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Why λ-additive (fuzzy) measures?

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WHY λ-ADDITIVE (FUZZY) MEASURES?

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The paper is concerned with generalized (i. e. monotone and possibly non-additive) measures. A discussion concerning the classification of these measures, according to the type and amount of non-additivity, is done. It is proved that λ-additive measures appear naturally as solutions of functional equations generated by the idea of (possible) non additivity.

Keywords: generalized measure (probability), λ-additive measure, functional equation

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1. INTRODUCTION

Classical measure theory is based on the concept of additivity (or which is more, countable additivity). Recently, new necessities (theoretical and practical) imposed the study of (possibly) non-additive measures. These measures play an increasing role in the description of all kind of phenomena and therefore are intensively studied now.

A major role in the history of generalized (i. e. monotone and possibly non-additive) measures was played by the Japanese scholar M. Sugeno, who formally introduced them in his doctoral thesis [2], under the name of fuzzy measures. Trying to study non additivity in the same thesis, he introduced the concept of $\lambda-$additivity (λ-rule), which appears to be extremely important, raising many future developments. Among these developments, we can nominate the important fact that any λ-additive measure (for $\lambda > -1$) is generated by a classical measure (i. e. it is representable, using the terminology in the present paper) as shown by Z. Wang in [3].

The main goal of our paper is to *discuss* about the concept of $\lambda-$additivity as (perhaps) the most natural and efficient (from computational point of view) instrument of measure of non-additivity. We do this in two steps (see the paragraph "Results").

The first step consists in a preliminary discussion pertaining a classification of generalized measures, according to two criteria: their degree of $\lambda-$additivity and their possible representability as homeomorphic images of classical measures. We took into consideration also the marginal value $\lambda = -1$, when speaking about $\lambda-$additivity.

The second step consists in applying functional equations methods for searching (possibly) non-additive measures. Using classical results (see [1]) we arrive at the conclusion that the λ-additive measures appear even in this framework, proving to be extremely

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natural and adequate. So, the answer to the question in the title should be: Because they are good!

We lay stress upon the fact that, for simplicity reasons, throughout the paper we are concerned only with monotone and normalized measures, which we call **measures**. The main theoretical tool we use is the monograph [4].

2. PRELIMINARY FACTS

As usual, the set $[0, \infty)$ of all positive real numbers, will be denoted by $\mathbb{R}_+$.

For any set T we shall write $\mathcal{P}(T)$ to denote the set of all subsets of T. Throughout the paper we shall consider a *class of sets* $\mathcal{T}$, i.e. $\emptyset \in \mathcal{T} \subset \mathcal{P}(T)$ and $T \in \mathcal{T}$.

A function $\mu : \mathcal{T} \to \mathbb{R}_+$ will be called a **measure** (or a **generalized probability**) if $\mu(\emptyset) = 0$, is monotone (i.e. $\mu(A) \leq \mu(B)$ whenever A, B are in $\mathcal{T}$ and $A \subset B$) and normalized (i.e. $\mu(T) = 1$). So, from now on, we shall consider measures $\mu : \mathcal{T} \to [0, 1]$.

If $\lambda \in [-1, \infty)$ and $\mu : \mathcal{T} \to [0, 1]$ is a measure, we say that μ is **λ-additive** (**satisfies the λ-rule**, according to [4]) if, for any A, B in $\mathcal{T}$ with $A \cap B = \emptyset$ and $A \cup B \in \mathcal{T}$ one has

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B) \tag{1}$$

(compare with [4], where the value $\lambda = -1$ is not taken into consideration).

In case $\lambda = 0$, μ is *additive.*

If there exists $\lambda \in [-1, \infty)$ (respectively $\lambda \in (-1, \infty)$) such that μ is λ-additive, we shall say that μ **is some λ-additive** (respectively **some λ^*-additive**).

Condition $\lambda \geq -1$ appears to be natural because it is compatible with the monotonicity of μ and is necessary in case μ is decomposable (see Definition 3.5).

Remark. Non-additive measures can describe, sometimes, better than additive measures different phenomena. Let us notice that superadditive measures ($\mu(A \cup B) \geq \mu(A) + \mu(B)$, if $A \cap B = \emptyset$) can express a cooperative action in terms of the measured property, while subadditive measures ($\mu(A \cup B) \leq \mu(A) + \mu(B)$, if $A \cap B = \emptyset$) can express non cooperation or mutual inhibition. So, additive (classical) measures can express no interaction. It is easy to see that λ-additive measures are particular cases of superadditive measures (in case $\lambda \geq 0$) or of subadditive measures (in case $\lambda \leq 0$). Of course, λ-additive measures cannot encompass all monotone measures.

Here is an example (due to one of the anonymous referees) of a subadditive measure which is not some λ-additive in the sense of the present paper. Let a, b, c be three different elements and take: $T = \{a, b, c\}$, $\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\mu : \mathcal{T} \to [0, 1]$ given via $\mu(\emptyset) = 0$, $\mu(\{a\}) = 1/4$, $\mu(\{b\}) = 1/2$, $\mu(\{a, b\}) = 1/2$, $\mu(\{a, b, c\}) = 1$. One can see that μ is "-2-additive", i.e. $\mu(A \cup B) = \mu(A) + \mu(B) - 2\mu(A)\mu(B)$ for any $A, B \in \mathcal{T}$, $A \cap B = \emptyset$, $A \cup B \in \mathcal{T}$.

3. RESULTS

3.1. Preliminary discussion

Definition 3.1. A function $h : [0, 1] \to [0, 1]$ which is strictly increasing, continuous and such that $h(0) = 0$, $h(1) = 1$, will be called a $T-$**function**.

It follows that h^{-1} is a T-function too.

Example 3.2. (Examples of T-functions)

a) The canonical T-function is defined as follows. For any $0 \neq \lambda \in (-1, \infty)$, $h_\lambda : [0,1] \to [0,1]$ acts via (see [4])

$$h_\lambda(x) = \frac{(\lambda+1)^x - 1}{\lambda}. \tag{2}$$

b) The inverse T-function of h_λ from a) is $\theta_\lambda : [0,1] \to [0,1]$, given via (see [4])

$$\theta_\lambda(y) = \frac{\ln(1+\lambda y)}{\ln(1+\lambda)}. \tag{3}$$

c) We can consider a natural non null number n which generates the T-function $h : [0,1] \to [0,1]$, given via

$$h(x) = x^n. \tag{4}$$

Definition 3.3. We shall say that the measure μ is **representable** if there exists an additive measure $m : \mathcal{T} \to [0,1]$ and a T-function $h : [0,1] \to [0,1]$, such that $\mu = h \circ m$. In this case we say that the pair (m, h) **represents** μ.

Remarks.

1. Of course, any additive measure is representable.

2. It follows that $m = h^{-1} \circ \mu$. The classical terminology (see [4]) refers to h^{-1}. More precisely, one says that μ is **quasi-additive** and h^{-1} is called **the T-function of** μ.

3. It is possible to have more than one pair (m, h) which represents μ as the following example shows.

Example 3.4. Take $T = \{1,2\}$, $\mathcal{T} = \mathcal{P}(T)$ and $\mu : \mathcal{T} \to [0,1]$ given via $\mu(\emptyset) = 0$, $\mu(T) = 1$, $\mu(\{1\}) = \alpha$, $\mu(\{2\}) = \beta$, where $0 < \alpha < \beta < 1$ (of course μ is a measure).

We can consider an additive measure $m : \mathcal{T} \to [0,1]$, given via $m(\emptyset) = 0$, $m(T) = 1$, $m(\{1\}) = a$, $m(\{2\}) = b$, where $0 < a < 1 - a = b < 1$.

Any strictly increasing continuous function $h : [0,1] \to [0,1]$ such that $h(0) = 0$, $h(1) = 1$, $h(a) = \alpha$, $h(b) = \beta$ generates the pair (m, h) which represents μ.

Remark. Assume $0 \neq \lambda \in (-1, \infty)$ and $\mu : \mathcal{T} \to [0,1]$ is λ-additive. Then, according to [3], μ is representable: there exists an additive measure $m : \mathcal{T} \to [0,1]$ such that $\mu = h_\lambda \circ m$ (see (2)).

Standard Terminology

In case $\mathcal{T}$ is a $\sigma-$algebra and the measure μ can be represented by a pair (m,h), where m is σ-additive (i.e. m is a classical probability), we say that μ is a **Sugeno measure** (see [4]) or a **fuzzy measure** (according to M. Sugeno who introduced the notion in his doctoral thesis [2]).

Definition 3.5. A measure μ will be called **decomposable** if there exist A, B in $\mathcal{T}$ such that $A \cap B = \emptyset$, $A \cup B = T$ and $\mu(A) > 0$, $\mu(B) > 0$.

Example 3.6. We shall exhibit an example of measure μ which is $-1-$additive and decomposable.

Take $T = \{1,2\}$, $\mathcal{T} = \mathcal{P}(T)$, $\mu : \mathcal{T} \to [0,1]$ defined via $\mu(\emptyset) = 0$, $\mu(T) = 1$, $\mu(\{1\}) = \alpha$ (where $0 < \alpha \leq 1$) and $\mu(\{2\}) = 1$ and check the aforementioned properties.

Remark. Any -1-additive measure μ has the following property: for any A, B in $\mathcal{T}$ such that $A \cap B = \emptyset$, $A \cup B = T$, one has either $\mu(A) = 1$ or $\mu(B) = 1$ (possibly $\mu(A) = \mu(B) = 1$). Indeed, if $\mu(A) = a$, $\mu(B) = b$, one has:

$1 - a = b(1-a) \Leftrightarrow 1 - a - b(1-a) = 0 \Leftrightarrow (1-a)(1-b) = 0 \Leftrightarrow (a = 1 \vee b = 1)$.

Proposition 3.7. If μ is a measure which is decomposable and $-1-$additive, it follows that μ is not representable.

P r o o f. Let A, B in $\mathcal{T}$ with $A \cap B = \emptyset$, $A \cup B = T$ and $\mu(A) = \alpha > 0$, $\mu(B) = \beta > 0$. Assume $\alpha = 1$ (see the preceding remark). Accepting the existence of a pair (m,h) which represents μ, we shall arrive at a contradiction. Indeed, $1 = \alpha = \mu(A) = h(m(A))$ implies $m(A) = 1$ and $\beta = h(m(B)) > 0$ implies $m(B) > 0$.

Hence $1 = m(A \cup B) = m(A) + m(B)$, false. □

Example 3.8. There exist representable measures which are not some λ-additive.

Indeed, take $T = [0,1]$, $\mathcal{T}$ =the Lebesgue measurable subsets of T, $m : \mathcal{T} \to [0,1]$ =the Lebesgue measure on T.

Define $\mu : \mathcal{T} \to [0,1]$ via $\mu(A) = m(A)^2$. Hence $\mu = h \circ m$ where $h : [0,1] \to [0,1]$ is the T-function given via $h(t) = t^2$, see (4).

The pair(m,h) represents μ.

Now, take $\lambda \in [-1,\infty)$. We shall show that μ is not λ-additive. To this end, let A, B in $\mathcal{T}$, $A \cap B = \emptyset$ and accept that

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B)$$

i.e.

$$(m(A) + m(B))^2 = m(A)^2 + m(B)^2 + \lambda m(A)^2 m(B)^2$$

which means

$$2m(A)m(B) = \lambda m(A)^2 m(B)^2.$$

It follows that, for any disjoint A, B in $\mathcal{T}$ with $m(A) > 0$, $m(B) > 0$, one must have

$$\lambda m(A)m(B) = 2$$

which is not possible (take e.g. $A_1 = [0, \frac{1}{a}]$, $B_1 = (\frac{1}{a}, \frac{2}{a}]$ and $A_2 = [0, \frac{1}{b}]$, $B_2 = (\frac{1}{b}, \frac{1}{2b}]$, with $a \neq b$, $a \geq 2$, $b \geq 2$.

We are now in position to state the following synthesis result.

Theorem 3.9. We have the following strict inclusions:

$$\begin{gathered}\text{the some } \lambda^*\text{-additive measures}\\ \subset\\ \text{the representable measures}\\ \subset\\ \text{all measures.}\end{gathered}$$

Proof. The first inclusion is a classical result (see [4] and [3]). Indeed, let $\lambda \in (-1, \infty)$ and take a λ-additive measure μ.

In case $\lambda = 0$, it is clear that μ is representable. In case $\lambda \neq 0$, we consider $m = \theta_\lambda \circ \mu$ (see (3)) which is additive. Hence $\mu = h_\lambda \circ m$ (see (2)) and (m, h_λ) represents μ. Actually, everything follows from the fact that the map $m \mapsto h_\lambda \circ m$ is a bijection with domain = the additive measures and codomain = the $\lambda-$measures.

The first inclusion is strict (Example 3.8) and the second inclusion is strict (Example 3.6 and Proposition 3.7). $\square$

We finish this preliminary discussion by presenting some strange measures which appear in a natural way.

Example 3.10. (The Marginal Measures) We shall consider an additive measure $m : \mathcal{T} \to [0, 1]$ and, for any $0 \neq \lambda \in (-1, \infty)$, the canonical T-function $h_\lambda : [0, 1] \to [0, 1]$ (see (3)). Then, for any $0 \neq \lambda \in (-1, \infty)$, we can construct the λ-additive function (see [4]) $\mu(m, \lambda) = h_\lambda \circ m$. Hence, for any $A \in \mathcal{T}$, one has

$$\mu(m, \lambda)(A) = \frac{(\lambda + 1)^{m(A)} - 1}{\lambda}.$$

It is seen that, for any $A \in \mathcal{T}$, the following limits exist

$$\mu(m, -1)(A) = \lim_{\lambda \to -1+} \mu(m, \lambda)(A) = \begin{cases} 1, & \text{if } \ m(A) > 0, \\ 0, & \text{if } \ m(A) = 0, \end{cases}$$

$$\mu(m, \infty)(A) = \lim_{\lambda \to \infty} \mu(m, \lambda)(A) = \begin{cases} 1, & \text{if } \ m(A) = 1, \\ 0, & \text{if } \ m(A) < 1. \end{cases}$$

We defined $\mu(m, -1) : \mathcal{T} \to [0, 1]$, $\mu(m, \infty) : \mathcal{T} \to [0, 1]$, which are measures and we shall call them **marginal measures**.

In order to better study their properties, we shall assume that $\mathcal{T}$ is an algebra of sets.

In case m is not decomposable, which is equivalent to the fact $T$is an **atom of** m (i. e. either $m(A) = 0$ or $m(A) = 1$ for any $A \in \mathcal{T}$), one can see that $\mu(m, -1) = \mu(m, \infty) = m = \mu(m, \lambda)$, for any $0 \neq \lambda \in (-1, \infty)$.

The interesting situation is when m is decomposable. In this case, one can see that:

a) $\mu(m, -1)$ is -1-additive and not representable, due to the fact that $\mu(m, -1)$ is decomposable too (see Proposition 7). Supplementarily, one can see that $\mu(m, -1)$ is not λ-additive, if $\lambda > -1$.

b) $\mu(m, \infty)$ is not representable and not λ-additive for any $\lambda \geq -1$. Concerning the non representability of $\mu = \mu(m, \infty)$: if (n, h) represents μ, let A, B in $\mathcal{T}$ with $A \cap B = \emptyset$, $A \cup B = T$ and $0 < m(A) < 1$, $0 < m(B) < 1$. Then $\mu(A) = 0 = h(n(A))$, $\mu(B) = 0 = = h(n(B))$, hence $n(A) = n(B) = 0$ and $n(T) = n(A) + n(B) = 0$, contradiction.

3.2. The functional equations point of view. Adequacy of λ-dditivity

The most "popular" measures are the classical ones, i. e. the additive measures. The increasing importance of the non-additive measures led to the study of the "amount of non-additivity" of a given measure μ. It is natural to study the problem as follows.

Define first the set $\Delta = [0, 1]^2$.

Consider a function $\varphi : \Delta \to \mathbb{R}$. We study the measures $\mu : \mathcal{T} \to [0, 1]$ having the property that, for any A, B in $\mathcal{T}$, such that $A \cap B = \emptyset$ and $A \cup B \in \mathcal{T}$, one has

$$\mu(A \cup B) = \mu(A) + \mu(B) + \varphi(\mu(A), \mu(B)) \tag{5}$$

(in (1), one has $\varphi(\mu(A), \mu(B)) = \lambda\mu(A)\mu(B)$).

So, the function φ is the instrument for measuring the amount of non-additivity of μ.

One can write (5) in a more general way, considering a function $F : \Delta \to \mathbb{R}$ and the condition (for any A, B in $\mathcal{T}$, $A \cap B = \emptyset$, $A \cup B \in \mathcal{T}$)

$$\mu(A \cup B) = F(\mu(A), \mu(B)) \tag{6}$$

(in (1): $F(\mu(A), \mu(B)) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B)$).

This point of view, concerning the measure of the amount of non-additivity of μ, will be called here *the functional equations point of view.* We shall see that, from this point of view, the $\lambda-$additivity of μ appears in a natural way.

Considering the equalities (5) and (6), the following problem is natural.

Problem A

Which type of function φ (or of function F) is more suitable? Here "suitable" means two facts:

a) the function φ (or the function F) should measure the amount of non-additivity of μ in an appropriate (realistic) way;

b) the function φ (or the function F) should be acceptable from computational point of view.

To be honest, point b) is decisive for the choice of φ (or F).

In this form, the problem seems to be too general, consequently too difficult. In order to increase the possibility of finding a solution, we shall reduce the investigation area. Namely, *we shall be concerned only with the study of representable measures* μ.

Assume that μ is represented by the pair (m, h). Hence, for any A, B in $\mathcal{T}$, $A \cap B = \emptyset$, $A \cup B \in \mathcal{T}$, relations (5) and (6) become respectively:

$$\begin{aligned} h(m(A \cup B)) &= h(m(A) + m(B)) \\ &= h(m(A)) + h(m(B)) + \varphi(h(m(A)), h(m(B))) \end{aligned} \tag{5'}$$

$$h(m(A \cup B)) = h(m(A) + m(B)) = F(h(m(A)), h(m(B))). \tag{6'}$$

Writing $m(A) = x$, $m(B) = y$, we have $(x, y) \in \Delta$. If is natural to impose the following conditions which should be fulfilled by h for any $(x, y) \in \{(s, t) \in \Delta \mid s + t \leq 1\}$:

$$h(x + y) = h(x) + h(y) + \varphi(h(x), h(y)) \tag{5''}$$

$$h(x + y) = F(h(x), h(y)). \tag{6''}$$

We shall be mainly concerned with the particular situation

$$F(u, v) = u + v + \varphi(u, v). \tag{7}$$

Consequently, we shall consider Problem A within this new (particular) framework, in the following form

Problem A$'$

Which type of φ (or of F) is more suitable? Supplementarily: considering φ (or of F) to be known, find h (viewing (5$''$) or (6$''$) as functional equations with unknown function h).

From now on, we shall be concerned with Problem A$'$. Before proceeding further, we shall change a little the framework, considering that $\varphi : \mathbb{R}^2 \to \mathbb{R}$, $F : \mathbb{R}^2 \to \mathbb{R}$ (i. e. φ and F are defined everywhere). This gives us the possibility of considering (6$''$) as a functional equation with F known and h unknown and to use the general theory (see [1]). On the other hand, to give φ means to give F, when working in case (7).

Using (5$'$), (5$''$), (6$'$), (6$''$) and $h(0) = 0$, we propose the following reasonable assumptions:

$$\varphi(u, v) = \varphi(v, u); \quad F(u, v) = F(v, u) \tag{8}$$

(for any $(u, v) \in \Delta$);

$$\varphi(u, 0) = \varphi(0, u) = 0; \quad F(u, 0) = F(0, u) = u \tag{9}$$

(for any $u \in [0, 1]$).

To continue, we assume (*Major Assumption*) that the most suitable type of F is the polynomial type. Hence, we shall consider that $F : \mathbb{R}^2 \to \mathbb{R}$ is a *polynomial function.*

General theory of the equation (6$''$) (see [1, §2.2.4]) says that the only possibilities for F are the following:

Type 1:

$$F(u,v) = u + v + C$$

with C real constant.

Type 2:

$$F(u,v) = \lambda uv + Bu + Bv + \frac{B^2 - B}{\lambda}$$

with λ, B real constants, $\lambda \neq 0$.

We shall work in the natural case (7).

For Type 1, it follows that $\varphi(u,v) = C$ for any real u and v. Due to (9) it follows that $\varphi = 0$, hence $F(u,v) = u + v$ and ($5''$) becomes the Cauchy equation

$$h(x+y) = h(x) + h(y)$$

for any real x and y. All the continuous solutions $h : \mathbb{R} \to \mathbb{R}$ are of the form $h(x) = ax$, with a =real constant. Due to $h(1) = 1$, one has $a = 1$, hence $h(x) = x$ and $\mu = h \circ m = m$ *is additive.*

For Type 2, we use (7) to get that, for any real u and v:

$$\varphi(u,v) = \lambda uv + (B-1)(u+v) + \frac{B^2 - B}{\lambda}.$$

The control conditions (8) and (9): (8) is automatically verified and condition (9) becomes (for any real u):

$$(B-1)u + \frac{B^2 - B}{\lambda} = 0$$

hence $B = 1$, $\varphi(u,v) = \lambda uv$ and $F(u,v) = u + v + \lambda uv$. So, (5) (or (6)) becomes: for any A, B in $\mathcal{T}$, $A \cap B = \emptyset$, $A \cup B \in \mathcal{T}$, one has

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B)$$

hence μ is λ-*additive* (we saw that condition $\lambda \geq -1$ is compulsory).

The functional equation ($5''$) for h (valid for any real x, y) becomes

$$h(x+y) = h(x) + h(y) + \lambda h(x)h(y). \tag{10}$$

Here $\lambda \neq 0$ is a real constant.

For the sake of completeness, let us recall how to solve (10). Define $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \lambda h(x) + 1$, hence $h(x) = \frac{f(x)-1}{\lambda}$ and (10) becomes

$$\frac{f(x+y) - 1}{\lambda} = \frac{f(x) + f(y) - 2}{\lambda} + \frac{(f(x)-1)(f(y)-1)}{\lambda}$$

$$\Leftrightarrow f(x+y) = f(x)f(y).$$

This classical modified Cauchy equation has all the continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = \exp(ax)$, with a =real constant. Hence, $h : \mathbb{R} \to \mathbb{R}$ has the form

$$h(x) = \frac{\exp(ax) - 1}{\lambda}.$$

We have automatically $h(0)=0$. Condition $h(1)=1$ means

$$\exp(a)-1=\lambda \Leftrightarrow \lambda+1=\exp(a) \Leftrightarrow a=\ln(\lambda+1).$$

So, we *are obliged to assume* $\lambda>-1$. Finally, we get $h:\mathbb{R}\to\mathbb{R}$, defined via

$$h(x)=\frac{(\lambda+1)^x-1}{\lambda}. \tag{11}$$

This result is "good", because, practically (2) and (11) coincide. We got, using h, the canonical T-function $h_\lambda:[0,1]\to[0,1]$ (see (2)), with $h_\lambda(x)=h(x)$ for $x\in[0,1]$. It follows that (m,h_λ) represents μ and that μ is λ-additive.

Remark. We could solve Problem A$'$, finding a pair (m,h_λ) which represents μ, *only in case* $\lambda>-1$. The case $\lambda=-1$, generating "pathological" measures (e. g. non representable ones), could not be treated in the aforementioned way.

At the same time, we could see that the functional equations point of view led us naturally to consider λ-additive measures which are natural and adequate.

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