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NESTED MATRICES AND INVERSE M-MATRICES

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Abstract. Given a sequence of real or complex numbers, we construct a sequence of nested, symmetric matrices. We determine the LU- and QR-factorizations, the determinant and the principal minors for such a matrix. When the sequence is real, positive and strictly increasing, the matrices are strictly positive, inverse M-matrices with symmetric, irreducible, tridiagonal inverses.

Keywords: nested matrix; tridiagonal matrix; inverse M-matrix; principal minor; determinant; QR-factorization

MSC 2010: 15A15, 15A09, 15B05

1. Basic results for nested matrices

For a positive integer n, and a sequence of complex numbers $a_1, a_2, \ldots, a_n$, the $n \times n$ *nested matrix* $M = M(a_1, a_2, \ldots, a_n)$ is defined by

$$M = \begin{bmatrix} a_1 & a_1 & a_1 & a_1 & \ldots & a_1 & a_1 \\ a_1 & a_2 & a_2 & a_2 & \ldots & a_2 & a_2 \\ a_1 & a_2 & a_3 & a_3 & \ldots & a_3 & a_3 \\ a_1 & a_2 & a_3 & a_4 & & a_4 & a_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & a_2 & a_3 & a_4 & \ldots & a_{n-1} & a_{n-1} \\ a_1 & a_2 & a_3 & a_4 & \ldots & a_{n-1} & a_n \end{bmatrix}.$$

We observe that M is symmetric, so that when all of the a_j are real, the spectrum of M must be real.

Theorem 1. *The $n\times n$ matrix $M(a_1,a_2,\ldots,a_n)$ has an LU-factorization in which L does not depend on the sequence $a_1,a_2,\ldots,a_n$.*

$$L=\begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0\\ 1 & 1 & 0 & 0 & \ldots & 0 & 0\\ 1 & 1 & 1 & 0 & \ldots & 0 & 0\\ 1 & 1 & 1 & 1 & & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & 1 & 1 & 1 & \ldots & 1 & 0\\ 1 & 1 & 1 & 1 & \ldots & 1 & 1 \end{bmatrix}$$

$$U=\begin{bmatrix} a_1 & a_1 & a_1 & a_1 & \ldots & a_1 & a_1\\ 0 & a_2-a_1 & a_2-a_1 & a_2-a_1 & \ldots & a_2-a_1 & a_2-a_1\\ 0 & 0 & a_3-a_2 & a_3-a_2 & \ldots & a_3-a_2 & a_3-a_2\\ 0 & 0 & 0 & a_4-a_3 & \ldots & a_4-a_3 & a_4-a_3\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \ldots & a_{n-1}-a_{n-2} & a_{n-1}-a_{n-2}\\ 0 & 0 & 0 & 0 & \ldots & 0 & a_n-a_{n-1} \end{bmatrix}.$$

P r o o f. The result follows directly from the fact that for $2\leqslant k\leqslant n$,

$$a_1+(a_2-a_1)+(a_3-a_2)+\ldots+(a_k-a_{k-1})=a_k.$$

□

Corollary 2. *The $n\times n$ matrix $M(a_1,a_2,\ldots,a_n)$ has an LDL^{T}-factorization where L is given in Theorem 1 and D is the $n\times n$ diagonal matrix $D=\operatorname{diag}(a_1,a_2-a_1,a_3-a_2,\ldots,a_n-a_{n-1})$.*

P r o o f. By direct computation, $DL^{\mathrm{T}}=U$ where U is given in Theorem 1. □

2. Determinants of nested matrices

The next theorem follows immediately from Theorem 1.

Theorem 3. *The determinant of $M(a_1,a_2,\ldots,a_n)$ is*

$$\det M(a_1,a_2,\ldots,a_n)=a_1(a_2-a_1)(a_3-a_2)\ldots(a_n-a_{n-1}).$$

Corollary 4. *Let c be a real or complex number, and let the real or complex sequence $a_0, a_{1,}, a_{2,} \ldots, a_n$ satisfy the second order recursion $a_n = a_{n-1} + ca_{n-2}$ for $n \geqslant 2$. Then*

$$\det M(a_1, a_2, \ldots, a_n) = a_1\, c^{n-1} \prod_{j=0}^{n-2} a_j.$$

In particular, when $n \geqslant 2$ and $c = a_0 = a_1 = 1$, each a_j is the j^{th} Fibonacci number F_j, which yields,

$$\det M(F_1, F_2, \ldots, F_n) = \prod_{j=1}^{n-2} F_j.$$

The product of the first $n+1$ Fibonacci numbers, $F_0 F_1 \ldots F_{n-1} F_n$ is sometimes called the n^{th} Fibonacci generalized factorial, the n^{th} fibotorial, or the n^{th} fibonorial. (See [3], and also the sequence A003266 in [2].)

Corollary 5. *Let k be a positive integer, and let $M_{n,k}$ be the $n \times n$ matrix $M_{n,k} = M\left(\binom{k}{k}, \binom{k+1}{k}, \ldots, \binom{k+n-1}{k}\right)$. Then*

$$d_{n,k} = \det M_{n,k} = \prod_{j=k}^{k+n-2} \binom{j}{k-1}.$$

P r o o f. By the previous theorem,

$$\det M_{n,k} = \binom{k}{k} \prod_{j=k}^{k+n-2} \left(\binom{j+1}{k} - \binom{j}{k} \right) = \prod_{j=k}^{k+n-2} \binom{j}{k-1}$$

using the basic binomial coefficient recursion,

$$\binom{j+1}{k} = \binom{j}{k-1} + \binom{j}{k}.$$

$\square$

Example 6. For $k = 1$ and $n \geqslant 1$, $M_{n,1} = M\left(1, 2, 3, \ldots, \binom{n}{1}\right)$ and $d_{n,1} = 1$. For $k = 2$ and $n \geqslant 1$, $M_{n,2} = M\left(1, 3, 6, 10, \ldots, \binom{n+1}{2}\right)$ and $d_{n,2} = n!$. For $k = 3$ and $n \geqslant 1$, $M_{n,3} = M\left(1, 4, 10, 20, \ldots, \binom{n+2}{3}\right)$ and $d_{n,3} = n!(n+1)!/(2!)^n$.

Proposition 7. *For $k \geqslant 4$ and $n \geqslant 1$, if $d_{n,k} = \det M_{n,k}$, then*

$$d_{n,k} = \frac{n!(n+1)!\dots(n+k-2)!}{((n-1)!)^n \prod\limits_{j=2}^{k-2} j!}.$$

P r o o f. Use Corollary 5, substitute using the equivalence $\binom{p}{q} = p!/q!(p-q)!$, and cancel factorials that obviously appear in both the numerator and denominator. $\square$

3. The inverse of a nested matrix

Theorem 8. *When $M = M(a_1, a_2, \dots, a_n)$ is invertible, its inverse is tridiagonal. Further,*

$$(M^{-1})_{ij} = \begin{cases} \dfrac{a_2}{a_1(a_2 - a_1)} & \text{when } i = j = 1, \\ \dfrac{a_{j+1} - a_{j-1}}{(a_{j+1} - a_j)(a_j - a_{j-1})} & \text{when } 1 < i = j < n, \\ \dfrac{1}{a_n - a_{n-1}} & \text{when } i = j = n, \\ \dfrac{-1}{a_{j+1} - a_j} & \text{when } |i - j| = 1, \\ 0 & \text{when } |i - j| > 1. \end{cases}$$

P r o o f. The tridiagonality of M^{-1} arises from the fact that the (i, j)-cofactors have a repeated row or column when $|j - i| > 1$. For $1 < j < n$, the (j, j)-cofactor is $\det M(a_1, a_2, \dots, a_{j-1}, a_{j+1}, \dots, a_n)$. The $(1, 1)$-cofactor is $\det M(a_2, a_3, \dots, a_n)$. The (n, n)-cofactor is $\det M(a_1, a_2, \dots, a_{n-1})$. For $1 \leqslant j < n$, the the $(j, j+1)$-minor is the determinant of

$$\begin{bmatrix} a_1 & \dots & a_1 & a_1 & a_1 & \dots & a_1 & a_1 \\ a_1 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & \dots & a_{j-1} & a_{j-1} & a_{j-1} & \dots & a_{j-1} & a_{j-1} \\ a_1 & \dots & a_{j-1} & a_j & a_{j+1} & \dots & a_{j+1} & a_{j+1} \\ a_1 & \dots & a_{j-1} & a_j & a_{j+2} & \dots & a_{j+2} & a_{j+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & \dots & a_{j-1} & a_j & a_{j+2} & \dots & a_{n-1} & a_{n-1} \\ a_1 & \dots & a_{j-1} & a_j & a_{j+2} & \dots & a_{n-1} & a_n \end{bmatrix}.$$

So by iteratively subtracting the first $j-1$ rows from each of the following rows, we obtain

$$\begin{bmatrix} a_1 & a_1 & \dots & a_1 & a_1 & \dots & \dots & a_1 & a_1 \\ 0 & a_2-a_1 & \dots & a_2-a_1 & a_2-a_1 & \dots & \dots & a_2-a_1 & a_2-a_1 \\ 0 & 0 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & a_{j-1}-a_{j-2} & a_{j-1}-a_{j-2} & a_{j-1}-a_{j-2} & \dots & a_{j-1}-a_{j-2} & a_{j-1}-a_{j-2} \\ 0 & 0 & \dots & 0 & a_j-a_{j-1} & a_{j+1}-a_{j-1} & \dots & a_{j+1}-a_{j-1} & a_{j+1}-a_{j-1} \\ 0 & 0 & \dots & 0 & a_j-a_{j-1} & a_{j+2}-a_{j-1} & \dots & a_{j+2}-a_{j-1} & a_{j+2}-a_{j-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_j-a_{j-1} & \dots & & a_{n-1}-a_{j-1} & a_{n-1}-a_{j-1} \\ 0 & 0 & \dots & 0 & a_j-a_{j-1} & \dots & & a_{n-1}-a_{j-1} & a_n-a_{j-1} \end{bmatrix}.$$

Now subtracting the j^{th} row from the all of the subsequent rows yields

$$\begin{bmatrix} a_1 & a_1 & \dots & a_1 & a_1 & \dots & \dots & a_1 & a_1 \\ 0 & a_2-a_1 & \dots & a_2-a_1 & a_2-a_1 & \dots & \dots & a_2-a_1 & a_2-a_1 \\ 0 & 0 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & a_{j-1}-a_{j-2} & a_{j-1}-a_{j-2} & a_{j-1}-a_{j-2} & \dots & a_{j-1}-a_{j-2} & a_{j-1}-a_{j-2} \\ 0 & 0 & \dots & 0 & a_j-a_{j-1} & a_{j+1}-a_{j-1} & \dots & a_{j+1}-a_{j-1} & a_{j+1}-a_{j-1} \\ 0 & 0 & \dots & 0 & 0 & a_{j+2}-a_{j+1} & \dots & a_{j+2}-a_{j+1} & a_{j+2}-a_{j+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & a_{j+2}-a_{j+1} & \dots & a_{n-1}-a_{j+1} & a_{n-1}-a_{j+1} \\ 0 & 0 & \dots & 0 & 0 & a_{j+2}-a_{j+1} & \dots & a_{n-1}-a_{j+1} & a_n-a_{j+1} \end{bmatrix}.$$

Thus the $(j, j+1)$-minor is the product

$$\begin{aligned} & a_1(a_2-a_1)(a_3-a_2)\dots(a_j-a_{j-1}) \cdot \det M(a_{j+2}-a_{j+1}, a_{j+3}-a_{j+1}, \dots, a_n-a_{j+1}) \\ & = a_1(a_2-a_1)(a_3-a_2)\dots(a_j-a_{j-1}) \cdot (a_{j+2}-a_{j+1})(a_{j+3}-a_{j+2})\dots(a_n-a_{n-1}) \\ & = \frac{\det M(a_2, a_3, \dots, a_n)}{a_{j+1}-a_j}. \end{aligned}$$

Since the matrix is symmetric, the $(j, j+1)$-cofactor is

$$\frac{(-1)^{j+j+1} \det M(a_2, a_3, \dots, a_n)}{a_{j+1}-a_j}.$$

Now use the fact that $(M^{-1})_{ij}$ is the ratio of the (i,j)-cofactor of M to the determinant of M, and apply Corollary 3. $\square$

4. The QR-factorization of a nested matrix

Lemma 9. *The $n \times n$ matrix L in Theorem* 1 has QR-factorization $L = Q_L R_L$ where

$$Q_L = \begin{bmatrix} \frac{1}{\sqrt{n}} & -\sqrt{\frac{n-1}{n}} & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{(n-1)n}} & -\sqrt{\frac{n-2}{n-1}} & 0 & \dots & 0 & 0 \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{(n-1)n}} & \frac{1}{\sqrt{(n-2)(n-1)}} & -\sqrt{\frac{n-3}{n-2}} & \dots & 0 & 0 \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{(n-1)n}} & \frac{1}{\sqrt{(n-2)(n-1)}} & \frac{1}{\sqrt{(n-3)(n-2)}} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & -\sqrt{\frac{2}{3}} & 0 \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{(n-1)n}} & \frac{1}{\sqrt{(n-2)(n-1)}} & \frac{1}{\sqrt{(n-3)(n-2)}} & \dots & \frac{1}{\sqrt{2\cdot 3}} & -\sqrt{\frac{1}{2}} \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{(n-1)n}} & \frac{1}{\sqrt{(n-2)(n-1)}} & \frac{1}{\sqrt{(n-3)(n-2)}} & \dots & \frac{1}{\sqrt{2\cdot 3}} & \frac{1}{\sqrt{1\cdot 2}} \end{bmatrix}$$

and

$$(R_L)_{ij} = \begin{cases} 0 & \text{if } i > j, \\ \dfrac{n-j+1}{\sqrt{n}} & \text{if } j \geqslant i = 1, \\ \dfrac{n-j+1}{\sqrt{(n-i+1)(n-i+2)}} & \text{if } j \geqslant i \geqslant 2. \end{cases}$$

P r o o f. Clearly, the columns of Q_L are pairwise orthogonal, unit vectors. A simple induction shows that the j^{th} column of L is a linear combination of the first j columns of Q_L for $1 \leqslant j \leqslant n$. Since $(Q_L)^{-1} = (Q_L)^{\mathrm{T}}$, $R_L = (Q_L)^{\mathrm{T}} L$, so the entries of R_L are obtained as unweighted sums of entries from the columns of Q_L. Specifically, $(R_L)_{ij}$ is the sum of the entries in column i of Q_L from $\max\{i-1, j\}$ to n for $2 \leqslant i \leqslant n$, and from j to n for $i = 1$. The orthogonality of the all 1's column (parallel to the first column of Q_L) to all of the other columns of Q_L, and the fact that the entries in each column of Q_L are constant on and below the diagonal, leads directly to the formula stated. $\square$

Theorem 10. *The $n \times n$ matrix $M(a_1, a_2, \dots, a_n)$ has a QR-factorization in which $Q = Q_L$, and hence, does not depend on the sequence $a_1, a_2, \dots, a_n$. For this choice of Q, the corresponding R is $R_L U$ where U is the upper triangular matrix in Theorem* 1*. Equivalently, $R = R_L D L^{\mathrm{T}}$ where D is the diagonal matrix in Corollary* 2*.*

P r o o f. $M = M(a_1, a_2, \dots, a_n)$ has an LU-decomposition as $M = LU$ where L and U are given in Theorem 1. By Lemma 9, $L = Q_L R_L$, so $M = (Q_L R_L)U =$

$Q_L(R_L U)$. Since R_L and U are both upper triangular, their product is upper triangular. $\square$

5. Strictly increasing, positive sequences and inverse M-matrices

Theorem 11. *When the a_j are real with $0 < a_1 < a_2 < \ldots < a_n$, $M = M(a_2, a_3, \ldots, a_n)$ has all principal minors positive. Further, M^{-1} an irreducible, tridiagonal M-matrix. That is, M is an entrywise positive, inverse M-matrix.*

Proof. The positive, increasing values of the a_j together with Theorem 3 guarantee that the principal minors are all positive and that M is entrywise positive. Theorem 8 and the positive, increasing values of the a_j guarantee that the inverse is an irreducible Z-matrix. An invertible Z-matrix with an entrywise positive inverse is an M-matrix, see [1], Theorem 6.2.3, Condition N_{38}. $\square$

Example 12. When $M = M(1, 2, 3, \ldots, n)$, M^{-1} is the symmetric, irreducible, tridiagonal M-matrix

$$M^{-1} = \begin{bmatrix} 2 & -1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & 0 & \ddots & 0 & 0 \\ 0 & -1 & 2 & -1 & \ddots & \ddots & \vdots \\ 0 & 0 & -1 & 2 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & 0 \\ 0 & 0 & \ddots & 0 & -1 & 2 & -1 \\ 0 & 0 & \ldots & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Theorem 13. *When the a_j are real with $0 < a_1 < a_2 < \ldots < a_n$, $M = M(a_2, a_3, \ldots, a_n)$ has Cholesky factorization $M = CC^{\mathrm{T}}$ with $C = LD^{1/2}$ where L is given in Theorem 1 and where*

$$D^{1/2} = \operatorname{diag}(\sqrt{a_1}, \sqrt{a_2 - a_1}, \sqrt{a_3 - a_2}, \ldots, \sqrt{a_n - a_{n-1}}).$$

Proof. Since the a_j are positive and strictly increasing, the entries of $D^{1/2}$ are well-defined. Clearly, $CC^{\mathrm{T}} = (LD^{1/2})(LD^{1/2})^{\mathrm{T}} = LDL^{\mathrm{T}} = M$ by Corollary 2. $\square$

6. Related matrices

Consider the nested matrix M given by

$$M = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix}$$

and the related matrices

$$N = \begin{bmatrix} c & b & a \\ b & b & a \\ a & a & a \end{bmatrix}, \quad S = \begin{bmatrix} a & a & a \\ b & b & a \\ c & b & a \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} a & b & c \\ a & b & b \\ a & a & a \end{bmatrix}.$$

Let J be

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

so $J^{-1} = J$. Then $N = JMJ$, $S = MJ$ and $T = JM$. Since $\det(J) = -1$, $\det(N) = \det(M)$, and $\det(S) = \det(T) = -\det(M)$. Interestingly, although N is permutation similar to M, the LU-decomposition and QR-decomposition of N do not have the simple structure that the corresponding decompositions of M have. Less surprisingly, neither S nor T has the nice decomposition properties that M possesses.

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