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# LAGRANGE APPROXIMATION IN BANACH SPACES

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*Abstract.* Starting from Lagrange interpolation of the exponential function $e^z$ in the complex plane, and using an integral representation formula for holomorphic functions on Banach spaces, we obtain Lagrange interpolating polynomials for representable functions defined on a Banach space $E$. Given such a representable entire funtion $f\colon E \to \mathbb{C}$, in order to study the approximation problem and the uniform convergence of these polynomials to $f$ on bounded sets of $E$, we present a sufficient growth condition on the interpolating sequence.

*Keywords*: Lagrange interpolation; Lagrange approximation; Kergin interpolation; Kergin approximation; Banach space

*MSC 2010*: 46G20, 30E10, 30E20

## 1. Introduction

In [13] it was shown that for a Banach space $E$ with a separable dual (or with a separable predual) the Cauchy integral formula takes the form

$$f(x) = \int_{E'} e^{\gamma(x)} \tilde{f}(\gamma)\, dW(\gamma),$$

where $W$ is a Wiener measure on $E'$ and $\tilde{f}$ a transform of $f$ involving the covariance operator $S\colon E \to E'$ of the measure. For any given $f$ this transform $\tilde{f}$ is in $L^p(W)$ for some $p > 1$. The formula holds for a wide class of holomorphic functions $f\colon E \to \mathbb{C}$ called the representable functions. This class includes the functions verifying the growth condition

$$|f(x)| \leqslant c e^{\sigma \|S(x)\|}.$$

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The same formula was extended to the context of fully nuclear spaces with basis in [4].

When $E = \mathbb{C}^n$, the measure $W$ is simply the standard Gaussian measure on the complex $n$-dimensional space:

$$W(A) = \frac{1}{\pi^n} \int_A \mathrm{e}^{-\|z\|^2} \,\mathrm{d}z_1 \ldots \mathrm{d}z_n.$$

On infinite-dimensional spaces there is no standard Gaussian measure. Modifications must be introduced so that the variances in different directions are summable [11]. This gives rise to the covariance operator of the measure $S\colon E \to E'$ defined by

$$S(x)(y) = \int_{E'} \gamma(x)\gamma(y) \,\mathrm{d}W(\gamma),$$

which has a (densely-defined) inverse $T : E' \to E$. The transform $\tilde{f}$ referred to above is (essentially) $\tilde{f}(\gamma) = \overline{(f \circ T)(\gamma)}$.

In the finite-dimensional case the covariance operator is the identity, and the integral formula reduces to

$$f(z) = \int_{\mathbb{C}^n} \mathrm{e}^{\gamma(z)} \overline{f(\gamma)} \,\mathrm{d}W(\gamma).$$

Note that in this case the class of representable functions includes all functions of order 2 and exponential type $\varepsilon < 1$, i.e.,

$$|f(z)| \leqslant c\mathrm{e}^{\varepsilon\|z\|^2}, \quad \text{with } \varepsilon < 1.$$

For representable functions on Banach spaces $f\colon E \to \mathbb{C}$, the Taylor series of $\mathrm{e}^z$ at 0 automatically yields the Taylor series of $f$ at 0:

$$\begin{aligned} f(x) &= \int_{E'} \mathrm{e}^{\gamma(x)} \tilde{f}(\gamma) \,\mathrm{d}W(\gamma) = \int_{E'} \sum_{k=0}^{\infty} \frac{\gamma(x)^k}{k!} \tilde{f}(\gamma) \,\mathrm{d}W(\gamma) \\ &= \sum_{k=0}^{\infty} \int_{E'} \frac{\gamma(x)^k}{k!} \tilde{f}(\gamma) \,\mathrm{d}W(\gamma) = \sum_{k=0}^{\infty} P_k(x), \end{aligned}$$

where $P_k$ is the $k$-th Taylor polynomial of $f$. It is therefore natural to ask if other approximations of the exponential function will give rise to the corresponding approximations of representable holomorphic functions $f\colon E \to \mathbb{C}$.

The main purpose of this paper is to show that this is indeed the case for the Lagrange approximation.

Lagrange interpolation has been generalized to several variables by the work of Kergin [10], Andersson and Passare [1], and the approximation problem has been studied by Bloom [2], Filipsson [7] and others.

Petersson [12], Filipsson [6] and Simon [14] have extended Kergin interpolation and approximation to the Banach space setting. Hence, the results obtained in this paper are not entirely new. However, we believe there is some value to our method, since it allows a jump from Lagrange interpolation of one function in one complex variable, $\mathrm{e}^z$, to functions on Banach spaces. We note also that in the finite-dimensional case, we obtain Gaussian-integral formulas for the Kergin polynomials.

We recall that given a sequence of complex numbers $a_0, a_1, \dots, a_n, \dots$ the Lagrange interpolating polynomial of order $k$ for $\mathrm{e}^z$ is the unique polynomial $p_k\colon \mathbb{C} \to \mathbb{C}$ of degree not exceeding $k$ which takes the values

$$p_k(a_j) = \mathrm{e}^{a_j}$$

for $j = 0, 1, \dots, k$. We allow the points $a_j$ to repeat themselves: if $a$ appears $m$ times in the series $a_0, a_1, \dots, a_k$, then

$$p_k(a) = \mathrm{e}^a,\ \ p_k'(a) = \mathrm{e}^a,\ \ \dots,\ \ p_k^{(m-1)}(a) = \mathrm{e}^a.$$

It is not always the case that the polynomials $p_k$ approximate $\mathrm{e}^z$. In order to obtain $p_k(z) \to \mathrm{e}^z$ uniformly on compact subsets of $\mathbb{C}$, a growth condition must be imposed on the sequence $(a_n)$:

$$\limsup_k \frac{|a_k|}{k} < \ln(2).$$

It is well-known that the polynomials $p_k$ may be written in terms of Newton's formula

$$p_k(z) = \sum_{j=0}^{k} [a_0 \dots a_j](z - a_0) \dots (z - a_{j-1}),$$

where the coefficients $[a_0 \dots a_j]$ are inductively defined as the divided differences:

$$[a_0] = \mathrm{e}^{a_0},$$
$$[a_0 \dots a_j] = \begin{cases} \dfrac{[a_0 \dots a_{j-1}] - [a_1 \dots a_j]}{a_0 - a_j}, & \text{if } a_0 \neq a_j, \\ [a_1 \dots a_{j-1} z]'(a_0), & \text{if } a_0 = a_j. \end{cases}$$

The errors $E_k(z)$ may also be expressed in terms of divided differences:

$$E_k(z) = \mathrm{e}^z - p_k(z) = [z a_0 \dots a_k](z - a_0) \dots (z - a_k).$$

We will need to express $[a_0 \dots a_j]$ as

$$[a_0 \dots a_j] = \int_{t_0+\dots+t_j=1} \mathrm{e}^{t_0 a_0 + \dots + t_j a_j}$$

where the integration is over the simplex

$$S_j = \{(t_0, \dots, t_j) \in \mathbb{R}^{j+1} \colon\ t_i \geqslant 0 \text{ and } t_0 + \dots + t_j = 1\}$$

with its standard $j$-dimensional Lebesgue measure. This formula is the Hermite-Genocchi formula (see [3], [9]) in the special case of the exponential function.

## 2. Approximation on a Banach space

Let $E$ be a Banach space with a separable dual (or with a separable predual). In this section, given a sequence of points $x_0, x_1, \dots, x_n, \dots$ in $E$ we will define Lagrange interpolating polynomials for any representable entire function $f \colon\ E \to \mathbb{C}$, and study the convergence of the Lagrange polynomials to the function $f$.

To this end, given the sequence of points and a function $f$, for each $\gamma \in E'$, we define $L_{k,\gamma}$ to be the $k$-th Lagrange polynomial for $\mathrm{e}^z$ interpolating the sequence $\gamma(x_0), \gamma(x_1), \dots, \gamma(x_n), \dots$ We now define the $k$-th Lagrange interpolant of $f$ as

$$L_k(x) = \int_{E'} L_{k,\gamma}(\gamma(x)) \tilde{f}(\gamma) \, \mathrm{d}W(\gamma).$$

Our goal in this section is to prove the following:

i) the $L_k$'s are well-defined (i.e., $L_{k,\gamma}(\gamma(x)) \in L^q(W)$ for all $q < \infty$),
ii) the $L_k$'s are continuous polynomials of degree at most k on $E$,
iii) $L_k$ interpolates $f$ on $x_0, x_1, \dots, x_k$, and
iv) with a suitable growth condition on $x_0, x_1, \dots, x_n, \dots$, the $L_k$'s converge to $f$ uniformly on bounded subsets of $E$.

We begin with the first three points of our program.

**Theorem 2.1.** *Each Lagrange polynomial $L_k$ is well-defined, continuous and interpolates $f$ on $x_0, x_1, \dots, x_k$.*

Proof. Define $M_k = \max\{\|x_j\| \colon\ j \leqslant k\}$ and note that since

$$L_{k,\gamma}(\gamma(x)) = \sum_{j=0}^{k} [\gamma(x_0) \dots \gamma(x_j)](\gamma(x) - \gamma(x_0)) \dots (\gamma(x) - \gamma(x_{j-1})),$$

we have

$$
\begin{aligned}
|L_{k,\gamma}(\gamma(x))| &\leqslant \sum_{j=0}^{k} \left| \int_{t_0+\ldots+t_j=1} \mathrm{e}^{\gamma(t_0x_0+\ldots+t_jx_j)} \right| |\gamma(x-x_0)| \ldots |\gamma(x-x_{j-1})| \\
&\leqslant \sum_{j=0}^{k} \frac{\mathrm{e}^{\|\gamma\| M_j}}{(j+1)!} \|\gamma\|^j (\|x\| + M_{j-1})^j \\
&\leqslant \mathrm{e}^{\|\gamma\| M_k} \sum_{j=0}^{k} \frac{[\|\gamma\|(\|x\| + M_k)]^j}{j!} \\
&< \mathrm{e}^{\|\gamma\| M_k} \mathrm{e}^{\|\gamma\|(\|x\|+M_k)} \\
&= \mathrm{e}^{\|\gamma\|(\|x\|+2M_k)}.
\end{aligned}
$$

Thus for all $q < \infty$, $|L_{k,\gamma}(\gamma(x))|^q \leqslant \mathrm{e}^{q\|\gamma\|(\|x\|+2M_k)}$, which is bounded by $\mathrm{e}^{\varepsilon\|\gamma\|^2}$ for $\|\gamma\|$ large enough. But this function is $W$-integrable by Fernique's theorem [5]. Thus since $\tilde{f}$ is in $L^p(W)$ for some $p > 1$, the integral in the definition of $L_k$ exists.

$L_{k,\gamma}(\gamma(x))$ is a finite sum of terms of the form $a_{j,\gamma}\gamma(x)^j$, each of which is in $L^q(W)$ as above, and produces a $j$-homogeneous continuous polynomial corresponding to the symmetric $j$-linear form on $E$:

$$
(y_1, \ldots, y_j) \mapsto \int_{E'} a_{j,\gamma}\gamma(y_1) \ldots \gamma(y_j) \tilde{f}(\gamma) \,\mathrm{d}W(\gamma).
$$

Finally, note that for $j = 0, 1, \ldots, k$

$$
\begin{aligned}
L_k(x_j) &= \int_{E'} L_{k,\gamma}(\gamma(x_j)) \tilde{f}(\gamma) \,\mathrm{d}W(\gamma) \\
&= \int_{E'} \mathrm{e}^{\gamma(x_j)} \tilde{f}(\gamma) \,\mathrm{d}W(\gamma) \\
&= f(x_j).
\end{aligned}
$$

Now, a growth condition on $M_k = \max\{\|x_j\| \colon j \leqslant k\}$ allows approximation of the original function $f$. □

**Theorem 2.2.** *Let $f \colon E \to \mathbb{C}$ be a representable function, and $x_0, x_1, \ldots, x_n, \ldots$ a sequence of points in $E$ verifying*

$$
\lim_k \frac{M_k}{\sqrt{k}} = 0.
$$

*Then the Lagrange polynomials $L_k$ converge to $f$ uniformly on bounded subsets of $E$.*

P r o o f. We consider the error terms

$$\mathrm{e}^{\gamma(x)} - L_{k,\gamma}(\gamma(x)) = [\gamma(x)\gamma(x_0)\dots\gamma(x_k)](\gamma(x)-\gamma(x_0))\dots(\gamma(x)-\gamma(x_k)).$$

Thus, as above, we obtain

$$\begin{aligned}|\mathrm{e}^{\gamma(x)} - L_{k,\gamma}(\gamma(x))| &\leqslant \left|\int_{s+t_0+\ldots+t_k=1} \mathrm{e}^{\gamma(sx+t_0x_0+\ldots+t_kx_k)}\right| |\gamma(x-x_0)|\dots|\gamma(x-x_k)| \\ &\leqslant \frac{\mathrm{e}^{\|\gamma\|(\|x\|+M_k)}}{(k+2)!}\|\gamma\|^{k+1}(\|x\|+M_k)^{k+1}.\end{aligned}$$

When $\|x\|$ is bounded and $k$ is large, we may consider this to be of the form

$$\frac{M_k^k\|\gamma\|^k\mathrm{e}^{M_k\|\gamma\|}}{k!}.$$

Now if $\tilde{f} \in L^p(W)$, for $p > 1$, is such that

$$f(x) = \int_{E'} \mathrm{e}^{\gamma(x)}\tilde{f}(\gamma)\,\mathrm{d}W(\gamma),$$

let $q$ be the conjugate exponent of $p$, and using Fernique's Theorem [5] we fix $\varepsilon > 0$ small enough to enssure that

$$\int_{E'} \mathrm{e}^{\varepsilon\|\gamma\|^2}\,\mathrm{d}W(\gamma) < \infty.$$

Then, using Young's inequality, we obtain

$$\begin{aligned}\frac{M_k^k\|\gamma\|^k\mathrm{e}^{M_k\|\gamma\|}}{k!} &\leqslant \frac{M_k^k\|\gamma\|^k\mathrm{e}^{qM_k^2/(2\varepsilon)}\mathrm{e}^{\varepsilon\|\gamma\|^2/(2q)}}{k!} \\ &= \frac{\mathrm{e}^{qM_k^2/(2\varepsilon)}q^{k/2}M_k^k}{\varepsilon^{k/2}\sqrt{k!}}\cdot\frac{\mathrm{e}^{\varepsilon\|\gamma\|^2/(2q)}\varepsilon^{k/2}\|\gamma\|^k}{q^{k/2}\sqrt{k!}}.\end{aligned}$$

The second factor is bounded by

$$\mathrm{e}^{\varepsilon\|\gamma\|^2/(2q)}\left(\mathrm{e}^{\varepsilon\|\gamma\|^2/q}\right)^{1/2} = \mathrm{e}^{\varepsilon\|\gamma\|^2/q}.$$

We check that the first factor tends to zero as $k$ grows. In fact, the sequence $\left(\frac{\mathrm{e}^{qM_k^2/(2\varepsilon)}q^{k/2}M_k^k}{\varepsilon^{k/2}\sqrt{k!}}\right)_k$ is summable: by the root test and Stirling's formula we have

$$\lim_k \frac{\mathrm{e}^{qM_k^2/(2k\varepsilon)}\sqrt{q}\,M_k}{\sqrt{\varepsilon}\;\sqrt[2k]{k!}} = \lim_k \frac{\mathrm{e}^{qM_k^2/(2k\varepsilon)}\sqrt{q\mathrm{e}}\,M_k}{\sqrt{\varepsilon}\;\sqrt[4k]{2\pi k}\,\sqrt{k}} = 0.$$

Therefore,

$$
\begin{aligned}
|f(x)-L_k(x)| &\leqslant \int_{E'} |\mathrm{e}^{\gamma(x)} - L_{k,\gamma}(\gamma(x))||\tilde{f}(\gamma)| \,\mathrm{d}W(\gamma) \\
&\leqslant \frac{\mathrm{e}^{qM_k^2/(2\varepsilon)} q^{k/2} M_k^k}{\varepsilon^{k/2}\sqrt{k!}} \int_{E'} \mathrm{e}^{\varepsilon\|\gamma\|^2/q} \|\tilde{f}(\gamma)\| \,\mathrm{d}W(\gamma).
\end{aligned}
$$

Now, using Hölder's inequality, we obtain

$$
\begin{aligned}
|f(x)-L_k(x)| &\leqslant \frac{\mathrm{e}^{qM_k^2/(2\varepsilon)} q^{k/2} M_k^k}{\varepsilon^{k/2}\sqrt{k!}} \left(\int_{E'} \mathrm{e}^{\varepsilon\|\gamma\|^2} \,\mathrm{d}W(\gamma)\right)^{1/q} \left(\int_{E'} \|\tilde{f}(\gamma)\|^p \,\mathrm{d}W(\gamma)\right)^{1/p} \\
&= \frac{\mathrm{e}^{qM_k^2/(2\varepsilon)} q^{k/2} M_k^k}{\varepsilon^{k/2}\sqrt{k!}} \|\mathrm{e}^{\varepsilon\|\gamma\|^2}\|_1^{1/q} \|\tilde{f}\|_p \to 0 \quad \text{as } k \to \infty.
\end{aligned}
$$

Thus $L_k$ converges to $f$ uniformly on bounded subsets of $E$.

Two final comments regarding the growth condition in Theorem 1.2:

First, in the finite-dimensional case something more can be said. In this case, the measure $W$ is the standard Gaussian measure on $\mathbb{C}^n$, and the $k$-th Lagrange interpolant of $f$ is

$$
L_k(x) = \int_{\mathbb{C}^n} L_{k,\gamma}(\gamma(x)) \overline{f(\gamma)} \mathrm{e}^{-\|\gamma\|^2} \,\mathrm{d}\gamma_1 \ldots \mathrm{d}\gamma_n.
$$

It is not hard to see then that if $f \in L^p(W)$, the Lagrange polynomials will converge uniformly to $f$ on bounded subsets of $\mathbb{C}^n$ if

$$
\limsup_k \frac{M_k}{\sqrt{k}} \leqslant \frac{0.6}{\sqrt{q\mathrm{e}}}
$$

where $q$ is the conjugate exponent of $p$.

Secondly, as was mentioned in Introduction, growth conditions must be imposed in order to enssure convergence of the Lagrange polynomials to the interpolated function $f$. The condition mentioned there corresponds to the case of the exponential function. The condition in Theorem 1.2 is independent of the function $f$ and thus must cover all representable functions. For a function of order 2 and exponential type $\varepsilon < 1$, Gelfond's classical condition in one complex variable [8] requires that for large $k$,

$$
|x_k| < C\, k^{1/2},
$$

where $C = \sqrt{\varepsilon^{-1}(\ln 2 - 1/2)}$. Thus our condition seems close to being the best possible. □

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