Archivum Mathematicum

A. A. Estaji; A. Karimi Feizabadi; M. Abedi

Strongly fixed ideals in $C(L)$ and compact frames

*Archivum Mathematicum*, Vol. 51 (2015), No. 1, 1–12

Persistent URL: http://dml.cz/dmlcz/144228

ARCHIVUM MATHEMATICUM (BRNO)
Tomus 51 (2015), 1–12

# STRONGLY FIXED IDEALS IN $C(L)$ AND COMPACT FRAMES

A. A. Estaji, A. Karimi Feizabadi, and M. Abedi

Abstract. Let $C(L)$ be the ring of real-valued continuous functions on a frame $L$. In this paper, strongly fixed ideals and characterization of maximal ideals of $C(L)$ which is used with strongly fixed are introduced. In the case of weakly spatial frames this characterization is equivalent to the compactness of frames. Besides, the relation of the two concepts, fixed and strongly fixed ideals of $C(L)$, is studied particularly in the case of weakly spatial frames. The concept of weakly spatiality is actually weaker than spatiality and they are equivalent in the case of conjunctive frames. Assuming Axiom of Choice, compact frames are weakly spatial.

## 1. Introduction

Characterizing maximal ideals of a ring is an important problem. Let $C(X)$ be the ring of real valued continuous functions on a completely regular Hausdorff space $X$. In the ring $C(X)$ the maximal ideals are precisely the fixed ones for a compact Hausdorff space $X$. Conversely, if every maximal ideal is fixed, then $X$ is compact. Also, for every Hausdorff completely regular space $X$ the following are equivalent:

- $X$ is a compact.
- Every proper ideal in $C(X)$ is fixed.
- Every maximal ideal in $C(X)$ is fixed.
- Every proper ideal in $C^*(X)$ is fixed.
- Every maximal ideal in $C^*(X)$ is fixed.

For more detailed information, see [13].

In this note, we investigate these results in the pointfree topology for a frame $L$ to replace a topological space $X$.

The necessary background on frames (pointfree topology) is given in Section 2.

The concept of a weakly spatial frame is introduced and the necessary tools for the main results of the paper are given in Section 3. The weakly spatial frames play an important role in this note. For regular frames they are equivalent with spatial frames (Corollary 3.8). There are many examples of frames which are weakly

---

2010 *Mathematics Subject Classification*: primary 06D22; secondary 13A15, 13C99.

*Key words and phrases*: frame, ring of real-valued continuous functions, weakly spatial frame, fixed and strongly fixed ideal.

Received April 15, 2014, revised October 2014. Editor J. Rosický.

DOI: 10.5817/AM2015-1-1

spatial but they are not spatial (Remark 3.3). Using the Axiom of Choice, compact frames are weakly spatial (Proposition 3.4).

In the last section, we introduce strongly fixed ideals which are actually stronger than fixed ideals (Proposition 4.5). In the case of weakly spatial frames they are equivalent (Proposition 4.7). Also If $L$ is a completely regular frame, then $L$ is a spatial frame if and only if for every ideal $I$ in $C(L)$, $I$ is a fixed ideal of $C(L)$ if and only if $I$ is a strongly fixed ideal of $C(L)$ (Proposition 4.10). The concept of fixed ideals in $C(L)$ was defined and studied by T. Dube in [9, 8].

Finally, in Proposition 4.12 it is proven that for a compact frame $L$ every maximal ideal of $C(L)$ is of the form $M_p$, for some prime element $p \in L$. Conversely, if every maximal ideal of $C(L)$ is of the form $M_p$, for some prime element $p \in L$, then $L$ is compact, as shown in Proposition 4.13.

## 2. Preliminaries

Here, we recall some definitions and results from the literature on frames and the pointfree version of the ring of continuous real valued functions. For more details see the appropriate references given in [1, 3, 13, 14, 17].

A *frame* is a complete lattice $L$ in which the distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s \colon s \in S\}$$

holds for all $x \in L$ and $S \subseteq L$. We denote the top element and the bottom element of $L$ by $\top$ and $\bot$ respectively. The frame of open subsets of a topological spase $X$ is denoted by $\mathfrak{O}X$.

A *frame homomorphism* (*or frame map*) is a map between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element.

An element $a$ of a frame $L$ is said to be *rather below* an element $b$, written $a \prec b$, in case there is an element $s$, called a *separating element*, such that $a \wedge s = \bot$ and $s \vee b = \top$. On the other hand, $a$ is *completely below* $b$, written $a \prec\prec b$, if there are elements $(c_q)$ indexed by the rational numbers $\mathbb{Q} \cap [0, 1]$ such that $c_0 = a$, $c_1 = b$, and $c_p \prec c_q$ for $p < q$. A frame $L$ is said to be *regular* if $a = \bigvee\{x \in L \mid x \prec a\}$ for each $a \in L$, and *completely regular* if $a = \bigvee\{x \in L \mid x \prec\prec a\}$ for each $a \in L$.

An element $a \in L$ is said to be *compact* if $a = \bigvee S$, $S \subseteq L$, implies $a = \bigvee T$ for some finite subset $T \subseteq S$. A frame $L$ is said to be *compact* whenever its top element $\top$ is compact.

An element $p \in L$ is said to be *prime* if $p < \top$ and $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. An element $m \in L$ is said to be *maximal* (*or dual atom*) if $m < \top$ and $m \leq x \leq \top$ implies $m = x$ or $x = \top$. As it is well known, every maximal element is prime.

Recall the contravariant *functor* $\Sigma$ from **Frm** to the category **Top** of topological spaces which assigns to each frame $L$ its *spectrum* $\Sigma L$ of prime elements with $\Sigma_a = \{p \in \Sigma L \mid a \not\leq p\}$ $(a \in L)$ as its open sets. Also, for a frame map $h \colon L \to M$, $\Sigma h \colon \Sigma M \to \Sigma L$ takes $p \in \Sigma M$ to $h_*(p) \in \Sigma L$, where $h_* \colon M \to L$ is the *right adjoint* of $h$ characterized by the condition $h(a) \leq b$ if and only if $a \leq h_*(b)$ for all

$a \in L$ and $b \in M$. Note that $h_*$ preserves primes and arbitrary meets. For more details about *functor* $\Sigma$ and its properties which are used in this note see [17].

Recall [3] that the frame $\mathfrak{R}$ of reals is obtained by taking the ordered pairs $(p,q)$ of rational numbers as generators and imposing the following relations:

(R1) $(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$.

(R2) $(p,q) \vee (r,s) = (p,s)$ whenever $p \leq r < q \leq s$.

(R3) $(p,q) = \bigvee\{(r,s) \mid p < r < s < q\}$.

(R4) $\top = \bigvee\{(p,q) \mid p,q \in \mathbb{Q}\}$.

It is well known that the pairs $(p,q)$ in $\mathfrak{R}$ and the open intervals $\langle p,q\rangle = \{x \in \mathbb{R} \colon p < x < q\}$ in the frame $\mathfrak{O}\mathbb{R}$ of open sets have the same role; in fact there is a frame isomorphism $\lambda\colon \mathfrak{R} \to \mathfrak{O}\mathbb{R}$ such that $\lambda(p,q) = \langle p,q\rangle$.

The set $C(L)$ of all frame homomorphisms from $\mathfrak{R}$ to $L$ has been studied as an $f$-ring in [2, 3].

Corresponding to every continuous operation $\diamond\colon \mathbb{Q}^2 \to \mathbb{Q}$ (in particular $+, \cdot, \wedge, \vee$) we have an operation on $C(L)$, denoted by the same symbol $\diamond$, defined by:

$$\alpha \diamond \beta(p,q) = \bigvee\{\alpha(r,s) \wedge \beta(u,w) : (r,s) \diamond (u,w) \leq (p,q)\}\,,$$

where $(r,s) \diamond (u,w) \leq (p,q)$ means that for each $r < x < s$ and $u < y < w$ we have $p < x \diamond y < q$. For every $r \in \mathbb{R}$, define the constant frame map $\mathbf{r} \in C(L)$ by $\mathbf{r}(p,q) = \top$, whenever $p < r < q$, and otherwise $\mathbf{r}(p,q) = \bot$.

The *cozero map* is the map $coz\colon C(L) \to L$, defined by

$$coz\,(\alpha) = \bigvee\{\alpha(p,0) \vee \alpha(0,q) \colon p,q \in \mathbb{Q}\} = \alpha\big((-,0) \vee (0,-)\big)\,,$$

where

$$(0,-) = \bigvee\{(0,q)) \colon q \in \mathbb{Q}, q > 0\}$$

and

$$(-,0) = \bigvee\{(p,0)) \colon p \in \mathbb{Q}, p < 0\}\,.$$

For $A \subseteq C(L)$, let $Coz\,(A) = \{coz\,(\alpha) : \alpha \in A\}$ with the cozero part of a frame L, $Coz\,(C(L))$, called $Coz\ L$ by previous authors. It is known that $L$ is completely regular if and only if $Coz\,(C(L))$ generates $L$.

For any $\alpha, \beta \in C(L)$, we have:

- $coz\,(\mathbf{0}) = \bot$, and $coz\,(\mathbf{1}) = \top$,
- $coz\,(\alpha + \beta) \leq coz\,(\alpha) \vee coz\,(\beta)$, and if $\alpha, \beta \geq \mathbf{0}$ the equality holds,
- $coz\,(|\alpha|) = coz\,(\alpha)$,
- $coz\,(\alpha\beta) = coz\,(\alpha) \wedge coz\,(\beta)$, and
- if $\alpha, \beta \geq \mathbf{0}$ then $coz\,(\alpha \wedge \beta) = coz\,(\alpha) \wedge coz\,(\beta)$.

For more details about *cozero map* and its properties which are used in this note see [3, 4].

For $A \subseteq Coz(L)$, we write $Coz^{\leftarrow}[A]$ to designate the family of frame maps $\{\alpha \in C(L) \colon\ coz\,(\alpha) \in A\}$.

An element $f$ of $C(L)$ is said to be bounded if there exists $n \in \mathbb{N}$ such that $f(-n,n) = \top$. The set of all bounded elements of $C(L)$ is denoted by $C^*(L)$ which is a sub $f$-ring of $C(L)$.

An ideal $I$ of $C(L)$ or $C^*(L)$ is *fixed* if $\bigvee_{\alpha \in I} coz\,(\alpha) < \top$ [9, 8]. This is the exact counterpart of the familiar classical notion concerning ideals of $C(X)$ and $C^*(X)$.

Here we recall necessary notations, definitions and results form [10]. Let $a \in L$, and $\alpha \in C(L)$. The sets $\{r \in \mathbb{Q} : \alpha(-,r) \leq a\}$ and $\{s \in \mathbb{Q} : \alpha(s,-) \leq a\}$ are denoted by $L(a,\alpha)$ and $U(a,\alpha)$, respectively.

For $a \neq \top$ it is obvious that for each $r \in L(a,\alpha)$ and $s \in U(a,\alpha)$, $r \leq s$. In fact, we have:

**Proposition 2.1** ([10])**.** *Let $L$ be a frame. If $p \in \Sigma L$ and $\alpha \in C(L)$, then $(L(p,\alpha), U(p,\alpha))$ is a Dedekind cut for a real number which is denoted by $\widetilde{p}(\alpha)$.*

To learn more about Dedekind cut see [12].

**Proposition 2.2** ([10])**.** *If $p$ is a prime element of a frame $L$, then there exists a unique map $\widetilde{p}\colon C(L) \longrightarrow \mathbb{R}$ such that for each $\alpha \in C(L)$, $r \in L(p,\alpha)$, and $s \in U(p,\alpha)$ we have $r \leq \widetilde{p}(\alpha) \leq s$.*

By the following proposition, $\tilde{p}$ is an $f$-ring homomorphism.

**Proposition 2.3** ([10])**.** *If $p$ is a prime element of frame $L$, then $\widetilde{p}\colon C(L) \longrightarrow \mathbb{R}$ is an onto $f$-ring homomorphism. Also, $\widetilde{p}$ is a linear map with $\widetilde{p}(\mathbf{1}) = 1$.*

Let $L$ be a frame and $p$ is a prime element of $L$. Throughout this paper for every $f \in C(L)$ we define $f[p] = \widetilde{p}(f)$.

## 3. Weakly spatial frames

Weakly spatial frames play a key role the present argument. The weakly spatiality is indeed weaker than spatiality.

**Definition 3.1.** A frame $L$ is said to be *weakly spatial* if $a < \top$ implies $\Sigma_a \neq \Sigma_\top$.

**Lemma 3.2.** *A frame $L$ is weakly spatial if and only if there is a prime element $p \in L$ such that $a \leq p < \top$, for every $a < \top$.*

**Proof.** Suppose that $L$ is weakly spatial, and $a < \top$. Hence $\Sigma_a \neq \Sigma_\top = \Sigma L$, so there is a prime element $p \in \Sigma L \setminus \Sigma_a$. Therefore $a \leq p$. Conversely, let $a < \top$. So there is a prime element $p \in L$ such that $a \leq p < \top$, hence $p \in \Sigma L \setminus \Sigma_a$. Therefore $L$ is weakly spatial. □

*Remark* 3.3. It is clear that if $L$ is spatial, then $L$ is weakly spatial. The inverse is clearly not true. In fact the spatiality and the weakly spatiality are very much different. As an example, let $L$ be a nonspatial frame and $M = L \cup \{\top_M\}$, where the order of $M$ is the same as in $L$ for the elements of $L$ and for every $x \in L$, $x < \top_M$. The top element $\top_L$ of $L$ is a prime element of $M$, so $M$ is weakly spatial for all $L$. Now since $\Sigma M = \Sigma L \cup \{\top_L\}$, $M$ is nonspatial.

The following proposition explains that compact frames are weakly spatial. It is necessary to say that the proof is inspired from Lemma III, 1.9 in [14].

**Proposition 3.4.** *Every compact frame is weakly spatial.*

**Proof.** Let $L$ be a compact frame and $a \in L$ such that $a < \top$. Using the Axiom of Choice, there exists a maximal ideal $P \subset L$ such that $a \in P$. Since $L$ is a compact frame, we conclude that $p = \bigvee P \neq \top$, and by the maximality of $P$ we have $\downarrow p = \{x \in L \mid x \leq p\} = P$. Since $P$ is also a prime ideal, $p$ is a prime element and $a \leq p < \top$. It follows that $\Sigma_a \neq \Sigma_\top$. Therefore $L$ is weakly spatial. □

**Lemma 3.5.** *Let $L$ be weakly spatial and $\alpha \in C(L)$. If $\Sigma_{coz\,(\alpha)} = \emptyset$, then $coz\,(\alpha) = \bot$.*

**Proof.** Let $r,\ s \in \mathbb{Q}$ such that $r < 0 < s$ and $p \in \Sigma L$. So we have $p \notin \Sigma_{coz\,(\alpha)}$ hence $coz\,(\alpha) \leq p$. Now, we claim that $\alpha(r,s) \not\leq p$. Because if $\alpha(r,s) \leq p$, then $\top = coz\,(\alpha) \vee \alpha(r,s) \leq p$, which is a contradiction. So $\Sigma_{\alpha(r,s)} = \Sigma L$, since $L$ is weakly spatial, we conclude that $\alpha(r,s) = \top$. On the other hand

$$\begin{aligned}\bot &= (\alpha(-,r) \vee \alpha(s,-)) \wedge \alpha(r,s)\\ &= (\alpha(-,r) \vee \alpha(s,-)) \wedge \top\\ &= \alpha(-,r) \vee \alpha(s,-)\,.\end{aligned}$$

Therefore, $coz\,(\alpha) = \bigvee\{\alpha(-,r) \vee \alpha(s,-) : r < 0 < s\} = \bot$. □

**Corollary 3.6.** *Let $L$ be a compact frame, and $\alpha \in C(L)$. If $\Sigma_{coz\,(\alpha)} = \emptyset$, then $coz\,(\alpha) = 0$.*

**Proof.** Obvious. □

Recall that a frame $L$ is *conjunctive* if for any $a,\ b \in L$ with $a \not\leq b$ there is an element $c \in L$ such that $a \vee c = \top$, $b \vee c \neq \top$. For more details about *conjunctive frames* and separation Axioms, see [15, 17, 18].

It is known that a frame $L$ is spatial if and only if for each $a,\ b \in L$ with $a \not\leq b$ there exists a prime element $p$ of $L$ such that $a \not\leq p$, $b \leq p$.

**Proposition 3.7.** *Let $L$ be a conjunctive frame. Then the following statements are equivalent:*

(1) *$L$ is a spatial frame.*

(2) *$L$ is a weakly spatial frame.*

**Proof.** (1) ⇒ (2). Obvious.

(2) ⇒ (1). Let $a,\ b \in L$ such that $a \not\leq b$. Then there exists $c \in L$ such that $a \vee c = \top$, $b \vee c \neq \top$. Since $L$ is a weakly spatial frame, we conclude by Lemma 3.2 that there exists a prime element $p \in L$ such that $c \vee b \leq p$. If $a \leq p$, then $c \vee a = \top \leq p$, which is a contradiction. Hence $a \not\leq p$ and $b \leq p$, which follows that $L$ is spatial. □

It is clear that any regular frame is a conjunctive frame [16]. So, by the previous proposition we have:

**Corollary 3.8.** *For regular frames, the notion of spatiality and weak spatiality coincide.*

Also, to see another version of the Corollary 3.8, see [7].

Recall that a frame $L$ is dually atomic if for any $\top \neq a \in L$, there is a maximal element $m \in L$ such that $a \leq m$ [15, 16]. This show that $m \notin \Sigma_a$. So any dually atomic frame is a weakly spatial frame. Also, a compact frame $L$ is dually atomic. Because if $\top \neq a \in L$, then there exists a maximal element $m \in L$ such that $a \leq m$. Therefore we have:

*Remark* 3.9. For compact frames, the notion of dual atomicity and weak spatiality coincide.

Notice that by Proposition 3.7 and Remark 3.9 we can conclude that for compact conjunctive frames, the notion of spatiality, weak spatiality and dual atomicity coincide.

## 4. Maximal, fixed and strongly fixed ideals of $C(L)$

Recall that in [11] we introduced the pointfree version of zero set $f \in C(X)$ given by $Z(f) = \{x \in X : f(x) = 0\}$. In the pointfree version we use prime elements $p \in L$ to replace points $x \in X$ as following definition:

**Definition 4.1.** Let $\alpha \in C(L)$. We define

$$Z(\alpha) = \{p \in \Sigma L : \alpha[p] = 0\}\,.$$

Such a set is said to be a zero-set in $L$. For $A \subseteq C(L)$, we write $Z[A]$ to designate the family of zero-sets $\{Z(\alpha) : \alpha \in A\}$. The family $Z[C(L)]$ of all zero-sets in $L$ will also be denoted, for simplicity, by $Z[L]$.

The following lemma plays an important role in this note.

**Lemma 4.2.** *Let $p$ be a prime element of $L$. For $\alpha \in C(L)$, $\alpha[p] = 0$ if and only if $coz\,(\alpha) \leq p$.*

**Proof.** Suppose that $\alpha[p] \neq 0$. If $\alpha[p] > 0$, then there exists a rational number $r$ such that $\alpha[p] \geq r > 0$. Thus, by Proposition 2.1, $r \in L(p, \alpha)$, and so by definition of $L(p, \alpha)$, $\alpha(-, r) \leq p$. Now, if $coz(\alpha) \leq p$, we have $\top = \alpha(0, -) \vee \alpha(-, r) \leq coz\,(\alpha) \vee p \leq p \vee p = p$ and obtain a contradiction. Therefore $coz\,(\alpha) \not\leq p$. In the case $\alpha[p] < 0$, the proof is similar.

Conversely, suppose that $\alpha[p] = 0$. So, by Proposition 2.1, for every two rational numbers $r < 0 < s$, we have $r \in L(\alpha, p)$ and $s \in U(\alpha, p)$, and hence $\alpha(-, r) \vee \alpha(s, -) \leq p$. Thus,

$$coz\,(\alpha) = \bigvee\{\alpha(-, r) \vee \alpha(s, -) : r < 0 < s\} \leq p\,.$$

□

Naturally, we have the following proposition for this definition.

**Proposition 4.3** ([11])**.** *For every $\alpha$, $\beta \in C(L)$, we have*

1. *For every $n \in \mathbb{N}$, $Z(\alpha) = Z(|\alpha|) = Z(\alpha^n)$.*
2. $Z(\alpha) \cap Z(\beta) = Z(|\alpha| + |\beta|) = Z(\alpha^2 + \beta^2)$.
3. $Z(\alpha) \cup Z(\beta) = Z(\alpha\beta)$.

(4) *If $\alpha$ is a unit of $C(L)$, then $Z(\alpha) = \emptyset$.*

(5) *$Z(L)$ is closed under countable intersection.*

It is known that an ideal $I$ in $C(X)$ or $C^*(X)$ is a fixed ideal if and only if $\bigcap Z[I]$ is nonempty. Also, $I$ is called a *free ideal* if $\bigcap Z[I] = \emptyset$ (see [13]). But in $C(L)$, being a fixed ideal is not equivalent to the condition $\bigcap Z[I] \neq \emptyset$ (see Example 4.6). Therefore, we define strongly fixed ideal in $C(L)$, as follows:

**Definition 4.4.** Let $I$ be any ideal in $C(L)$ or $C^*(L)$. If $\bigcap Z[I]$ is nonempty, we call $I$ a *strongly fixed ideal*; if $\bigcap Z[I] = \emptyset$, then $I$ is a strongly free ideal.

Evidently, if $\Sigma L \neq \emptyset$, then the zero ideal in $C(L)$ or $C^*(L)$ is strongly fixed. More generally, if $Z(\alpha)$ is nonempty, then the principal ideal $(\alpha)$ is strongly fixed, because clearly $\bigcap Z[(\alpha)] = Z(\alpha)$. Moreover, if $L$ is a weakly spatial frame, then every strongly free ideal $I$ in $C(L)$ or $C^*(L)$ contains nonzero strongly fixed ideals. In fact, if $I$ contains a nonzero function $\beta$ whose zero set is nonempty, then $I$ contains the nonzero strongly fixed ideal $(\beta)$. On the other hand, it is manifest that no strongly fixed ideal can contain a strongly free ideal. Also, if $\emptyset \neq S \subseteq \Sigma L$, then $\{\alpha : S \subseteq Z(\alpha)\}$ is strongly fixed ideal.

**Proposition 4.5.** *Every strongly fixed ideal in $C(L)$ or $C^*(L)$ is a fixed ideal in $C(L)$ or $C^*(L)$.*

**Proof.** Let $I$ be a strongly fixed ideal in $C(L)$. Then there exists a prime element $p \in \bigcap Z[I]$. By Lemma 4.2, $\bigvee_{\alpha\in I} coz\,(\alpha) \leq p < \top$, that is, $I$ is a fixed ideal in $C(L)$. □

**Example 4.6.** (a) Let $L$ be a completely regular frame such that $\Sigma L = \emptyset$. Then, every ideal in $C(L)$ or $C^*(L)$ is strongly free.

(b) If $\alpha \in C(L)$ such that $coz\,(\alpha) < \top$ and the ideal $I$ of $C(L)$ is generated by $\alpha$, then $\bigvee_{\beta\in I} coz\,(\beta) \leq coz\,(\alpha) < \top$, and so $I$ is a fixed ideal in $C(L)$.

**Proposition 4.7.** *If $L$ is a weakly spatial frame, then every fixed ideal in $C(L)$ or $C^*(L)$ is a strongly fixed ideal in $C(L)$ or $C^*(L)$.*

**Proof.** Let $I$ be a fixed ideal in $C(L)$. Since $L$ is a weakly spatial frame and $\bigvee_{\alpha\in I} coz\,(\alpha) < \top$, we can conclude by Lemma 3.2 that there exists $p \in \Sigma L$ such that $\bigvee_{\alpha\in I} coz\,(\alpha) \leq p < \top$. Then, by Lemma 4.2, $p \in \bigcap Z[I]$, that is, $I$ is a strongly fixed ideal in $C(L)$. □

Define $M_p = \{f \in C(L) : f[p] = 0\}$ for every prime element $p \in L$. In the following proposition, we show that the strongly fixed maximal ideals are precisely the ideals $M_p$.

We regard the Stone–Čech compactification of $L$, denoted $\beta L$, as the frame of completely regular ideals of $L$. We denote the right adjoint of the join map $j_L \colon \beta L \to L$ by $r_L$ and recall that $r_L(a) = \{x \in L : x \prec\!\prec a\}$. We define $M^I = \{\alpha \in C(L) : r_L(coz\,(\alpha)) \subseteq I\}$, for all $1_{\beta L} \neq I \in \beta L$. If $M^I = M^J$ then $I = J$ (see [5]).

**Proposition 4.8.** *Let $L$ be a completely regular frame.*

(1) *The strongly fixed maximal ideals of $C(L)$ are precisely the ideals $M_p$, for $p \in \Sigma L$. The ideals $M_p$ are distinct for distinct $p \in \Sigma L$. For each $p \in \Sigma L$, $C(L)/M_p$ is isomorphic with the real field $\mathbb{R}$; in fact, the mapping $\alpha + M_p \to \alpha[p]$ is the unique isomorphism of $C(L)/M_p$ onto $\mathbb{R}$.*

(2) *The strongly fixed maximal ideals of $C^*(L)$ are precisely the ideals*

$$M_p^* = \{\alpha \in C^*(L) : \alpha[p] = 0\} \ (p \in \Sigma L).$$

*The ideals $M_p^*$ are distinct for distinct $p \in \Sigma L$. For each $p \in \Sigma L$, $C^*(L)/M_p^*$ is isomorphic with the real field $\mathbb{R}$; in fact, the mapping $\alpha + M_p^* \to \alpha[p]$ is the unique isomorphism of $C^*(L)/M_p^*$ onto $\mathbb{R}$.*

**Proof.** $M_p$ is the kernel of the homomorphism $\widetilde{p}\colon C(L) \longrightarrow \mathbb{R}$. Since by Proposition 2.3 the homomorphism $\widetilde{p}$ is onto the field $\mathbb{R}$, $C(L)/M_p \simeq \mathbb{R}$. Hence its kernel $M_p$ is a maximal ideal. It is clear that $M_p$ is a strongly fixed ideal for every prime $p \in L$. Therefore, $M_p$ is a strongly fixed maximal ideal. On the other hand, if $M$ is any strongly fixed maximal ideal in $C(L)$, then there exists a point $p$ in $\bigcap Z[M]$. Evidently, $M \subseteq M_p$, which has just been shown to be a ideal. Hence since $M$ is maximal, we must have $M = M_p$.

Now, suppose that $p$, $q \in \Sigma L$ and $M_p = M_q$. So, $M^{r_L(p)} = M_p = M_q = M^{r_L(q)}$, i.e., $r_L(p) = r_L(q)$. Therefore, we conclude that $p = q$. Thus the ideals $M_p$ are distinct for distinct $p \in \Sigma L$. The proof of (2) is identical to (1). □

**Corollary 4.9.** *If $L$ is a completely regular frame and $M$ is a maximal ideal in $C(L)$, then $M$ is a fixed maximal ideal in $C(L)$ if and only if $M$ is a strongly fixed maximal ideal in $C(L)$.*

**Proof.** As in Proposition 3.3 in [9], we have that the fixed maximal ideals of $C(L)$ are precisely the ideals $M_p$ for prime elements $p \in \Sigma L$. Now, by Proposition 4.8, the proof is complete. □

It is easy to see that every strongly fixed ideal of $C(L)$ is contained in a strongly fixed maximal ideal, but for fixed ideals we have the following:

**Proposition 4.10.** *Let $L$ be a completely regular frame. Then the following statements are equivalent:*

(1) *$L$ is a spatial frame.*

(2) *For every ideal $I$ in $C(L)$, $I$ is a fixed ideal of $C(L)$ if and only if $I$ is a strongly fixed ideal of $C(L)$.*

(3) *Every fixed ideal of $C(L)$ is contained in a fixed maximal ideal.*

**Proof.** (1) $\Leftrightarrow$ (3). See Corollary 3.5 in [9].

(1) $\Rightarrow$ (2). It follows from Proposition 4.7.

(2) $\Rightarrow$ (1). Let $\top \neq a \in L$. Since $L$ is a completely regular frame, we conclude that there exists $\{\alpha_j\}_{j \in J} \subseteq C(L)$ such that $a = \bigvee_{j \in J} coz\,(\alpha_j)$. Put $I = \langle \alpha_j : j \in J \rangle$. Then $\bigvee_{\alpha \in I} coz\,(\alpha) = a < \top$, that is, $I$ is a fixed ideal of $C(L)$. By hypothesis, $I$ is a strongly fixed ideal of $C(L)$, and so there exists $p \in \Sigma L$ such that $p \in \bigcap Z[I]$.

Thus, by Lemma 4.2, $a = \bigvee_{\alpha \in I} coz\,(\alpha) \leq p < \top$. Therefore, by Lemma 3.2, $L$ is a weakly spatial frame. Now, by Corollary 3.8, the proof is now complete. □

**Proposition 4.11.** *Let $L$ be a weakly spatial frame. Then $L$ is a compact frame if and only if $\Sigma L$ is a compact space.*

**Proof.** Suppose that $L$ is a compact frame, and $\bigcup_{j \in J} \Sigma_{a_j} = \Sigma L$. So $\Sigma_{\bigvee a_j} = \Sigma_\top$ since $L$ is weakly spatial, $\bigvee a_j = \top$. Hence, by compactness of $L$, there exist $j_1, \ldots, j_n \in J$ such that $a_{j_1} \vee \cdots \vee a_{j_n} = \top$, and so $\Sigma_{a_{j_1}} \cup \cdots \cup \Sigma_{a_{j_n}} = \Sigma_\top$. Conversely, suppose that $\Sigma L$ is a compact space and $\bigvee a_j = \top$. Hence, $\bigcup \Sigma_{a_j} = \Sigma_{\bigvee a_j} = \Sigma_\top = \Sigma L$. Thus, by compactness of $\Sigma L$, there exist $j_1, \ldots, j_n \in J$ such that $\Sigma_{a_{j_1}} \cup \cdots \cup \Sigma_{a_{j_n}} = \Sigma_\top$. So, $\Sigma_{a_{j_1} \vee \cdots \vee a_{j_n}} = \Sigma_\top$. Hence since $L$ is weakly spatial, $a_{j_1} \vee \cdots \vee a_{j_n} = \top$. Therefore $L$ is compact. □

**Proposition 4.12.** *If $L$ is compact and $M$ is a maximal ideal of $C(L)$, then there exists a prime element $p \in L$ such that $M = M_p$.*

**Proof.** Assume that for every prime element $p$, $M \not\subseteq M_p$. We have that for every $p \in L$ there exists $f_p \in M$ such that $f_p \notin M_p$. So, by Lemma 4.2, $coz\,(f_p) \not\leq p$, and hence $p \in \Sigma_{coz\,(f_p)}$. Therefore, $\Sigma_{\bigvee_p coz\,(f_p)} = \bigcup_p \Sigma_{coz\,(f_p)} = \Sigma L = \Sigma_\top$. Hence, by weakly spatiality, $\bigvee_p coz\,(f_p) = \top$. So, since $L$ is compact, there are $p_1, \ldots, p_n \in \Sigma L$ such that $coz\,(f_{p_1}) \vee \cdots \vee coz\,(f_{p_n}) = \top$. Thus, by the property of *cozero map*, $coz\,(f_{p_1}^2 + \cdots + f_{p_n}^2) = \top$, and hence $h = f_{p_1}^2 + \cdots + f_{p_n}^2 \in M$ is invertible, which is a contradiction. Therefore, $M \subseteq M_p$ for some $p \in \Sigma L$. Since $M$ is maximal, we conclude that $M = M_p$. □

**Proposition 4.13.** *If $L$ is compact and $M$ is a maximal ideal of $C^*(L)$, then there exists a prime element $p \in L$ such that $M = M_p^*$.*

**Proof.** It is similar to Proposition 4.12. □

There is a homeomorphism $\tau\colon \Sigma\Re \to \mathbb{R}$ such that $r < \tau(p) < s$ if and only if $(r, s) \not\leq p$ for all prime element $p$ of $\Re$ and all $r, s \in \mathbb{Q}$ (see Proposition 1 of [3, page 12]).

**Lemma 4.14.** *Every prime (maximal) element of $\Re$ is of the form $p_x = \bigvee\{(-, r) \vee (s, -) : r, s \in \mathbb{Q}, r \leq x \leq s\}$ for some $x \in \mathbb{R}$, and $\tau(p_x) = x$. In particular for every $r \in \mathbb{Q}$, $p_r = (-, r) \vee (r, -)$ and $\tau((-, r) \vee (r, -)) = r$.*

**Proof.** Since $\Re$ is a completely regular frame, the prime elements are precisely the maximal elements, and maximal elements are of the form $p_x$ for some $x \in \mathbb{R}$. □

*Remark* 4.15. In Lemma 4.7 from [6], compact completely regular frames are characterized exactly as Proposition 4.16 that characterize compact weakly spatial frames, with *strongly fixed* instead of *fixed*. Note that, by Proposition 4.10, strongly fixed is equivalent to fixed if and only if $L$ is spatial. In addition there exist compact frames that are nonspatial. So, in our topic strongly fixed is not equivalent to fixed. Then the following proposition is stronger version of Lemma 4.7 in [6].

**Theorem 4.16.** *Let $L$ be a weakly spatial frame. Then the following statements are equivalent:*

(1) *$L$ is a compact frame.*

(2) *Every proper ideal in $C(L)$ is strongly fixed.*

(3) *Every maximal ideal in $C(L)$ is strongly fixed.*

(4) *Every proper ideal in $C^*(L)$ is strongly fixed.*

(5) *Every maximal ideal in $C^*(L)$ is strongly fixed.*

**Proof.** (1) $\Rightarrow$ (2). Let $I$ be a proper ideal in $C(L)$. By Proposition 4.12, there exists a prime element $p \in L$ such that $I \subseteq M_p$. So, $p \in \bigcap Z[M_p] \subseteq \bigcap Z[I]$. It follows that $I$ is a strongly fixed ideal.

(1) $\Rightarrow$ (4) is similar to (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are trivial.

First we show that $\Sigma L$ is a compact space to prove (3) $\Rightarrow$ (1). For this, we prove that every maximal ideal $M$ in $C(\Sigma L)$ is of the form $M_x$ for some $x \in \Sigma L$. Define $\phi\colon C(L) \to C(\Sigma L)$ by $\phi(f) = \tau \circ \Sigma f = \tau \circ f_\star$, where $\tau\colon \Sigma\Re \to \mathbb{R}$ is the homeomorphism discussed in Lemma 4.14 and $f_\star\colon L \to \Re$ is a right adjoint of $f$.

By hypothesis, there is a prime element $p \in L$ such that $\phi^{-1}(M) \subseteq M_p$, so $M \subseteq \phi(M_p)$. Hence $\bigcap\{Z(f) : f \in \phi(M_p)\} \subset \bigcap\{Z(f) : f \in M\}$. Now, it is enough to show that $\bigcap\{Z(f) : f \in \phi(M_p)\} \neq \emptyset$. Let $f \in M_p$. Then $f[p] = 0$, and hence, by Lemma 4.2, $coz\,(f) \leq p$, that is to say, $f((0,-) \vee (-,0)) \leq p$. So $(0,-) \vee (-,0) \leq f_\star(p)$. Thus since $(0,-) \vee (-,0)$ is a maximal element of $\Re$ and $f_\star(p)$ is a prime element, $(0,-) \vee (-,0) = f_\star(p)$. Now, by Lemma 4.14, we have $0 = \tau((0,-) \vee (-,0)) = \tau f_\star(p) = \phi(f)$. Therefore

$$x = p \in \bigcap\{Z(f) : f \in \phi(M_p)\} \subset \bigcap\{Z(f) : f \in M\}\,.$$

So $M = M_x$. Hence every maximal ideal of $C(\Sigma L)$ is fixed, thus $\Sigma L$ is compact. Since $L$ is weakly spatial, by Proposition 4.11, $L$ is compact.

(5) $\Rightarrow$ (1) is similar to (3) $\Rightarrow$ (1). □

*Remark* 4.17. Let $\mathcal{M}(C(L))$ denote the set of all maximal ideals in $C(L)$. We make $\mathcal{M}(C(L))$ into a topological space by taking, as a base for the closed sets, all sets of the form

$$\mathfrak{F}(\alpha) = \{M \in \mathcal{M}(C(L)) : \alpha \in M\} \quad (\alpha \in C(L))\,.$$

Define $\Theta\colon \Sigma L \to \mathcal{M}(C(L))$ by $\Theta(p) = M_p$. If $L$ is a compact completely regular frame, then by Proposition 4.8 and Theorem 4.16, $\Theta$ is one-one and onto, respectively. Also $\Theta^{-1}(\mathfrak{F}(\alpha)) = Z(\alpha)$ and $\Theta(Z(\alpha)) = \mathfrak{F}(\alpha)$. Therefore, $\Sigma L$ and $\mathcal{M}(C(L))$ are homeomorphic.

**Proposition 4.18.** *Suppose that $L$ and $L'$ are two compact completely regular frames. Then the following statements are equivalent:*

(1) *$L \cong L'$.*

(2) *$\Sigma L$ and $\Sigma L'$ are homeomorphic.*

(3) *$C(L)$ and $C(L')$ are isomorphic.*

**Proof.** (1) $\Leftrightarrow$ (2). Since every compact completely regular frame is spatial, we conclude that $L \cong O\Sigma L$ and $L' \cong O\Sigma L'$.

(1) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (2). Let $\phi\colon C(L) \to C(L')$ be an isomorphism. Consider $\varphi\colon \Sigma L \to \mathcal{M}(C(L))$ and $\psi\colon \Sigma L' \to \mathcal{M}(C(L'))$ to be the homeomorphisms corresponding to $L$ and $L'$ given in Remark 4.17. It is clear that $\overline{\phi}\colon \mathcal{M}(C(L)) \to \mathcal{M}(C(L'))$ with $\overline{\phi}(M_p) = M_{\phi(p)}$ is one-one and onto. Hence $\psi^{-1}\overline{\phi}\varphi\colon \Sigma L \to \Sigma L'$ is a homeomorphism. □

**Acknowledgement.** The authors thank the kind hospitality of Hakim Sabzevari University during several times we stayed there. We also gratefully thank Professor M. Mehdi Ebrahimi for the topic and support. The authors are deeply appreciated Professor T. Dube for useful suggestions. We appreciate the referee for comments and for taking the time and effort to review our manuscript.

## References

[1] Banaschewski, B., *Prime elements from prime ideals*, Order **2** (2) (1985), 211–213.

[2] Banaschewski, B., *Pointfree topology and the spectra of f-rings*, Ordered algebraic structures, (Curacoa, 1995), Kluwer Acad. Publ., Dordrecht, 1997, pp. 123–148.

[3] Banaschewski, B., *The real numbers in pointfree topology*, Textos de Mathemática (Série B), Vol. 12, University of Coimbra, Departmento de Mathemática, Coimbra, 1997.

[4] Banaschewski, B., Gilmour, C.R.A., *Pseudocompactness and the cozero part of a frame*, Comment. Math. Univ. Carolin. **37** (1996), 577–587.

[5] Dube, T., *Some ring-theoretic properties of almost P-frames*, Algebra Universalis **60** (2009), 145–162.

[6] Dube, T., *On the ideal of functions with compact support in pointfree function rings*, Acta Math. Hungar. **129** (2010), 205–226.

[7] Dube, T., *A broader view of the almost Lindelf property*, Algebra Universalis **65** (2011), 263–276.

[8] Dube, T., *Real ideal in pointfree rings of continuous functions*, Bull. Asut. Math. Soc. **83** (2011), 338–352.

[9] Dube, T., *Extending and contracting maximal ideals in the function rings of pointfree topology*, Bull. Math. Soc. Sci. Math. Roumanie **55** (103) (4) (2012), 365–374.

[10] Ebrahimi, M.M., Karimi, A., *Pointfree prime representation of real Riesz maps*, Algebra Universalis **2005** (54), 291–299.

[11] Estaji, A.A., Feizabadi, A. Karimi, Abedi, M., *Zero sets in pointfree topology and strongly z-ideals*, accepted in Bulletin of the Iranian Mathematical Society.

[12] Garcáa, J. Gutiérrez, Picado, J., *How to deal with the ring of (continuous) real-valued functions in terms of scales*, Proceedings of the Workshop in Applied Topology WiAT'10, 2010, pp. 19–30.

[13] Gillman, L., Jerison, M., *Rings of continuous functions*, Springer Verlag, 1979.

[14] Johnstone, P.T., *Stone Spaces*, Cambridge Univ. Press, 1982.

[15] Paseka, J., *Conjunctivity in quantales*, Arch. Math. (Brno) **24** (4) (1988), 173–179.

[16] Paseka, J., Šmarda, B., *$T_2$-frames and almost compact frames*, Czechoslovak Math. J. **42** (3) (1992), 385–402.

[17] Picado, J., Pultr, A., *Frames and Locales: topology without points*, Frontiers in Mathematics, Springer, Basel, 2012.

[18] Simmons, H., *The lattice theoretical part of topological separation properties*, Proc. Edinburgh Math. Soc. (2) **21** (1978), 41–48.

A.A. Estaji and M. Abedi,
Faculty of Mathematics and Computer Sciences,
Hakim Sabzevari University,
Sabzevar, Iran
*E-mail*: aaestaji@hsu.ac.ir ms.abedi@hsu.ac.ir

A. Karimi Feizabadi,
Department of Mathematics, Gorgan Branch,
Islamic Azad University,
Gorgan, Iran
*E-mail*: akarimi@gorganiau.ac.ir