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ON THE EXISTENCE OF PARABOLIC ACTIONS IN CONVEX DOMAINS OF $\mathbb{C}^{k+1}$

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Abstract. We prove that the one-parameter group of holomorphic automorphisms induced on a strictly geometrically bounded domain by a biholomorphism with a model domain is parabolic. This result is related to the Greene-Krantz conjecture and more generally to the classification of domains having a non compact automorphisms group. The proof relies on elementary estimates on the Kobayashi pseudo-metric.

Keywords: parabolic boundary point; convex domain; automorphism group

MSC 2010: 32M05, 32H02, 32H50

1. Main results

It is a standard and classical result of Cartan that if Ω is a bounded domain in $\mathbb{C}^n$ whose automorphism group $\operatorname{Aut}(\Omega)$ is not compact then there exist a point $x \in \Omega$, a point $p \in \partial\Omega$, and automorphisms $\varphi_j \in \operatorname{Aut}(\Omega)$ such that $\varphi_j(x) \to p$. Such a point p is called a *boundary orbit accumulation point*.

The classification of domains with non-compact automorphism groups deeply relies on the geometry of the boundary at an orbit accumulation point p. For instance, Wong and Rosay [15], [16] showed that if p is a strongly pseudoconvex point, then the domain is biholomorphic to the euclidean ball. In their works [1]–[3], Bedford and Pinchuk introduced a scaling technique to analyse the case of a weakly pseudoconvex boundary orbit accumulation point. In particular, they characterized the pseudoconvex and finite type domains in $\mathbb{C}^2$ having a non-compact automorphism group. The papers [4]–[6] deal with a local version, in the spirit of Wong-Rosay, of this result.

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On the other hand, Greene and Krantz [8] suggested the following conjecture.

Greene-Krantz Conjecture. *If the automorphism group* $\mathrm{Aut}(\Omega)$ *of a smoothly bounded pseudoconvex domain* $\Omega \Subset \mathbb{C}^n$ *is non-compact, then any orbit accumulation point is of finite type.*

This conjecture is still open, even for convex domains, despite a quite large number of partial results: Greene and Krantz [8], Kim [11], Kim and Krantz [12], [13], Kang [10], Landucci [14], and Byun and Gaussier [7]. We refer to the survey [9] for a more precise discussion of the above conjecture and for a general presentation of the subject.

The scaling technique applied to a bounded and strictly geometrically convex domain $\Omega \subset \mathbb{C}^{k+1}$ produces a biholomorphism $\psi\colon D \to \Omega$ where D is of the form $D = \{(w,z) \in \mathbb{C}^{k+1}\colon \operatorname{Re} w + \sigma(z) < 0\}$ for some smooth convex function σ on $\mathbb{C}^{k+1}$. In view of the above conjecture, it seems relevant to show that the one-parameter group of biholomorphic mappings induced on Ω by the translations $(w,z) \mapsto (w+\mathrm{i}t, z)$ is parabolic. This is what we establish in this short note:

Theorem 1. *Let* Ω *be a* $\mathcal{C}^1$*-smooth bounded strictly geometrically convex domain in* $\mathbb{C}^{k+1}$*. Let* $\psi\colon \Omega \to D$ *be a biholomorphism, where* $D := \{(w,z) \in \mathbb{C}^{k+1}\colon \operatorname{Re} w + \sigma(z) < 0\}$ *and* σ *is a* $\mathcal{C}^1$*-smooth nonnegative convex function on the complex plane such that* $\sigma(0) = 0$*. Then there exists a point* $a_\infty \in \partial\Omega$ *such that* $\lim_{t\to\infty} \psi^{-1}(w \pm \mathrm{i}t, z) = a_\infty$ *for any* $(w,z) \in D$*.*

We now start to prove the above theorem and first recall some notation and definitions.

For two domains D, Ω in $\mathbb{C}^n$, we denote by $\mathrm{Hol}(D,\Omega)$ the set of all holomorphic maps from D into Ω. We denote by $d(z,\partial\Omega)$ the distance from a point $z \in \Omega$ to $\partial\Omega$ and by Δ the open unit disk in the complex plane.

Let p, q be two points in a domain Ω in $\mathbb{C}^n$ and let X be a vector in $\mathbb{C}^n$. The *Kobayashi infinitesimal pseudometric* $F_\Omega(p,X)$ is defined by

$$F_\Omega(p,X) = \inf\{\alpha > 0;\ \exists g \in \mathrm{Hol}(\Delta,\Omega),\ g(0) = p,\ g'(0) = X/\alpha\}.$$

The *Kobayashi pseudodistance* $k_\Omega(p,q)$ is defined by

$$k_\Omega(p,q) = \inf \int_a^b F_\Omega(\gamma(t), \gamma'(t))\, \mathrm{d}t,$$

where the infimum is taken over all differentiable curves $\gamma\colon [a,b] \to \Omega$ such that $\gamma(a) = p$ and $\gamma(b) = q$.

Before proceeding to prove Theorem 1, we establish a few lemmas.

Lemma 2. *Let Ω be a $\mathcal{C}^1$-smooth bounded strictly geometrically convex domain in $\mathbb{C}^{k+1}$. Then there exists $\varepsilon_0 > 0$ such that for any $\eta \in \partial\Omega$ and for any $\varepsilon \in (0, \varepsilon_0]$ there is a constant $K(\varepsilon) > 0$ such that*

$$k_\Omega(z, w) \geqslant -\frac{1}{2} \ln d(z, \partial\Omega) - K(\varepsilon)$$

holds for any $z, w \in \Omega$ with $|z - \eta| < \varepsilon,\ |w - \eta| > 3\varepsilon$.

P r o o f. Since $\partial\Omega$ is strictly geometrically convex, there exists a family of holomorphic peak functions

$$F\colon\ \Omega \times \partial\Omega \to \mathbb{C}, \quad (z, \eta) \mapsto F(z, \eta)$$

such that

(i) F is continuous and $F(., \eta)$ is holomorphic;
(ii) $|F| < 1$;
(iii) there exist a positive constant A and a positive constant ε_0 such that $|1 - F(\eta + t\vec{n}_\eta, \eta)| \leqslant At$ for $t \in [0, \varepsilon_0]$, where $\vec{n}_\eta$ is the normal to $\partial\Omega$ at η.

Taking $\varepsilon_0 > 0$ small enough, we may assume that $\partial B(\eta, 3\varepsilon) \cap \partial\Omega \neq \emptyset$ for $\varepsilon \leqslant \varepsilon_0$ and for any $\eta \in \partial\Omega$.

Let γ be a smooth path in Ω such that $\gamma(0) = z,\ \gamma(1) = w,\ \int_0^1 F_\Omega(\gamma(t), \gamma'(t))\,\mathrm{d}t \leqslant k_\Omega(z, w) + 1$. Let $z_0 \in \gamma$ be such that $|z_0 - \eta| = 3\varepsilon$. We have

$$k_\Omega(z, w) \geqslant \int_0^1 F_\Omega(\gamma(t), \gamma'(t))\,\mathrm{d}t - 1 \geqslant k_\Omega(z, z_0) - 1. \tag{1.1}$$

Let $\widetilde{\eta} \in \partial\Omega$ be such that $z = \widetilde{\eta} + t\vec{n}_\eta$, $t > 0$. We set $u_0 := F(z_0, \widetilde{\eta})$ and $u := F(z, \widetilde{\eta})$, u and u_0 are in the unit disk Δ. Then we have

$$k_\Omega(z, z_0) \geqslant k_\Delta(u, u_0) = \frac{1}{2} \ln \frac{1 + |\tau_{u_0}(u)|}{1 - |\tau_{u_0}(u)|} \geqslant -\frac{1}{2} \ln(1 - |\tau_{u_0}(u)|), \tag{1.2}$$

where $\tau_{u_0}(u) = \dfrac{u - u_0}{1 - \bar{u}_0 u}$. One easily checks that

$$1 - |\tau_{u_0}(u)| \leqslant \frac{2|1 - u|}{1 - |u_0|}. \tag{1.3}$$

Using the properties of F we obtain

$$|1 - u| = |1 - F(z, \widetilde{\eta})| \leqslant At = Ad(z, b\Omega). \tag{1.4}$$

Since $|\eta - \widetilde{\eta}| \leqslant |\eta - z| + |z - \widetilde{\eta}| < 2\varepsilon$ and $|z_0 - \eta| = 3\varepsilon$, we have $|z_0 - \widetilde{\eta}| \geqslant \varepsilon$.

Setting $M(\varepsilon) := \sup\limits_{\substack{\eta\in\partial\Omega,\, z\in\Omega \\ |z-\eta|\geqslant\varepsilon}} |F(z,\eta)|$, $M(\varepsilon) < 1$ yields

$$1 - |u_0| = 1 - |F(z_0, \widetilde{\eta})| \geqslant 1 - M(\varepsilon) > 0. \tag{1.5}$$

From (1.3), (1.4), and (1.5) we get

$$1 - |\tau_{u_0}(u)| \leqslant \frac{2A}{1 - M(\varepsilon)} d(z, \partial\Omega). \tag{1.6}$$

Then from (1.1), (1.2), and (1.6) we obtain

$$k_\Omega(z, w) \geqslant -\frac{1}{2} \ln d(z, \partial\Omega) - \frac{1}{2} \ln \frac{2A}{1 - M(\varepsilon)} - 1 \tag{1.7}$$

and this completes the proof. □

Lemma 3. *Let Ω be a $\mathcal{C}^1$-smooth, bounded, strictly geometrically convex domain in $\mathbb{C}^{k+1}$ and let $\eta, \eta' \in \partial\Omega$ satisfy $\eta \neq \eta'$. Then there exist $\varepsilon > 0$ and a constant K such that*

$$k_\Omega(z, w) \geqslant -\frac{1}{2} \ln d(z, \partial\Omega) - \frac{1}{2} \ln d(w, \partial\Omega) - K$$

for any $z \in B(\eta, \varepsilon)$ and any $w \in B(\eta', \varepsilon)$.

Proof. Let η and η' be two distinct points on $\partial\Omega$. Suppose that $|z - \eta| < \varepsilon$ and $|w - \eta'| < \varepsilon$ and let γ be a C^1 path in Ω connecting z and w such that $k_\Omega(z, w) \geqslant \int_0^1 F_\Omega[\gamma(t), \gamma'(t)]\, \mathrm{d}t - 1$. If ε is small enough we may find $z_0 \in \gamma$ such that $|z_0 - \eta| > 3\varepsilon$ and $|z_0 - \eta'| > 3\varepsilon$. Let $z_0 = \gamma(t_0)$, then

$$\begin{aligned} k_\Omega(z, w) &\geqslant \int_0^{t_0} F_\Omega(\gamma(t), \gamma'(t))\, \mathrm{d}t + \int_{t_0}^1 F_\Omega(\gamma(t), \gamma'(t))\, \mathrm{d}t - 1 \\ &\geqslant k_\Omega(z, z_0) + k_\Omega(z_0, w) - 1 \\ &\geqslant -\frac{1}{2} \ln d(z, \partial\Omega) - \frac{1}{2} \ln d(w, \partial\Omega) - 2K(\varepsilon) - 1, \end{aligned}$$

where the last inequality is obtained by applying twice Lemma 2. □

We now recall the definition of horospheres. Let $a \in \Omega,\ \eta \in \partial\Omega,\ R > 0$. The *big horosphere with pole a, center η and radius R* in Ω is defined as follows:

$$F_a^\Omega(\eta, R) = \Big\{ z \in \Omega \colon \liminf_{w\to\eta} (k_\Omega(z, w) - k_\Omega(a, w)) < \frac{1}{2} \ln R \Big\}.$$

Lemma 4. *If Ω is a $\mathcal{C}^1$-smooth, bounded, strictly geometrically convex domain in $\mathbb{C}^{k+1}$, then $\overline{F_a^\Omega(\eta,R)}\cap\partial\Omega\subset\{\eta\}$ for any $a\in\Omega,\ \eta\in\partial\Omega,\ R>0$.*

P r o o f. If there exists $\eta'\in\partial\Omega\cap\overline{F_a^\Omega(\eta,R)}$ then we can find a sequence $\{z_n\}\subset\Omega$ with $z_n\to\eta'$ and a sequence $\{w_n\}\subset\Omega$ with $w_n\to\eta$ such that

$$k_\Omega(z_n,w_n)-k_\Omega(a,w_n)<\frac{1}{2}\ln R. \tag{1.8}$$

By Lemma 3, the following estimate holds if $\eta\neq\eta'$ and n is great enough:

$$k_\Omega(z_n,w_n)\geqslant-\frac{1}{2}\ln d(z_n,\partial\Omega)-\frac{1}{2}\ln d(w_n,\partial\Omega)-K, \tag{1.9}$$

where K is a constant which only depends on η, η' and Ω.

On the other hand, we have

$$k_\Omega(a,w_n)\leqslant-\frac{1}{2}\ln d(w_n,\partial\Omega)+K(a), \tag{1.10}$$

since $\partial\Omega$ is smooth.

From (1.8), (1.9), and (1.10) we get

$$-\frac{1}{2}\ln d(z_n,\partial\Omega)\lesssim 1, \tag{1.11}$$

which is absurd. □

P r o o f of Theorem 1. Set $a_n:=\psi^{-1}(-t_n,0)$ where $\lim t_n=\infty$. After taking a subsequence we may assume that $\lim a_n=a_\infty\in\partial\Omega$. We may also assume that a_∞ is the origin in $\mathbb{C}^{k+1}$.

Set $b_t:=\psi^{-1}(-1+\mathrm{i}t,0)$. According to Lemma 4, it suffices to show that there exists $R_0>0$ such that

$$\{b_t\colon\ t\in\mathbb{R}\}\subset F_{a_0}^\Omega(a_\infty,R_0). \tag{1.12}$$

Since $a_n\to a_\infty$, we have

$$\liminf_{w\to a_\infty}(k_\Omega(b_t,w)-k_\Omega(a_0,w))\leqslant\liminf_{n\to\infty}(k_\Omega(b_t,a_n)-k_\Omega(a_0,a_n)). \tag{1.13}$$

Then by the invariance of the Kobayashi metric and the convexity of D we have

$$\begin{aligned} k_\Omega(b_t,a_n)-k_\Omega(a_0,a_n)&=k_D((-1+\mathrm{i}t,0),(-t_n,0))-k_D((-t_0,0),(-t_n,0))\\ &=k_H(-1+\mathrm{i}t,-t_n)-k_H(-t_0,-t_n),\end{aligned} \tag{1.14}$$

where H is the left half plane $\{w\in\mathbb{C}\colon\ \operatorname{Re}w<0\}$.

Let $\sigma\colon H \to \Delta$ be a biholomorphism between H and the disk Δ given by $\sigma(w) = (w+1)/(w-1)$. Set $z_t := \sigma(-1+\mathrm{i}t) = \mathrm{i}t/(-2+\mathrm{i}t)$ and $x_n := \sigma(-t_n) = (-t_n+1)/(-t_n-1)$. Then we have

$$
\begin{aligned}
(1.15)\qquad k_H(-1+\mathrm{i}t,\,-t_n) - k_H(-t_0,-t_n) &= k_\Delta(z_t,x_n) - k_\Delta(x_0,x_n)\\
&= \ln\left(\frac{|1-x_nz_t|+|x_n-z_t|}{|1-x_nz_t|-|x_n-z_t|}\,\frac{|1-x_nx_0|+|x_n-x_0|}{|1-x_nx_0|-|x_n-x_0|}\right)\\
&= \ln\left(\frac{|1-x_nx_0|+|x_n-x_0|}{|1-x_nz_t|-|x_n-z_t|}\,\frac{|1-x_nz_t|+|x_n-z_t|}{|1-x_nx_0|-|x_n-x_0|}\right)\\
&= \ln\left(\frac{|1-x_nx_0|^2-|x_n-x_0|^2}{|1-x_nz_t|^2-|x_n-z_t|^2}\left(\frac{|1-x_nz_t|+|x_n-z_t|}{|1-x_nx_0|-|x_n-x_0|}\right)^2\right)\\
&= \ln\left(\frac{1-x_0^2}{1-|z_t|^2}\left(\frac{|1-x_nz_t|+|x_n-z_t|}{|1-x_nx_0|-|x_n-x_0|}\right)^2\right).
\end{aligned}
$$

From (1.14) and (1.15) we conclude

$$
(1.16)\qquad \lim_{n\to\infty}(k_\Omega(b_t,a_n) - k_\Omega(a_0,a_n)) = \ln\left(\frac{1-x_0^2}{1-|z_t|^2}\,\frac{|1-z_t|^2}{|1-x_0|^2}\right) = \ln\frac{1-x_0^2}{|1-x_0|^2}.
$$

Finally, (1.12) follows directly from (1.13) and (1.16) when $\ln(1-x_0^2)/|1-x_0|^2 < \frac{1}{2}\ln R_0$. □

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