*) - g(\alpha, y^*)| < 2\varepsilon.$$

Thus, for $y^* \in V$,

$$|\varphi(x^*) - \varphi(y^*)| = \left| \int_{\mathbb{T}} g(\alpha, x^*) - g(\alpha, y^*) \, \mathrm{d}\alpha \right| \leqslant \int_{\mathbb{T}} |g(\alpha, x^*) - g(\alpha, y^*)| \, \mathrm{d}\alpha < 2\varepsilon.$$

Hence φ is continuous at the point x^*.

(d) If f is bounded continuous on B, $\hom f$ is continuous on B by (c). The rest of the proof now follows by transfinite induction and the Lebesgue dominated convergence theorem. $\square$

Definition 2.6. The mapping $\hom\colon \mathcal{C}(B_{X^*}, \mathbb{C}) \to \mathcal{C}(B_{X^*}, \mathbb{C})$ induces a mapping (denoted likewise) $\hom\colon \mathcal{M}(B_{X^*}, \mathbb{C}) \to \mathcal{M}(B_{X^*}, \mathbb{C})$ defined as

$$(\hom \mu)(f) = \mu(\hom f), \quad f \in \mathcal{C}(B_{X^*}, \mathbb{C}), \quad \mu \in \mathcal{M}(B_{X^*}, \mathbb{C}).$$

Due to Lemma 2.5 (c), $\hom \mu$ is a well defined measure on B_{X^*}.

Lemma 2.7. *Let $F \subset B_{X^*}$ be a closed set. Then the set $\bigcup\limits_{\alpha \in \mathbb{T}} \alpha F$ is a closed homogeneous set in B_{X^*}.*

P r o o f. The assertion follows from the observation that $\bigcup\limits_{\alpha \in \mathbb{T}} \alpha F = \varphi(\mathbb{T} \times F)$, where

$$\varphi(\alpha, x^*) = \alpha x^*, \quad (\alpha, x^*) \in \mathbb{T} \times F.$$

Hence $\bigcup\limits_{\alpha \in \mathbb{T}} \alpha F$ is a continuous image of a compact set, and thus it is itself compact. Obviously, it is also homogeneous. $\square$

Lemma 2.8. *Let K be a compact space and $\mu \in \mathcal{M}(K, \mathbb{C})$. Then there exists a Baire function $\omega\colon K \to \mathbb{T}$ such that $\mathrm{d}|\mu| = \omega \, \mathrm{d}\mu$.*

P r o o f. Let μ be defined on a σ-algebra $\mathcal{S}$ containing all Borel subsets of K. By [21], Theorem 6.12, there exists an $\mathcal{S}$-measurable function $\varphi\colon K \to \mathbb{T}$ such that $\mathrm{d}|\mu| = \varphi \, \mathrm{d}\mu$. By Lusin's theorem, there exists a Baire function $\omega\colon K \to \mathbb{C}$ such that $\omega = \varphi$ holds $|\mu|$-almost everywhere. Finally, we adjust ω on a Baire $|\mu|$-null set such that ω has values in $\mathbb{T}$. $\square$

Analogously as in Lemma 2.5, now, we summarize the basic properties of the mapping hom.

Lemma 2.9. *Let B be a homogeneous universally measurable subset of B_{X^*}.*

(a) *If f is a bounded Baire function on B_{X^*} and $\mu \in \mathcal{M}(B_{X^*}, \mathbb{C})$, then* $(\hom \mu)(f) = \mu(\hom f)$.

(b) *If $\mu \in \mathcal{M}(B, \mathbb{C})$, then* $\hom \mu \in \mathcal{M}(B, \mathbb{C})$.

(c) *If $\mu \in \mathcal{M}(B_{X^*}, \mathbb{C})$ is boundary, then* $\hom \mu$ *is boundary.*

Proof. (a) Let

$$\mathcal{F} = \{f\colon B_{X^*} \to \mathbb{C}\colon f \text{ bounded Baire, } (\hom \mu)(f) = \mu(\hom f)\}.$$

The definition of the mapping hom provides $\mathcal{C}(B_{X^*}) \subset \mathcal{F}$. Obviously, $\mathcal{F}$ is closed with respect to taking pointwise limits of bounded sequences. Hence $\mathcal{F}$ contains all bounded Baire functions on B_{X^*}.

(b) Let $\mu \in \mathcal{M}(B, \mathbb{C})$ be a given nonzero measure. Let $K \subset B_{X^*} \setminus B$ be compact. We find compact sets $K_n \subset B$, $n \in \mathbb{N}$, such that $|\mu|(B \setminus K_n) \to 0$. Using Lemma 2.7 we may assume that K_n are homogeneous. Let $f_n\colon B_{X^*} \to [0,1]$ be continuous such that $f_n = 0$ on K_n and $f_n = 1$ on K. Let $\omega\colon B_{X^*} \to \mathbb{T}$ be a Baire function satisfying $\mathrm{d}|\hom \mu| = \omega \,\mathrm{d}(\hom \mu)$ (see Lemma 2.8). Then for each $n \in \mathbb{N}$ we have by (a)

$$\begin{aligned}
|\hom \mu|(K) &\leqslant |\hom \mu|(f_n) = \int_{B_{X^*}} f_n(x^*)\omega(x^*)\,\mathrm{d}(\hom \mu)\\
&= \left| \int_{\mathbb{T}} \left(\int_B f_n(\alpha x^*)\omega(\alpha x^*)\,\mathrm{d}\mu(x^*) \right) \mathrm{d}\alpha \right|\\
&\leqslant \int_{\mathbb{T}} \left(\int_B |f_n(\alpha x^*)|\,\mathrm{d}|\mu|(x^*) \right) \mathrm{d}\alpha\\
&= \int_{\mathbb{T}} \left(\int_{B\setminus K_n} |f_n(\alpha x^*)|\,\mathrm{d}|\mu|(x^*) \right) \mathrm{d}\alpha\\
&\leqslant |\mu|(B \setminus K_n).
\end{aligned}$$

Since $|\mu|(B \setminus K_n) \to 0$, we get $|\hom \mu|(K) = 0$. Thus $|\hom \mu|(B_{X^*} \setminus B) = 0$.

(c) For the proof see [3], Lemma 4.2, or [9], §23, Lemma 10. □

If X is a complex Banach space, then the following analogue of Lazar's characterization of real L_1-preduals (see [10], Theorem) is due to Effros:

A complex Banach space X is an L_1-predual if and only if, for any $x^* \in B_{X^*}$ and measures $\mu, \nu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^1_{\max}(B_{X^*})$, it holds that $\hom \mu = \hom \nu$ (see [3], Theorem 4.3, or [9], §23, Theorem 5).

Lemma 2.10. *Let K, L be K-analytic topological spaces and $r\colon K\to L$ be a continuous surjection. Let $g\colon L\to\mathbb{C}$. Then g is a Baire function on L if and only if $g\circ r$ is a Baire function on K.*

Proof. A function $g\colon L\to\mathbb{C}$ is Baire if and only if the real-valued functions $\operatorname{Re} g, \operatorname{Im} g\colon L\to\mathbb{R}$ are Baire, which is by [13], Lemma 2.1, equivalent to $(\operatorname{Re} g)\circ r$, $(\operatorname{Im} g)\circ r$ being Baire. This holds, again, if and only if

$$g\circ r=\operatorname{Re}(g\circ r)+\mathrm{i}\operatorname{Im}(g\circ r)=(\operatorname{Re} g)\circ r+\mathrm{i}(\operatorname{Im} g)\circ r$$

is a Baire function. $\square$

Lemma 2.11. *Let K be a compact convex set in a locally convex space such that $\operatorname{ext} K$ is Lindelöf. Let $f\colon \operatorname{ext} K\to\mathbb{R}$ be bounded and continuous. Then there exist a lower semicontinuous convex Baire function $l\colon K\to\mathbb{R}$ and upper semicontinuous concave Baire function $u\colon K\to\mathbb{R}$ such that $l\leqslant u$ and $l=u=f$ on $\operatorname{ext} K$.*

Proof. Using [15], Lemma 4.5, we find sequences (u_n) and (l_n) such that

- ▷ the functions u_n are continuous concave on K, l_n are continuous convex on K,
- ▷ $\inf f(\operatorname{ext} K)\leqslant \inf l_1(K)$, $\sup u_1(K)\leqslant \sup f(\operatorname{ext} K)$,
- ▷ $u_n\searrow f$, $l_n\nearrow f$ on $\operatorname{ext} K$.

We define $u=\inf\limits_{n\in\mathbb{N}} u_n$, $l=\sup\limits_{n\in\mathbb{N}} l_n$. Then we observe that $l\leqslant u$ by the minimum principle (see [1], Theorem I.4.10, or [16], Theorem 3.16), both functions are Baire, u is upper semicontinuous concave and l is lower semicontinuous convex. Apparently, $l=u=f$ on $\operatorname{ext} K$. This finishes the proof. $\square$

Lemma 2.12. *Let X be a complex Banach space such that $\operatorname{ext} B_{X^*}$ is Lindelöf. Let $f\in\mathcal{B}^{\alpha,b}(\operatorname{ext} K,\mathbb{C})$ be homogeneous. Then there exist a homogenous K-analytic set $B\supset \operatorname{ext} B_{X^*}$ and a homogeneous bounded Baire function g on B_{X^*} such that*

(a) *$g=f$ on $\operatorname{ext} B_{X^*}$,*
(b) *$\mu(g)=\nu(g)$ for any $\mu,\nu\in\mathcal{M}^1(B)$ with $\mu\prec\nu$,*
(c) *$\|g\|_{l^\infty(B_{X^*})}\leqslant 2\|f\|_{l^\infty(\operatorname{ext} B_{X^*})}$.*

Proof. We proceed by transfinite induction on the class of a function f.

We assume first that f is continuous. Let $f=f_1+\mathrm{i}f_2$ be decomposed into its real and imaginary part. By Lemma 2.11, there exist lower semicontinuous convex Baire functions l_1, l_2 on B_{X^*} and upper semicontinuous concave Baire functions u_1, u_2 on B_{X^*} such that $l_j\leqslant u_j$ and $l_j=u_j=f_j$ on $\operatorname{ext} B_{X^*}$, $j\in\{1,2\}$.

For $j\in\{1,2\}$, let

$$B_j=\{x\in K\colon\ u_j(x)=l_j(x)\}.$$

Since

$$B_j = \{x \in B_{X^*} \colon u_j(x) - l_j(x) \leqslant 0\} = \bigcap_{n=1}^{\infty} \Big\{x \in B_{X^*} \colon u_j(x) - l_j(x) < \frac{1}{n}\Big\},$$

the set B_j is a G_δ set containing $\operatorname{ext} B_{X^*}$ and, for $\mu, \nu \in \mathcal{M}^1(B_j)$ with $\mu \prec \nu$, we have by [16], Proposition 3.56,

$$\int_{B_j} u_j \, \mathrm{d}\mu \geqslant \int_{B_j} u_j \, \mathrm{d}\nu = \int_{B_j} l_j \, \mathrm{d}\nu \geqslant \int_{B_j} l_j \, \mathrm{d}\mu = \int_{B_j} u_j \, \mathrm{d}\mu.$$

Hence

$$\mu(u_j) = \nu(u_j) = \mu(l_j) = \nu(l_j).$$

The set $B_3 = B_1 \cap B_2$ is a G_δ set containing $\operatorname{ext} B_{X^*}$. Also, $l_j = u_j$ on B_3 for $j \in \{1, 2\}$. Thus for $\mu, \nu \in \mathcal{M}^1(B_3)$ with $\mu \prec \nu$ it holds that

$$\mu(u_j) = \nu(u_j) = \mu(l_j) = \nu(l_j). \tag{2.1}$$

We write $B_{X^*} \setminus B_3 = \bigcup F_n$, where F_n are closed sets in B_{X^*}. Then

$$H_n = \bigcup_{\alpha \in \mathbb{T}} \alpha F_n, \quad n \in \mathbb{N},$$

are homogeneous closed sets disjoint from $\operatorname{ext} B_{X^*}$ (see Lemma 2.7). For a given $n \in \mathbb{N}$ let $G_n = X \setminus H_n$. Then G_n is a homogeneous open set containing $\operatorname{ext} B_{X^*}$. Furthermore, $\bigcap G_n \subset B_3$.

Fix $n \in \mathbb{N}$. By the Lindelöf property of $\operatorname{ext} B_{X^*}$ there exists a countable cover of $\operatorname{ext} B_{X^*}$ by closed sets $\{K_{n,k} \colon k \in \mathbb{N}\}$ such that

$$\operatorname{ext} B_{X^*} \subset \bigcup_{k=1}^{\infty} K_{n,k} \subset G_n.$$

By replacing $K_{n,k}$ with $\bigcup\limits_{\alpha \in \mathbb{T}} \alpha K_{n,k}$, if necessary, we may assume that $K_{n,k}$ are homogeneous. Then $K_n = \bigcup\limits_{k=1}^{\infty} K_{n,k}$ is a homogeneous F_σ set satisfying

$$\operatorname{ext} B_{X^*} \subset K_n \subset G_n.$$

Thus $B = \bigcap K_n$ is a K-analytic homogeneous set satisfying

$$\operatorname{ext} B_{X^*} \subset B \subset B_3.$$

We set

$$g_1 = u_1, \quad g_2 = u_2, \quad g = \hom(g_1 + \mathrm{i} g_2) \quad \text{on } B_{X^*}.$$

By Lemma 2.5 (d), g is a Baire function on B_{X^*}. Further, $g = f$ on $\operatorname{ext} B_{X^*}$, since, for $x^* \in \operatorname{ext} B_{X^*}$,

$$\begin{aligned} g(x^*) = (\hom(g_1 + \mathrm{i} g_2))(x^*) &= \int_{\mathbb{T}} \alpha^{-1}(g_1 + \mathrm{i} g_2)(\alpha x^*)\,\mathrm{d}\alpha \\ &= \int_{\mathbb{T}} \alpha^{-1} f(\alpha x^*)\,\mathrm{d}\alpha = (\hom f)(x^*) = f(x^*). \end{aligned}$$

Next, let $\mu, \nu \in \mathcal{M}^1(B)$ with $\mu \prec \nu$ be given. For $\alpha \in \mathbb{T}$, let $\sigma_\alpha \colon B_{X^*} \to B_{X^*}$ denote the affine homeomorphism defined by $\sigma_\alpha(x^*) = \alpha x^*$, $x^* \in B_{X^*}$. Then $\sigma_\alpha \mu \prec \sigma_\alpha \nu$ for each $\alpha \in \mathbb{T}$, and thus employing (2.1)

$$\begin{aligned} \mu(g) = \mu(\hom(g_1 + \mathrm{i} g_2)) &= \int_{\mathbb{T}} \alpha^{-1} \left(\int_B (g_1 + \mathrm{i} g_2)(\alpha x^*)\,\mathrm{d}\mu(x^*) \right) \mathrm{d}\alpha \\ &= \int_{\mathbb{T}} \alpha^{-1} (\sigma_\alpha \mu)(g_1 + \mathrm{i} g_2)\,\mathrm{d}\alpha = \int_{\mathbb{T}} \alpha^{-1} (\sigma_\alpha \mu)(u_1 + \mathrm{i} u_2)\,\mathrm{d}\alpha \\ &= \int_{\mathbb{T}} \alpha^{-1} (\sigma_\alpha \nu)(u_1 + \mathrm{i} u_2)\,\mathrm{d}\alpha = \ldots = \nu(g). \end{aligned}$$

Finally, due to [16], Theorem 3.85,

$$\|g_j\|_{l^\infty(B)} = \|f_j\|_{l^\infty(\operatorname{ext} B_{X^*})}, \quad j \in \{1, 2\}.$$

Hence

$$\begin{aligned} \|g\|_{l^\infty(B_{X^*})} &\leqslant \|g_1 + \mathrm{i} g_2\|_{l^\infty(B_{X^*})} \leqslant \|f_1\|_{l^\infty(\operatorname{ext} B_{X^*})} + \|f_2\|_{l^\infty(\operatorname{ext} B_{X^*})} \\ &\leqslant 2\|f\|_{l^\infty(\operatorname{ext} B_{X^*})}. \end{aligned}$$

Hence g satisfies the conditions (a), (b) and (c), which concludes the proof for the case $\alpha = 0$.

Assume now that the claim holds true for all β smaller then some countable ordinal α. Given $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*}, \mathbb{C})$, let (f_n) be a bounded sequence of functions with $f_n \in \mathcal{B}^{\alpha_n,b}(\operatorname{ext} B_{X^*}, \mathbb{C})$ for some $\alpha_n < \alpha$, $n \in \mathbb{N}$, such that $f_n \to f$. We may assume that $\|f_n\|_{l^\infty(\operatorname{ext} B_{X^*})} \leqslant \|f\|_{l^\infty(\operatorname{ext} B_{X^*})}$ for $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we use the induction hypothesis and find a homogeneous K-analytic set $B_n \supset \operatorname{ext} B_{X^*}$ along with a homogeneous Baire function g_n on B_{X^*} that coincides with f_n on $\operatorname{ext} B_{X^*}$, and satisfies $\mu(g_n) = \nu(g_n)$ for any $\mu, \nu \in \mathcal{M}^1(B_n)$ with $\mu \prec \nu$ and also

$$\|g_n\|_{l^\infty(B_{X^*})} \leqslant 2\|f_n\|_{l^\infty(\operatorname{ext} B_{X^*})} \leqslant 2\|f\|_{l^\infty(\operatorname{ext} B_{X^*})}.$$

Let $g_n = |g_n| e^{\mathrm{i}\,\mathrm{Arg}\, g_n}$ be the polar decomposition of g_n (here $\mathrm{Arg}\colon \mathbb{C} \to (-\pi, \pi]$ denotes the principal value of a complex number, where we set $\mathrm{Arg}\, 0 = \pi$). Since the functions $z \mapsto |z|$ and $z \mapsto \mathrm{Arg}\, z$ are Baire on $\mathbb{C}$, the functions $x^* \mapsto |g_n(x^*)|$ and $x^* \mapsto e^{\mathrm{i}\,\mathrm{Arg}(g_n(x^*))}$ are Baire on B_{X^*}. We set

$$r(x^*) = \limsup_{n\to\infty} |g_n(x^*)|, \quad a(x^*) = \limsup_{n\to\infty} \mathrm{Arg}(g_n(x^*)), \quad x^* \in B_{X^*},$$

and

$$h(x^*) = r(x^*) e^{\mathrm{i}a(x^*)}, \quad x^* \in B_{X^*}.$$

Then h is a Baire function on B_{X^*} satisfying $\|h\|_{l^\infty(B_{X^*})} \leqslant 2\|f\|_{l^\infty(\mathrm{ext}\, B_{X^*})}$. Further, let

$$B = \Big\{ x^* \in \bigcap_{n=1}^\infty B_n \colon \ (g_n(x^*)) \text{ converges} \Big\}, \quad g(x^*) = (\hom h)(x^*), \quad x^* \in B_{X^*}.$$

Then B is a homogeneous K-analytic set containing $\mathrm{ext}\, B_{X^*}$, g is a bounded homogeneous Baire function on B_{X^*} (by Lemma 2.5 (a), (d)), $\|g\|_{l^\infty(B_{X^*})} \leqslant 2\|f\|_{l^\infty(\mathrm{ext}\, B_{X^*})}$ and

$$g(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f_n(x) = f(x), \quad x \in \mathrm{ext}\, B_{X^*}.$$

Finally, for $\mu, \nu \in \mathcal{M}^1(B)$ satisfying $\mu \prec \nu$ we have

$$\mu(g) = \int_B \Big(\lim_{n\to\infty} g_n\Big)\, \mathrm{d}\mu = \lim_{n\to\infty} \mu(g_n) = \lim_{n\to\infty} \nu(g_n) = \int_B \Big(\lim_{n\to\infty} g_n\Big)\, \mathrm{d}\nu = \nu(g).$$

This finishes the proof. $\square$

Lemma 2.13. *Let X be a complex Banach space, $B \supset \mathrm{ext}\, B_{X^*}$ be a homogeneous K-analytic set and $f\colon B_{X^*} \to \mathbb{C}$ be a function such that*

(a) *f is bounded and Baire,*
(b) *$\mu(f) = \nu(f)$ for every $\mu, \nu \in \mathcal{M}^1(B)$ with $\mu \prec \nu$,*
(c) *$\mu(f) = 0$ for every $\mu \in \mathcal{M}^{\mathrm{bnd}}(B_{X^*}, \mathbb{R}) \cap \mathcal{A}(B_{X^*}, \mathbb{R})^\perp$.*

Then there exists an affine bounded Baire function $h\colon K \to \mathbb{C}$ such that

(d) *$h = f$ on B,*
(e) *$\mu(h) = h(r(\mu))$ for any $\mu \in \mathcal{M}^1_{\max}(B_{X^*})$.*

P r o o f. Let $B \supset \mathrm{ext}\, B_{X^*}$ and $f\colon B_{X^*} \to \mathbb{R}$ be as in the hypothesis.
We set

$$h(x^*) = \nu(f), \quad \nu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^1_{\max}(B_{X^*}), \quad x^* \in B_{X^*}.$$

Then h is correctly defined because of (c).

Further, h is affine. Indeed, let $\alpha x^* + (1-\alpha)y^*$ be a convex combination of points $x^*, y^* \in B_{X^*}$. Pick $\nu_{x^*} \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^1_{\max}(B_{X^*})$ and $\nu_{y^*} \in \mathcal{M}^1_{y^*}(B_{X^*}) \cap \mathcal{M}^1_{\max}(B_{X^*})$. Since the set of maximal measures is a convex cone and the mapping r is affine,

$$\alpha\nu_{x^*} + (1-\alpha)\nu_{y^*} \in \mathcal{M}^1_{\alpha x^* + (1-\alpha)y^*}(B_{X^*}) \cap \mathcal{M}^1_{\max}(B_{X^*}).$$

Thus

$$\begin{aligned} h(\alpha x^* + (1-\alpha)y^*) &= (\alpha\nu_{x^*} + (1-\alpha)\nu_{y^*})(f) = \alpha\nu_{x^*}(f) + (1-\alpha)\nu_{y^*}(f) \\ &= \alpha h(x^*) + (1-\alpha)h(y^*), \end{aligned}$$

and h is affine.

Obviously, due to (b), the fact that any maximal measure is carried by B (see [1], Remark, page 38, or [16], Theorem 3.79 (a)) and the definition of h, we have

$$\begin{gathered} h(x^*) = \nu(f) = \varepsilon_{x^*}(f) = f(x^*), \quad \nu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^1_{\max}(B_{X^*}), \quad x^* \in B, \\ h(r(\mu)) = \mu(f) = \mu(h), \quad \mu \in \mathcal{M}^1_{\max}(B_{X^*}). \end{gathered}$$

Thus (d) and (e) hold.

Finally we show that h is Baire. The set B is K-analytic, and thus universally measurable by [19], Corollary 2.9.3. Further, it follows from [5], Theorem 1 and Theorem 3, that

$$\mathcal{M}^1(B) = \{\mu \in \mathcal{M}^1(B_{X^*})\colon\ \mu(B) = 1\}$$

is K-analytic.

Since f is a bounded Baire function on B, the function $\tilde{f}\colon \mathcal{M}^1(B) \to \mathbb{C}$ defined as

$$\tilde{f}(\mu) = \int_B f \,\mathrm{d}\mu, \quad \mu \in \mathcal{M}^1(B),$$

is a well defined Baire function on $\mathcal{M}^1(B)$. The mapping $r\colon \mathcal{M}^1(B) \to B_{X^*}$ is an affine continuous surjection (this follows from [1], page 12, or [16], Proposition 2.38) and $\tilde{f} = h \circ r$.

Indeed, let $\mu \in \mathcal{M}^1(B)$. We pick a maximal measure $\nu \in \mathcal{M}^1_{\max}(B_{X^*})$ with $\mu \prec \nu$. Then $\nu \in \mathcal{M}^1(B)$ and $r(\mu) = r(\nu)$, thus due to (b)

$$\tilde{f}(\mu) = \mu(f) = \nu(f) = h(r(\nu)) = h(r(\mu)) = (h \circ r)(\mu).$$

By Lemma 2.10, h is a Baire function on B_{X^*}. $\square$

Lemma 2.14. *Let K be a compact convex set and $f\colon K \to \mathbb{C}$ be a bounded Baire affine function such that $\mu(f) = f(r(\mu))$ for every $\mu \in \mathcal{M}^1_{\max}(K)$. Then f is strongly affine.*

P r o o f. The result is acquired by applying Lemma 2.5 from [13] to the real and imaginary part of the complex function f in the hypothesis. □

Lemma 2.15. *Let K be a topological space, $\mathcal{H} \subset \mathcal{C}(K, \mathbb{F})$, $\alpha \in [0, \omega_1)$, and $f \in \mathcal{B}^\alpha(\mathcal{H})$. Then there exists a countable set $\mathcal{F} \subset \mathcal{H}$ such that $f \in \mathcal{B}^\alpha(\mathcal{F})$.*

P r o o f. The assertion follows by transfinite induction. □

Lemma 2.16. *Let X be a complex L_1-predual such that $\operatorname{ext} B_{X^*}$ is Lindelöf. Then for every bounded homogeneous Baire function on $\operatorname{ext} B_{X^*}$ there exists its homogeneous Baire strongly affine extension on B_{X^*}.*

P r o o f. Let f be a homogeneous bounded Baire function on $\operatorname{ext} B_{X^*}$. By Lemma 2.12, there exist a homogeneous K-analytic set $B \supset \operatorname{ext} B_{X^*}$ and a bounded Baire homogeneous function $h\colon B_{X^*} \to \mathbb{C}$ such that

- ▷ $h = f$ on $\operatorname{ext} B_{X^*}$,
- ▷ for any $\mu, \nu \in \mathcal{M}^1(B)$ with $\mu \prec \nu$ it holds that $\mu(h) = \nu(h)$.

Let

$$\omega \in \mathcal{M}^{\mathrm{bnd}}(B_{X^*}, \mathbb{R}) \cap \mathcal{A}(B_{X^*}, \mathbb{R})^\perp$$

be given. Without loss of generality we may assume that $\omega = \mu - \nu$, where $\mu, \nu \in \mathcal{M}^1_{\max}(B_{X^*})$. Then $r(\mu) = r(\nu)$. By Effros' theorem [3], Theorem 4.3 (see also [9], §23, Theorem 5) and Lemma 2.9 (a),

$$\mu(h) = \mu(\hom h) = (\hom \mu)(h) = (\hom \nu)(h) = \nu(\hom h) = \nu(h).$$

Hence $\omega(h) = 0$. By Lemma 2.13, there exists an affine bounded Baire extension g of h satisfying $\mu(g) = g(r(\mu))$ for each $\mu \in \mathcal{M}^1_{\max}(B_{X^*})$. By Lemma 2.14, the extension g is strongly affine.

It remains to show that g is homogeneous. Given $x^* \in B_{X^*}$ and a maximal measure $\mu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^1_{\max}(B_{X^*})$, the measure $\sigma_\alpha\mu \in \mathcal{M}^1_{\alpha x^*}(B_{X^*}) \cap \mathcal{M}^1_{\max}(B_{X^*})$ for every $\alpha \in \mathbb{T}$. Due to [16], Theorem 3.79 (c), we have

$$g(\alpha x^*) = (\sigma_\alpha\mu)(g) = (\sigma_\alpha\mu)(h) = \alpha\mu(h) = \alpha\mu(g) = \alpha g(x^*).$$

This concludes the proof. □

Lemma 2.17. *Let K be a compact convex set with* $\operatorname{ext} K$ *being Lindelöf. Then any bounded Baire $\mathbb{F}$-valued function on* $\operatorname{ext} K$ *can be extended to a bounded Baire $\mathbb{F}$-valued function on K.*

Proof. The real variant is precisely [13], Lemma 2.8. For the complex version decompose the given function to its real and imaginary part and apply the real version. $\square$

Definition 2.18. Let X be a complex L_1-predual with $\operatorname{ext} B_{X^*}$ Lindelöf. For any bounded Baire function f on $\operatorname{ext} B_{X^*}$ we define

$$Tf(x^*) = (\operatorname{hom}\mu)(\tilde{f}), \quad \mu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^1_{\max}(B_{X^*}), \quad x^* \in B_{X^*},$$

where $\tilde{f}$ is an arbitrary bounded Baire function on B_{X^*} extending f.

We point out that Tf is well defined since

- ▷ $\operatorname{hom}\mu = \operatorname{hom}\nu$ for any $\mu, \nu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^1_{\max}(B_{X^*})$ and $x^* \in B_{X^*}$ by the mentioned Effros' theorem,
- ▷ f has a bounded Baire extension on B_{X^*} (see Lemma 2.17),
- ▷ given two bounded Baire extensions $\tilde{f}_1$, $\tilde{f}_2$ of f, they coincide on a Baire set containing $\operatorname{ext} B_{X^*}$, and thus $(\operatorname{hom}\mu)(\tilde{f}_1) = (\operatorname{hom}\mu)(\tilde{f}_2)$ for any $\mu \in \mathcal{M}^1_{\max}(B_{X^*})$.

The mapping T is defined analogously as in the real case (see [13], Definition 2.9). An obvious difference lies in using an operator hom instead of odd. It is also a natural generalization of the dilation mapping defined in the simplicial case, e.g. in [16], Definition 6.7.

Lemma 2.19. *Let X be a complex L_1-predual with* $\operatorname{ext} B_{X^*}$ *Lindelöf. Let f be a bounded Baire complex-valued function on* $\operatorname{ext} B_{X^*}$*. Then Tf is a bounded homogeneous Baire strongly affine function on B_{X^*} such that $Tf = \operatorname{hom} f$ on* $\operatorname{ext} B_{X^*}$*.*

Proof. Let $\tilde{f}$ be a bounded Baire function on B_{X^*} extending f (see Lemma 2.17). Since $\operatorname{hom}\tilde{f}$ is a homogeneous bounded Baire function on B_{X^*}, by Lemma 2.16 there exists a homogeneous Baire strongly affine function h on B_{X^*} satisfying $h = \operatorname{hom}\tilde{f}$ on $\operatorname{ext} B_{X^*}$. Let $x^* \in B_{X^*}$ be given and let $\mu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^1_{\max}(B_{X^*})$. Since $\operatorname{hom}\mu$ is boundary (see Lemma 2.9 (c)) and $h = \operatorname{hom}\tilde{f}$ on a Baire set containing $\operatorname{ext} B_{X^*}$, we obtain

$$Tf(x^*) = (\operatorname{hom}\mu)(\tilde{f}) = \mu(\operatorname{hom}\tilde{f}) = \mu(h) = h(x^*).$$

Thus Tf is a homogeneous Baire strongly affine function on B_{X^*}.

Finally, for a point $x^* \in \operatorname{ext} B_{X^*}$ we have

$$Tf(x^*) = h(x^*) = (\operatorname{hom}\tilde{f})(x^*) = (\operatorname{hom} f)(x^*).$$

The proof is finished. $\square$

Remark 2.20. Let X be a complex Banach space and $f\colon B_{X^*}\to\mathbb{C}$ a bounded affine homogeneous function. Then $f(0)=0$ and f can be extended to an element of X^{**}.

Lemma 2.21. *Let X be a complex L_1-predual with $\operatorname{ext} B_{X^*}$ Lindelöf. Let (f_n) be a bounded sequence of Baire complex-valued functions on $\operatorname{ext} B_{X^*}$ converging pointwise to f on $\operatorname{ext} B_{X^*}$. Then $Tf_n\to Tf$.*

P r o o f. For $n\in\mathbb{N}$, let $\tilde f_n$ be bounded Baire extensions of the functions f_n (see Lemma 2.17), obviously we may assume that they are bounded by the same constant. We set

$$h_1=\limsup_{n\to\infty}(\operatorname{Re}\tilde f_n),\quad h_2=\limsup_{n\to\infty}(\operatorname{Im}\tilde f_n)\quad\text{and}\quad \tilde f=h_1+\mathrm{i}h_2.$$

Then $\tilde f$ is a bounded Baire function extending f. The set

$$B=\bigl\{x^*\in B_{X^*}\colon\ \text{both }(\operatorname{Re}\tilde f_n(x^*))\text{ and }(\operatorname{Im}\tilde f_n(x^*))\text{ converge}\bigr\}$$

is a Baire set containing $\operatorname{ext} B_{X^*}$. Thus, for $x^*\in B_{X^*}$ and $\mu\in\mathcal{M}^1_{x^*}(B_{X^*})\cap\mathcal{M}^1_{\max}(B_{X^*})$, we have

$$\begin{aligned}\lim_{n\to\infty}(Tf_n)(x^*)&=\lim_{n\to\infty}(\hom\mu)(\tilde f_n)=\lim_{n\to\infty}\int_B\tilde f_n\,\mathrm{d}(\hom\mu)\\&=\lim_{n\to\infty}\int_B(\operatorname{Re}\tilde f_n+\mathrm{i}\operatorname{Im}\tilde f_n)\,\mathrm{d}(\hom\mu)=\int_B(h_1+\mathrm{i}h_2)\,\mathrm{d}(\hom\mu)\\&=(\hom\mu)(\tilde f)=Tf(x^*).\end{aligned}$$

This concludes the proof. $\square$

We recall that the validity of [2], Theorem II.1.2 (a), can be extended to complex Banach spaces (see [14], Proposition 3.1).

Proposition 2.22. *Let X be a complex Banach space and f a Baire-1 affine homogeneous function on B_{X^*}. Then $f\in X^{**}_1$.*

Lemma 2.23. *Let X be a complex L_1-predual with $\operatorname{ext} B_{X^*}$ Lindelöf and $\alpha\in[0,\omega_1)$. Let $f\in\mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*},\mathbb{C})$. Then*
- ▷ *$Tf\in X^{**}_{\alpha+1}$ if $\alpha\in[0,\omega_0)$,*
- ▷ *$Tf\in X^{**}_{\alpha}$ if $\alpha\in[\omega_0,\omega_1)$.*

P r o o f. If $\alpha=0$, then Tf is a homogeneous strongly affine function whose restriction to $\operatorname{ext} B_{X^*}$ is equal to a continuous function $\hom f$ (see Lemma 2.19). Thus

$Tf \in \mathcal{B}^{1,b}(B_{X^*})$ by Remark 2.20 and [15], Theorem 1.2. Using Proposition 2.22 we acquire that $Tf \in X_1^{**}$.

For $\alpha < \omega_0$ now the proof follows by induction using Lemma 2.21.

If $\alpha = \omega_0$, let $f_n \in \mathcal{B}^{\alpha_n,b}(\operatorname{ext} B_{X^*}, \mathbb{C})$, $\alpha_n < \alpha$, form a bounded sequence converging to $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*}, \mathbb{C})$. By Lemma 2.21, $Tf_n \to Tf$. By the first part of the proof, $Tf \in X_\alpha^{**}$.

For higher Baire classes we use again transfinite induction. $\square$

Lemma 2.24. *Let X be a complex L_1-predual with* $\operatorname{ext} B_{X^*}$ *being a Lindelöf H-set and $\alpha \in [1, \omega_1)$. Let $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*}, \mathbb{C})$. Then $Tf \in X_\alpha^{**}$.*

P r o o f. The proof is analogous to the proof of Lemma 2.23, we only use instead of [15], Theorem 1.2, as the starting point of transfinite induction the following fact from [15], Theorem 1.3: *If* $\operatorname{ext} B_{X^*}$ *is a Lindelöf H-set and $h \in X^{**}$ is a strongly affine function on B_{X^*} whose restriction on* $\operatorname{ext} B_{X^*}$ *is Baire-*1*, then h is Baire-*1 *on B_{X^*}.* Any such function is then in X_1^{**} by Proposition 2.22. $\square$

We conclude the paper with the proofs of the main results introduced at the beginning of this section.

P r o o f of Theorem 2.1. By Lemma 2.23, if $\alpha \in [0, \omega_0)$ then the function Tf is in $X_{\alpha+1}^{**}$, and if $\alpha \in [\omega_0, \omega_1)$ then $Tf \in X_\alpha^{**}$. Since $Tf = \hom f = f$ on $\operatorname{ext} B_{X^*}$ (see Lemma 2.19), the proof is finished. $\square$

P r o o f of Theorem 2.2. The proof is analogous to the proof of Theorem 2.1, only we use Lemma 2.24 instead of Lemma 2.23. $\square$

P r o o f of Corollary 2.3. A function $f \in X_\alpha^{**}$ is bounded, homogeneous, Baire-α and strongly affine. The restriction mapping $f \in X_\alpha^{**} \mapsto f|_{\operatorname{ext} B_{X^*}}$ is therefore an isometric isomorphism onto the space of all bounded homogeneous Baire-α functions on $\operatorname{ext} B_{X^*}$ due to Theorem 2.2.

The norm preservation is guaranteed by the following observation. Let $x^* \in B_{X^*}$ be arbitrary and $\mu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^1_{\max}(B_{X^*})$. The set

$$B = \bigl\{y^* \in B_{X^*} \colon\ |f(y^*)| \leqslant \|f|_{\operatorname{ext} B_{X^*}}\|_{l^\infty(\operatorname{ext} B_{X^*})}\bigr\}$$

is a Baire set containing $\operatorname{ext} B_{X^*}$, and thus $\mu(B) = 1$. Hence

$$|f(x^*)| = |\mu(f)| \leqslant \int_B |f| \,\mathrm{d}\mu \leqslant \|f|_{\operatorname{ext} B_{X^*}}\|_{l^\infty(\operatorname{ext} B_{X^*})}.$$

This concludes the proof. $\square$

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