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ON REPRESENTATIONS OF RESTRICTED LIE SUPERALGEBRAS

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Abstract. Simple modules for restricted Lie superalgebras are studied. The indecomposability of baby Kac modules and baby Verma modules is proved in some situation. In particular, for the classical Lie superalgebra of type $A(n|0)$, the baby Verma modules $Z_\chi(\lambda)$ are proved to be simple for any regular nilpotent p-character χ and typical weight λ. Moreover, we obtain the dimension formulas for projective covers of simple modules with p-characters of standard Levi form.

Keywords: restricted Lie superalgebra; χ-reduced representation; indecomposable module; simple module; p-character

MSC 2010: 17B10, 17B35, 17B50

1. Introduction

The finite-dimensional simple Lie superalgebras over the field of complex numbers were classified by Kac in the 1970s (cf. [8]). Although until now, the classification of finite-dimensional simple (restricted) Lie superalgebras over a field of prime characteristic has not yet been completed, there has been increasing interest in modular representation theory of restricted Lie superalgebras in recent years. W. Wang and L. Zhao [15], [16] initiated and developed systematically the modular representations of Lie superalgebras over an algebraically closed field of characteristic $p > 2$. In [15], the super version of the celebrated Kac-Weisfeiler Property was shown to hold for the basic classical Lie superalgebras, which by definition admit an even non-degenerate supersymmetric bilinear form and whose even subalgebras are reductive. There also has been increasing interest [1], [2], [3], [4], [9], [11] in modular representation theory of algebraic supergroups in connection with other areas in recent years. Indeed,

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the modular representation theory of supergroups and Lie superalgebras has found remarkable applications to classical mathematics (see [11] for references and some historical remarks). Representations of Cartan type Lie superalgebras of prime characteristic were studied in [12], [13], [14], [17], [18], [19].

The modular representations of restricted Lie algebras of prime characteristic have been developed over the years (see [7] for a review). The corresponding question for restricted Lie superalgebras naturally arises. The present work has been largely motivated by the representation theory of modular Lie algebras (cf. [5], [7]). Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie superalgebra over an algebraically closed field $\mathbb{F}$ of characteristic $p > 2$. It is obvious that for each $x \in \mathfrak{g}_{\bar{0}}$, the element $x^p - x^{[p]}$ is even and central in the universal enveloping superalgebra $U(\mathfrak{g})$. Let Z denote the central subalgebra of $U(\mathfrak{g})$ generated by all the elements $x^p - x^{[p]}$ with $x \in \mathfrak{g}_{\bar{0}}$, which is the so-called p-center. Since each irreducible $\mathfrak{g}$-module is finite-dimensional (cf. [15], [20]), the Lie superalgebra version of Schur's Lemma [8], §1.1.6, implies that the p-center Z acts by scalars on any irreducible $\mathfrak{g}$-module M. Then there exists a unique $\chi \in \mathfrak{g}_{\bar{0}}^*$ such that $x^p \cdot v - x^{[p]} \cdot v = \chi(x)^p v$, for all $x \in \mathfrak{g}_{\bar{0}}$, $v \in M$. Therefore, M is a module for the finite-dimensional superalgebra $U_\chi(\mathfrak{g}) = U(\mathfrak{g})/(x^p - x^{[p]} - \chi(x)^p | x \in \mathfrak{g}_{\bar{0}})$, where $(x^p - x^{[p]} - \chi(x)^p | x \in \mathfrak{g}_{\bar{0}})$ denotes the ideal of $U(\mathfrak{g})$ generated by all the elements $x^p - x^{[p]} - \chi(x)^p$ with $x \in \mathfrak{g}_{\bar{0}}$. The superalgebra $U_\chi(\mathfrak{g})$ is called the χ-reduced enveloping superalgebra. More generally, a $\mathfrak{g}$-module M is said to have a p-character χ provided that $x^p \cdot v - x^{[p]} \cdot v = \chi(x)^p v$, for all $x \in \mathfrak{g}_{\bar{0}}$, $v \in M$, or equivalently, it is a $U_\chi(\mathfrak{g})$-module.

This paper is structured as follows. In Section 2, we recall some basic notation and properties for restricted Lie superalgebras. Section 3 is devoted to studying representations of restricted Lie superalgebras with an admissible $\mathbb{Z}$-grading of depth one. The baby Kac modules are proved to be indecomposable and have simple socle. In the final section, we study representations of the classical Lie superalgebra $\mathfrak{sl}(n+1|1)$ with p-character of standard Levi form. The baby Verma modules are proved to be indecomposable. When χ is regular nilpotent and λ is typical, the baby Verma module $Z_\chi(\lambda)$ is simple. Moreover, we obtain the dimension formulas for projective covers of simple modules with p-characters of standard Levi form.

2. Preliminaries

Throughout this paper, $\mathbb{F}$ is assumed to be an algebraically closed field of prime characteristic $p > 2$. All modules (vector spaces) are over $\mathbb{F}$ and finite-dimensional.

The following notion of restricted Lie superalgebras is a generalization of the one for restricted Lie algebras (see [6]).

Definition 2.1 (cf. [10]). A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ is called a restricted Lie superalgebra if there is a so-called p-mapping $[p]$ on $\mathfrak{g}_{\overline{0}}$ satisfying the following conditions:

(i) $(\operatorname{ad} x)^p y = \operatorname{ad}(x^{[p]})y$, for all $x \in \mathfrak{g}_{\overline{0}}$ and $y \in \mathfrak{g}$,
(ii) $(kx)^{[p]} = k^p x^{[p]}$, for all $k \in \mathbb{F}$, $x \in \mathfrak{g}_{\overline{0}}$,
(iii) $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum\limits_{i=1}^{p-1} s_i(x,y)$, for all $x, y \in \mathfrak{g}_{\overline{0}}$,

where $is_i(x,y)$ is the coefficient of λ^{i-1} in $\operatorname{ad}(\lambda x+y)^{p-1}(x)$ and λ is an indeterminate.

Remarks 2.2.

(1) The condition (iii) in Definition 2.1 is equivalent to the following condition.
(iii′) The following relation in the universal enveloping superalgebra $U(\mathfrak{g})$ holds:

$$(x+y)^p - x^p - y^p = (x+y)^{[p]} - x^{[p]} - y^{[p]}, \quad \forall\, x, y \in \mathfrak{g}_{\overline{0}}.$$

(2) In short, a restricted Lie superalgebra is a Lie superalgebra whose even subalgebra is a restricted Lie algebra and the odd part is a restricted module over the even subalgebra by the adjoint action.

Examples 2.3.

(1) Let $A = A_{\overline{0}} \oplus A_{\overline{1}}$ be any associative $\mathbb{F}$-superalgebra. Then A admits the structure of a Lie superalgebra by defining the bracket operation as $[x,y] = xy - (-1)^{\overline{x}\overline{y}}yx$ for any homogeneous elements $x, y \in A$, where $\overline{x}, \overline{y}$ denote the parity of x and y, respectively. Furthermore, A becomes a restricted Lie superalgebra with the p-mapping given by $x^{[p]} = x^p$ for any $x \in A_{\overline{0}}$, i.e., the p-mapping is just taken as the p-th power in the superalgebra A.
(2) The Lie superalgebra of an algebraic supergroup is a restricted Lie superalgebra (see [11]).

Let $(\mathfrak{g}, [p])$ be a restricted Lie superalgebra and $\chi \in \mathfrak{g}_{\overline{0}}^*$. A $\mathfrak{g}$-module M is said to be χ-reduced if $x^p \cdot v - x^{[p]} \cdot v = \chi(x)^p v$ for all $x \in \mathfrak{g}_{\overline{0}}$, $v \in M$. In particular, it is called a restricted module if $\chi = 0$. As in the case of restricted Lie algebras, one can define the so-called χ-reduced enveloping superalgebra $U_\chi(\mathfrak{g})$ to be the quotient of the universal enveloping superalgebra $U(\mathfrak{g})$ by the ideal generated by $\{x^p - x^{[p]} - \chi(x)^p\,;\ x \in \mathfrak{g}_{\overline{0}}\}$, i.e., $U_\chi(\mathfrak{g}) = U(\mathfrak{g})/(x^p - x^{[p]} - \chi(x)^p | x \in \mathfrak{g}_{\overline{0}})$. If $\chi = 0$, the superalgebra $U_0(\mathfrak{g})$ is called the restricted enveloping superalgebra and denoted by $u(\mathfrak{g})$ for brevity. All the χ-reduced or restricted $\mathfrak{g}$-modules constitute a full subcategory of the $\mathfrak{g}$-module category, which coincides with the $U_\chi(\mathfrak{g})$ or the $u(\mathfrak{g})$-module category. Each simple $\mathfrak{g}$-module is a $U_\chi(\mathfrak{g})$-module for a unique $\chi \in \mathfrak{g}_{\overline{0}}^*$ (cf. [15], [20]).

3. Representations of restricted Lie superalgebras with an admissible $\mathbb{Z}$-grading of depth one

In this section, we always assume that $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a restricted Lie superalgebra with an admissible $\mathbb{Z}$-grading of depth one, i.e., $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ so that $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0$, $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ and $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$, for all $i, j \in \{-1, 0, 1\}$ where we make the convention that $\mathfrak{g}_k = 0$ for $k \notin \{-1, 0, 1\}$. Set $\mathfrak{g}^+ = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ which is a restricted subalgebra of $\mathfrak{g}$. We have the following easy observation.

Lemma 3.1. *Any irreducible $\mathfrak{g}_0$-module can be regarded as an irreducible $\mathfrak{g}^+$-module with trivial $\mathfrak{g}_1$-action. Moreover, any irreducible $\mathfrak{g}^+$-module is an irreducible $\mathfrak{g}_0$-module with trivial $\mathfrak{g}_1$-action.*

Proof. The first statement is obvious, since $\mathfrak{g}_1$ is an ideal in $\mathfrak{g}^+$.

To prove the second statement, let M be any irreducible $\mathfrak{g}^+$-module. Note that $x^2 = [x, x]/2 = 0$ holds in $U(\mathfrak{g})$ for any $x \in \mathfrak{g}_1$. Then the superspace $M^{\mathfrak{g}_1} := \{v \in M;\ xv = 0,\ \forall\, x \in \mathfrak{g}_1\} \neq 0$ by [21], Proposition 2.1, Chapter 3. It is easy to check that $M^{\mathfrak{g}_1}$ is a $\mathfrak{g}^+$-submodule of M. Hence $M^{\mathfrak{g}_1} = M$ by the simplicity of M as a $\mathfrak{g}^+$-module, i.e., $\mathfrak{g}_1$ acts trivially on M. The proof is completed. $\square$

By Lemma 3.1, any irreducible $\mathfrak{g}_0$-module can be viewed as an irreducible $\mathfrak{g}^+$-module with trivial $\mathfrak{g}_1$-action.

Definition 3.2. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a restricted Lie superalgebra with an admissible $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ and let M be an irreducible $U_\chi(\mathfrak{g}_{\bar{0}})$-module. Set $K_\chi(M) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{g}^+)} M$, where M is viewed as a $U_\chi(\mathfrak{g}^+)$-module with trivial $\mathfrak{g}_1$-action. The induced $\mathfrak{g}$-module $K_\chi(M)$ is called a baby Kac module associated with M.

Proposition 3.3. *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a restricted Lie superalgebra with an admissible $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ and let M be an irreducible $U_\chi(\mathfrak{g}_{\bar{0}})$-module. Then the baby Kac module $K_\chi(M)$ is an indecomposable $U_\chi(\mathfrak{g})$-module. Moreover, $K_\chi(M)$ has a simple socle. For any irreducible $U_\chi(\mathfrak{g}_{\bar{0}})$-modules M and N, we have $K_\chi(M) \cong K_\chi(N)$ as $U_\chi(\mathfrak{g})$-modules if and only if $M \cong N$ as $U_\chi(\mathfrak{g}_{\bar{0}})$-modules.*

Proof. Fix an ordered basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{g}_{-1}$. For a subset $I = \{i_1 < i_2 < \ldots < i_s\} \subseteq \{1, \ldots, n\}$, let x_I denote $x_{i_1} \ldots x_{i_s}$ in $U_\chi(\mathfrak{g})$. Then as a vector space, $K_\chi(M)$ has a basis $\{x_I \otimes v_i;\ I \subseteq \{1, \ldots, n\}, i = 1, \ldots, \dim M\}$, where $\{v_i;\ i = 1, \ldots, \dim M\}$ is a basis of M.

Suppose $\mathscr{M}$ is a nonzero submodule of $K_\chi(M)$. Let $0 \neq v = \sum_{I,i} a_{I,i} x_I \otimes v_i \in \mathscr{M}$ where $a_{I,i} \in \mathbb{F}$. We can apply some action of x_i on v, so that we get $x_1 \ldots x_n \otimes w \in \mathscr{M}$, where $0 \neq w \in M$. Note that M is a simple $U_\chi(\mathfrak{g}_{\bar{0}})$-module, hence $U_\chi(\mathfrak{g}_{\bar{0}})w = M$. It is a routine to check that $x_1 \ldots x_n \otimes M = U_\chi(\mathfrak{g}_{\bar{0}})x_1 \ldots x_n \otimes w \in \mathscr{M}$, i.e., any submodule of $K_\chi(M)$ contains the subspace $x_1 \ldots x_n \otimes M$ which is isomorphic to M^σ as a $\mathfrak{g}_{\bar{0}}$-module, where M^σ is a twist of M with the same underlining space and the module structure given by $x \circ v = xv + \operatorname{tr}(\operatorname{ad} x|_{\mathfrak{g}_{-1}})v$ for $x \in \mathfrak{g}_{\bar{0}}$, $v \in M^\sigma$. Therefore $K_\chi(M)$ is an indecomposable $U_\chi(\mathfrak{g})$-module.

By the discussion above, $K_\chi(M)$ has a unique irreducible submodule $U_\chi(\mathfrak{g})x_1 \ldots x_n \otimes M$, i.e., $K_\chi(M)$ has a simple socle $U_\chi(\mathfrak{g})x_1 \ldots x_n \otimes M$.

Let M and N be two simple $U_\chi(\mathfrak{g}_{\bar{0}})$-modules. Suppose $K_\chi(M) \cong K_\chi(N)$ as $U_\chi(\mathfrak{g})$-modules. Let $\varphi\colon K_\chi(M) \longrightarrow K_\chi(N)$ be the isomorphism. Note that $0 = \varphi(x_i \cdot x_1 \ldots x_n \otimes w) = x_i\varphi(x_1 \ldots x_n \otimes w)$ for any $i = 1, \ldots, n, w \in M$. We have $\varphi(x_1 \ldots x_n \otimes M) \subseteq x_1 \ldots x_n \otimes N$. Similarly, we have $\varphi^{-1}(x_1 \ldots x_n \otimes N) \subseteq x_1 \ldots x_n \otimes M$. Consequently, $\varphi(x_1 \ldots x_n \otimes M) = x_1 \ldots x_n \otimes N$. Since $x_1 \ldots x_n \otimes M \cong M^\sigma$ and $x_1 \ldots x_n \otimes N \cong N^\sigma$ as $\mathfrak{g}_{\bar{0}}$-modules, it follows that $M \cong N$ as $\mathfrak{g}_{\bar{0}}$-modules.

We completed the proof. □

The following result asserts that any simple $U_\chi(\mathfrak{g})$-module is a homomorphic image of a baby Kac module $K_\chi(M)$ for some simple $U_\chi(\mathfrak{g}_{\bar{0}})$-module M.

Proposition 3.4. *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a restricted Lie superalgebra with an admissible $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Let $\chi \in \mathfrak{g}_{\bar{0}}^*$. Then any simple $U_\chi(\mathfrak{g})$-module is a homomorphic image of a baby Kac module $K_\chi(M)$ for some simple $U_\chi(\mathfrak{g}_{\bar{0}})$-module M.*

Proof. Let $\mathscr{M}$ be any simple $U_\chi(\mathfrak{g})$-module. Take a simple $U_\chi(\mathfrak{g}^+)$-submodule M in $\mathscr{M}$. We then have the canonical $U_\chi(\mathfrak{g})$-module homomorphism

$$\Psi\colon\ K_\chi(M) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{g}^+)} M \longrightarrow \mathscr{M}$$

induced from the inclusion $M \hookrightarrow \mathscr{M}$.

Since $\mathscr{M}$ is a simple $U_\chi(\mathfrak{g})$-module, Ψ is epimorphism. Therefore, $\mathscr{M} \cong K_\chi(M)/\ker \Psi$. □

In the following, we assume that $\mathfrak{g}_{\bar{0}}$ is an algebraic Lie algebra, hence $\mathfrak{g}_{\bar{0}}$ has a triangular decomposition $\mathfrak{g}_{\bar{0}} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$. Set $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ and $\mathscr{B} = \mathfrak{b} \oplus \mathfrak{g}_1$. Let $\chi \in \mathfrak{g}_{\bar{0}}^*$, then under the coadjoint action of the automorphism group $\operatorname{Aut}(\mathfrak{g}_{\bar{0}})$ of $\mathfrak{g}_{\bar{0}}$, the character χ can be conjugated to another character $\chi' \in \mathfrak{g}_{\bar{0}}^*$ such that $\chi'(\mathfrak{n}) = 0$, i.e., there exists a $\sigma \in \operatorname{Aut}(\mathfrak{g}_{\bar{0}})$ such that $(\sigma \cdot \chi)(\mathfrak{n}) = 0$. Since $U_\chi(\mathfrak{g}) \cong U_{\sigma\cdot\chi}(\mathfrak{g})$, the representation theory of $U_\chi(\mathfrak{g})$ is equivalent to that of $U_{\sigma\cdot\chi}(\mathfrak{g})$. Therefore, in the sequel we always assume $\chi \in \mathfrak{g}_{\bar{0}}^*$ with $\chi(\mathfrak{n}) = 0$. Set $\Lambda_\chi = \{\lambda \in \mathfrak{h}^*;\ \lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p\}$. For any $\lambda \in \Lambda_\chi$, one can define a one-dimensional $\mathfrak{b}$-module $\mathbb{F}_\lambda$ with a basis

v_λ such that $h \cdot v_\lambda = \lambda(h)v_\lambda$ for any $h \in \mathfrak{h}$, and $\mathfrak{n} \cdot v_\lambda = 0$. It is easy to check that $\mathbb{F}_\lambda$ is a $U_\chi(\mathfrak{b})$-module. Moreover, any $U_\chi(\mathfrak{b})$-module is of the form $\mathbb{F}_\lambda$ for some $\lambda \in \Lambda_\chi$.

We have the following easy observation, the proof of which is straightforward and similar to that of Lemma 3.1.

Lemma 3.5. *Any irreducible $\mathfrak{b}$-module can be regarded as an irreducible $\mathscr{B}$-module with trivial $\mathfrak{g}_1$-action. Moreover, any irreducible $\mathscr{B}$-module is an irreducible $\mathfrak{b}$-module with trivial $\mathfrak{g}_1$-action.*

By Lemma 3.5, any irreducible $U_\chi(\mathfrak{b})$-module $\mathbb{F}_\lambda$ with $\lambda \in \Lambda_\chi$ can be viewed as an irreducible $\mathscr{B}$-module with trivial $\mathfrak{g}_1$-action.

Definition 3.6. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a restricted Lie superalgebra with an admissible $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_{\bar{0}}$ is an algebraic Lie algebra with a triangular decomposition $\mathfrak{g}_{\bar{0}} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$. Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ with $\chi(\mathfrak{n}) = 0$ and $\lambda \in \Lambda_\chi$. Set $Z_\chi(\lambda) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathscr{B})} \mathbb{F}_\lambda$. We call $Z_\chi(\lambda)$ a baby Verma module.

Proposition 3.7. *Any baby Kac module $K_\chi(M)$ is a quotient of some baby Verma module $Z_\chi(\lambda)$ for $\lambda \in \Lambda_\chi$.*

P r o o f. Any simple $U_\chi(\mathfrak{g}_{\bar{0}})$-module M is a quotient of some baby Verma $U_\chi(\mathfrak{g}_{\bar{0}})$-module $U_\chi(\mathfrak{g}_{\bar{0}}) \otimes_{U_\chi(\mathfrak{b})} \mathbb{F}_\lambda$ for $\lambda \in \Lambda_\chi$. Note that

$$Z_\chi(\lambda) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathscr{B})} \mathbb{F}_\lambda = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{g}^+)} (U_\chi(\mathfrak{g}^+) \otimes_{U_\chi(\mathscr{B})} \mathbb{F}_\lambda),$$

and $U_\chi(\mathfrak{g}^+) \otimes_{U_\chi(\mathscr{B})} \mathbb{F}_\lambda$ is isomorphic to $U_\chi(\mathfrak{g}_{\bar{0}}) \otimes_{U_\chi(\mathfrak{b})} \mathbb{F}_\lambda$ as a $U_\chi(\mathfrak{g}_{\bar{0}})$-module, so we then have the epimorphism of $U_\chi(\mathfrak{g}_{\bar{0}})$-modules

$$U_\chi(\mathfrak{g}^+) \otimes_{U_\chi(\mathscr{B})} \mathbb{F}_\lambda \twoheadrightarrow M,$$

which induces the epimorphism of $U_\chi(\mathfrak{g})$-modules

$$Z_\chi(\lambda) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{g}^+)} (U_\chi(\mathfrak{g}^+) \otimes_{U_\chi(\mathscr{B})} \mathbb{F}_\lambda) \twoheadrightarrow U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{g}^+)} M = K_\chi(M).$$

The proof is completed. $\square$

4. Representations of the classical Lie superalgebra of type $A(n|0)$ with p-characters of standard Levi form

In this section, we assume that $\mathfrak{g}=\mathfrak{sl}(n+1|1)$ is the classical Lie superalgebra of type $A(n|0)$. Then $\mathfrak{g}$ has a $\mathbb{Z}$-grading $\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$ with

$$\mathfrak{g}_{-1}=\operatorname{span}_{\mathbb{F}}\{e_{n+2,1},\dots,e_{n+2,n+1}\},\quad \mathfrak{g}_1=\operatorname{span}_{\mathbb{F}}\{e_{1,n+2},\dots,e_{n+1,n+2}\}$$

and

$$\mathfrak{g}_0=\operatorname{span}_{\mathbb{F}}\{e_{i,j},\,e_{k,k}+e_{n+2,n+2}\,;\ 1\leqslant i,j,k\leqslant n+1, i\neq j\}\cong\mathfrak{gl}_{n+1},$$

where $e_{i,j}$ stands for the $(n+2)\times(n+2)$ matrix with 1 appearing in the i-th row and j-th column, and 0 in the other positions. The even part $\mathfrak{g}_{\bar 0}$ of $\mathfrak{g}$ coincides with $\mathfrak{g}_0$, while the odd part is $\mathfrak{g}_{\bar 1}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_1$. We have a canonical triangular decomposition $\mathfrak{g}_{\bar 0}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}$ with $\mathfrak{n}^-=\operatorname{span}_{\mathbb{F}}\{e_{i,j}\,;\ 1\leqslant j<i\leqslant n+1\}$, $\mathfrak{h}=\operatorname{span}_{\mathbb{F}}\{h_k=e_{k,k}+e_{n+2,n+2}\,;\ k=1,\dots,n+1\}$, $\mathfrak{n}=\operatorname{span}_{\mathbb{F}}\{e_{i,j}\,;\ 1\leqslant i<j\leqslant n+1\}$. Set $\mathfrak{b}=\mathfrak{h}\oplus\mathfrak{n}$, which is a Borel subalgebra of $\mathfrak{g}_{\bar 0}$. Set $\mathscr{B}=\mathfrak{b}\oplus\mathfrak{g}_1$, which is a Borel subalgebra of $\mathfrak{g}^+=\mathfrak{g}_0\oplus\mathfrak{g}_1$.

Definition 4.1. A p-character $\chi\in\mathfrak{g}_{\bar 0}^*$ is said to have standard Levi form if $\chi(\mathfrak{h}\oplus\mathfrak{n})=0$ and there exists a subset $I\subseteq\{2,\dots,n+1\}$ such that $\chi(e_{l,l-1})\neq 0$ for all $l\in I$ and $\chi(e_{k,k-1})=0$ for all $k\in\{2,\dots,n+1\}\setminus I$, while $\chi(e_{i,j})=0$ for $i,j\in\{1,\dots,n+1\}$ with $i-j\geqslant 2$. In particular, when $I=\{2,\dots,n+1\}$, the p-character χ is called regular nilpotent.

Proposition 4.2. *Let $\mathfrak{g}=\mathfrak{sl}(n+1|1)$ be the classical Lie superalgebra of type $A(n|0)$. Let $\chi\in\mathfrak{g}_{\bar 0}^*$ have standard Levi form, $\lambda\in\Lambda_\chi$. Then the baby Verma module $Z_\chi(\lambda)$ is indecomposable and has a unique maximal submodule.*

Proof. Set $\mathfrak{N}^-=\mathfrak{n}^-\oplus\mathfrak{g}_{-1}$, which is a restricted subalgebra of $\mathfrak{g}$. Note that $x^2=[x,x]/2=0$ for any $x\in\mathfrak{g}_{-1}$, and $\mathfrak{g}_{-1}$ is an ideal in $\mathfrak{N}^-$, hence any simple $\mathfrak{N}^-$-module is a simple $\mathfrak{n}^-$-module with trivial $\mathfrak{g}_{-1}$-action by [21], Proposition 2.1, Chapter 3.

Since the p-character χ has standard Levi form, $\chi([\mathfrak{n}^-,\mathfrak{n}^-])=0$ and $\chi(x^{[p]})=0$ for any $x\in\mathfrak{n}^-$. Then the p-character χ defines a one-dimensional $\mathfrak{n}^-$-module $\mathbb{F}_\chi$ which is indeed a $U_\chi(\mathfrak{n}^-)$-module. More precisely, $\mathbb{F}_\chi$ has a basis v_χ such that $x\cdot v_\chi=\chi(x)v_\chi$ for any $x\in\mathfrak{n}^-$. Moreover, $\mathbb{F}_\chi$ is the unique simple $U_\chi(\mathfrak{n}^-)$-module. Therefore, $\mathbb{F}_\chi$ is also the unique simple $U_\chi(\mathfrak{N}^-)$-module. The projective cover of $\mathbb{F}_\chi$ in the $U_\chi(\mathfrak{N}^-)$-module category is just $U_\chi(\mathfrak{N}^-)$. Then $U_\chi(\mathfrak{N}^-)$ is an indecomposable $U_\chi(\mathfrak{N}^-)$-module and has a maximal submodule $\mathfrak{M}\subsetneqq U_\chi(\mathfrak{N}^-)$. Since $Z_\chi(\lambda)\cong U_\chi(\mathfrak{N}^-)$ as a $U_\chi(\mathfrak{N}^-)$-module, it is an indecomposable $U_\chi(\mathfrak{g})$-module and has a unique maximal submodule contained in $\mathfrak{M}$. $\square$

For a given p-character $\chi \in \mathfrak{g}_{\bar{0}}^*$ of standard Levi form and $\lambda \in \Lambda_\chi$, let $L_\chi(\lambda)$ denote the simple quotient of $Z_\chi(\lambda)$.

Proposition 4.3. *Let $\mathfrak{g} = \mathfrak{sl}(n+1|1)$ be the classical Lie superalgebra of type $A(n|0)$. Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ have standard Levi form. Then any simple $U_\chi(\mathfrak{g})$-module is of the form $L_\chi(\lambda)$ for some $\lambda \in \Lambda_\chi$.*

Proof. Let $\mathscr{M}$ be any simple $U_\chi(\mathfrak{g})$-module. Take a simple $U_\chi(\mathscr{B})$ submodule $\mathscr{W}$ in $\mathscr{M}$. Then $\mathscr{W}$ is one-dimensional and spanned by a basis w of weight $\lambda \in \Lambda_\chi$. We have the homomorphism

$$\begin{aligned} Z_\chi(\lambda) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathscr{B})} \mathbb{F}_\lambda &\longrightarrow \mathscr{M} \\ u \otimes v_\lambda &\longmapsto u \cdot w, \end{aligned}$$

which is surjective by the simplicity of $\mathscr{M}$ as a $U_\chi(\mathfrak{g})$-module. Consequently, $\mathscr{M} \cong L_\chi(\lambda)$. $\square$

Remark 4.4. If $\chi \in \mathfrak{g}_{\bar{0}}^*$ is regular nilpotent and $\lambda \in \Lambda_\chi$, then $U_\chi(\mathfrak{g}_{\bar{0}}) \otimes_{U_\chi(\mathfrak{b})} \mathbb{F}_\lambda$ is a simple $U_\chi(\mathfrak{g}_{\bar{0}})$-module (cf. [5]). Moreover, it is also a simple $U_\chi(\mathfrak{g}^+)$-module. Therefore, the baby Verma module

$$\begin{aligned} Z_\chi(\lambda) &= U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathscr{B})} \mathbb{F}_\lambda \\ &= U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{g}^+)} \big(U_\chi(\mathfrak{g}^+) \otimes_{U_\chi(\mathscr{B})} \mathbb{F}_\lambda\big) \\ &= U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{g}^+)} \big(U_\chi(\mathfrak{g}_{\bar{0}}) \otimes_{U_\chi(\mathfrak{b})} \mathbb{F}_\lambda\big) \end{aligned}$$

is also a baby Kac module. The converse is also true, since any simple $U_\chi(\mathfrak{g}_{\bar{0}})$-module is of the form $U_\chi(\mathfrak{g}_{\bar{0}}) \otimes_{U_\chi(\mathfrak{b})} \mathbb{F}_\lambda$ for some $\lambda \in \Lambda_\chi$.

Theorem 4.5. *Let $\mathfrak{g} = \mathfrak{sl}(n+1|1)$ be the classical Lie superalgebra of type $A(n|0)$. Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ be regular nilpotent and $\lambda, \mu \in \Lambda_\chi$. Then $Z_\chi(\lambda) \cong Z_\chi(\mu)$ if and only if $\lambda = w \cdot \mu$ for some $w \in W$, where W is the Weyl group of $\mathfrak{g}_{\bar{0}} \cong \mathfrak{gl}_{n+1}$.*

Proof. According to Remark 4.4,

$$Z_\chi(\lambda) \cong K_\chi(U_\chi(\mathfrak{g}_{\bar{0}}) \otimes_{U_\chi(\mathfrak{b})} \mathbb{F}_\lambda) \quad \text{and} \quad Z_\chi(\mu) \cong K_\chi(U_\chi(\mathfrak{g}_{\bar{0}}) \otimes_{U_\chi(\mathfrak{b})} \mathbb{F}_\mu),$$

where $U_\chi(\mathfrak{g}_{\bar{0}}) \otimes_{U_\chi(\mathfrak{b})} \mathbb{F}_\lambda$ and $U_\chi(\mathfrak{g}_{\bar{0}}) \otimes_{U_\chi(\mathfrak{b})} \mathbb{F}_\mu$ are simple $U_\chi(\mathfrak{g}_{\bar{0}})$-modules. Hence $Z_\chi(\lambda) \cong Z_\chi(\mu)$ as $U_\chi(\mathfrak{g})$-modules if and only if $U_\chi(\mathfrak{g}_{\bar{0}}) \otimes_{U_\chi(\mathfrak{b})} \mathbb{F}_\lambda \cong U_\chi(\mathfrak{g}_{\bar{0}}) \otimes_{U_\chi(\mathfrak{b})} \mathbb{F}_\mu$ as $U_\chi(\mathfrak{g}_{\bar{0}})$-modules by Proposition 3.3. On the other hand, since $\mathfrak{g}_{\bar{0}} \cong \mathfrak{gl}_{n+1}$, it follows from [7], Proposition 10.5, that $U_\chi(\mathfrak{g}_{\bar{0}}) \otimes_{U_\chi(\mathfrak{b})} \mathbb{F}_\lambda \cong U_\chi(\mathfrak{g}_{\bar{0}}) \otimes_{U_\chi(\mathfrak{b})} \mathbb{F}_\mu$ if and only if $\lambda = w \cdot \mu$ for some $w \in W$. The conclusion follows. $\square$

Definition 4.6. Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ and $\lambda \in \Lambda_\chi$. Let M be a $U_\chi(\mathfrak{g}^+)$-module. A nonzero vector $v \in M$ is called a maximal vector of weight λ if $h \cdot v = \lambda(h)v$ and $x \cdot v = 0$ for any $h \in \mathfrak{h}$ and $x \in \mathfrak{n} \oplus \mathfrak{g}_1$.

Remark 4.7. If $\chi(\mathfrak{n}) = 0$, then any $U_\chi(\mathfrak{g})$-module M contains at least a maximal vector. Indeed, any simple $U_\chi(\mathscr{B})$-submodule of M is generated by a maximal vector.

Definition 4.8. Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ and $\lambda \in \Lambda_\chi$. We call λ a typical weight if $\lambda(h_i) \neq i - n - 1$ for $i = 1, \ldots, n+1$.

Remark 4.9. Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ with $\chi(h_i) \neq 0$ for $i = 1, \ldots, n+1$. Then any $\lambda \in \Lambda_\chi$ is typical.

Theorem 4.10. *Let $\mathfrak{g} = \mathfrak{sl}(n+1|1)$ be the classical Lie superalgebra of type $A(n|0)$. Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ be regular nilpotent and $\lambda \in \Lambda_\chi$ be typical. Then the baby Verma module $Z_\chi(\lambda)$ is a simple $U_\chi(\mathfrak{g})$-module. Moreover, the baby Verma module $Z_\chi(w \cdot \lambda)$ is also a simple $U_\chi(\mathfrak{g})$-module and $Z_\chi(w \cdot \lambda) \cong Z_\chi(\lambda)$ for any $w \in W$.*

Proof. According to Remark 4.4, the baby Verma module $Z_\chi(\lambda)$ is also a baby Kac module $K_\chi(M)$, where $M = U_\chi(\mathfrak{g}_{\bar{0}}) \otimes_{U_\chi(\mathfrak{b})} \mathbb{F}_\lambda$ is a simple $U_\chi(\mathfrak{g}_{\bar{0}})$-module regarded as a simple $U_\chi(\mathfrak{g}^+)$-module with trivial $\mathfrak{g}_1$-action. Assume that $\mathbb{F}_\lambda = \mathbb{F}v_\lambda$ where v_λ is indeed a maximal vector of weight λ.

Let $\mathfrak{M}$ be any nonzero submodule of $Z_\chi(\lambda)$. According to the proof of Proposition 3.3, $\mathfrak{M}$ contains $e_{n+2,1} \ldots e_{n+2,n+1} \otimes v_\lambda$. Since λ is typical, $\lambda(h_i) \neq i - n - 1$ for $i = 1, \ldots, n+1$. We have the following computation:

$$
\begin{aligned}
&e_{1,n+2} \cdot (e_{n+2,1} \ldots e_{n+2,n+1} \otimes v_\lambda) \\
&\qquad = h_1 e_{n+2,2} \ldots e_{n+2,n+1} \otimes v_\lambda - e_{n+2,1} e_{1,n+2} e_{n+2,2} \ldots e_{n+2,n+1} \otimes v_\lambda \\
&\qquad = e_{n+2,2} \ldots e_{n+2,n+1} \otimes h_1 \cdot v_\lambda + [h_1, e_{n+2,2} \ldots e_{n+2,n+1}] \otimes v_\lambda \\
&\qquad\quad + e_{n+2,1} e_{n+2,2} e_{1,n+2} e_{n+2,3} \ldots e_{n+2,n+1} \otimes v_\lambda \\
&\qquad\quad - e_{n+2,1} e_{12} e_{n+2,3} \ldots e_{n+2,n+1} \otimes v_\lambda \\
&\qquad = (\lambda(h_1) + n) e_{n+2,2} \ldots e_{n+2,n+1} \otimes v_\lambda.
\end{aligned}
$$

It follows that $e_{n+2,2} \ldots e_{n+2,n+1} \otimes v_\lambda \in \mathfrak{M}$.

Letting $e_{2,n+2}, \ldots e_{n+1,n+2}$ consecutively act on $e_{n+2,2} \ldots e_{n+2,n+1} \otimes v_\lambda$, we finally get $1 \otimes v_\lambda \in \mathfrak{M}$. Hence $\mathfrak{M} = Z_\chi(\lambda)$ by the simplicity of M as a $U_\chi(\mathfrak{g}_{\bar{0}})$-module.

The second statement follows from Theorem 4.5. □

Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ be of standard Levi form and $\lambda \in \Lambda_\chi$. Let $Q_\chi(\lambda)$ denote the projective cover of $L_\chi(\lambda)$ in the $U_\chi(\mathfrak{g})$-module category. Let $[M : L]$ denote the multiplicity of a simple $U_\chi(\mathfrak{g})$-module L as a composition factor of a $U_\chi(\mathfrak{g})$-module M. We have the following dimension formula for the projective cover $Q_\chi(\lambda)$ of the simple $U_\chi(\mathfrak{g})$-module $L_\chi(\lambda)$.

Theorem 4.11. *Let $\mathfrak{g} = \mathfrak{sl}(n+1|1)$ be the classical Lie superalgebra of type $A(n|0)$. Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ be of standard Levi form and $\lambda \in \Lambda_\chi$. Then*

$$\dim Q_\chi(\lambda) = 2^{n+1} p^{n(n+1)/2} \sum_{\mu \in \Lambda_\chi} [Z_\chi(\mu) : L_\chi(\lambda)]. \tag{4.1}$$

Proof. Set $\mathfrak{N} = \mathfrak{n} + \mathfrak{g}_1$. Since $Q_\chi(\lambda)$ is a projective $U_\chi(\mathfrak{g})$-module and $U_\chi(\mathfrak{g})$ is free as a $U_\chi(\mathfrak{N})$-module, $Q_\chi(\lambda)$ is a projective $U_\chi(\mathfrak{N})$-module. Note that $\chi(\mathfrak{N}) = 0$ and $\mathfrak{N}$ is a nilpotent subalgebra, hence $U_\chi(\mathfrak{N})$ is a local superalgebra. Hence, projective $U_\chi(\mathfrak{N})$-modules are free $U_\chi(\mathfrak{N})$-modules. In particular, $Q_\chi(\lambda)$ is a free $U_\chi(\mathfrak{N})$-module. Then

$$\dim Q_\chi(\lambda) = \dim U_\chi(\mathfrak{N}) \cdot \dim Q_\chi(\lambda)^{\mathfrak{N}} = 2^{n+1} p^{n(n+1)/2} \dim Q_\chi(\lambda)^{\mathfrak{N}}. \tag{4.2}$$

We have the natural isomorphism

$$Q_\chi(\lambda)^{\mathfrak{N}} \cong \mathrm{Hom}_{\mathfrak{N}}(\mathbb{F}, Q_\chi(\lambda)) \tag{4.3}$$

where $\mathbb{F}$ denotes the trivial $\mathfrak{N}$-module. On the other hand, by Frobenius reciprocity, we have

$$\mathrm{Hom}_{\mathfrak{N}}(\mathbb{F}, Q_\chi(\lambda)) \cong \mathrm{Hom}_{\mathscr{B}}(U_\chi(\mathscr{B}) \otimes_{U_\chi(\mathfrak{N})} \mathbb{F}, Q_\chi(\lambda)). \tag{4.4}$$

Note that $\mathfrak{N}$ is an ideal in $\mathscr{B}$, so it acts on the induced module $U_\chi(\mathscr{B}) \otimes_{U_\chi(\mathfrak{N})} \mathbb{F}$ trivially. Regard $U_\chi(\mathscr{B}) \otimes_{U_\chi(\mathfrak{N})} \mathbb{F}$ as an $U_\chi(\mathfrak{h})$-module; it is isomorphic to $U_\chi(\mathfrak{h})$. Hence it can be decomposed as the direct sum of its one-dimensional $U_\chi(\mathfrak{h})$-modules $\mathbb{F}_\mu$ with $\mu \in \Lambda_\chi$, i.e., we have

$$\mathrm{Hom}_{\mathscr{B}}(U_\chi(\mathscr{B}) \otimes_{U_\chi(\mathfrak{N})} \mathbb{F}, Q_\chi(\lambda)) \cong \bigoplus_{\mu \in \Lambda_\chi} \mathrm{Hom}_{\mathscr{B}}(\mathbb{F}_\mu, Q_\chi(\lambda)). \tag{4.5}$$

By Frobenius reciprocity again, we have

$$\mathrm{Hom}_{\mathscr{B}}(\mathbb{F}_\mu, Q_\chi(\lambda)) \cong \mathrm{Hom}_{\mathfrak{g}}(Z_\chi(\mu), Q_\chi(\lambda)). \tag{4.6}$$

Since $\mathfrak{g}$ is a simple Lie superalgebra, $U_\chi(\mathfrak{g})$ is a symmetric superalgebra by [15], Proposition 2.7. Then the projective cover $Q_\chi(\lambda)$ of $L_\chi(\lambda)$ coincides with the injective hull of $L_\chi(\lambda)$. Therefore, we have

$$\dim \mathrm{Hom}_{\mathfrak{g}}(Z_\chi(\mu), Q_\chi(\lambda)) = [Z_\chi(\mu) : L_\chi(\lambda)]. \tag{4.7}$$

By (4.2)–(4.7), we finally get the dimension formula

$$\dim Q_\chi(\lambda) = 2^{n+1} p^{n(n+1)/2} \sum_{\mu \in \Lambda_\chi} [Z_\chi(\mu) : L_\chi(\lambda)].$$

□

Example 4.12. Let $\mathfrak{g} = \mathfrak{sl}(2|1)$ and let $\chi \in \mathfrak{g}_{\bar{0}}^*$ be regular nilpotent with $\chi(h_1)\chi(h_2) \neq 0$ and $\chi(h_1) \neq \chi(h_2)$. Then there is only one nonzero summand on the right hand side of (4.1) and $\dim Q_\chi(\lambda) = 4p$ for any $\lambda \in \Lambda_\chi$.

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