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Czechoslovak Mathematical Journal, Vol. 64 (2014), No. 2, 365–385

Persistent URL: `http://dml.cz/dmlcz/144004`

Czechoslovak Mathematical Journal, 64 (139) (2014), 365–385

COMMUTATORS OF SUBLINEAR OPERATORS GENERATED BY CALDERÓN-ZYGMUND OPERATOR ON GENERALIZED WEIGHTED MORREY SPACES

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(Received January 7, 2013)

Abstract. In this paper, the boundedness of a large class of sublinear commutator operators T_b generated by a Calderón-Zygmund type operator on a generalized weighted Morrey spaces $M_{p,\varphi}(w)$ with the weight function w belonging to Muckenhoupt's class A_p is studied. When $1 < p < \infty$ and $b \in \mathrm{BMO}$, sufficient conditions on the pair (φ_1, φ_2) which ensure the boundedness of the operator T_b from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ are found. In all cases the conditions for the boundedness of T_b are given in terms of Zygmund-type integral inequalities on (φ_1, φ_2), which do not require any assumption on monotonicity of $\varphi_1(x,r)$, $\varphi_2(x,r)$ in r. Then these results are applied to several particular operators such as the pseudo-differential operators, Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

Keywords: generalized weighted Morrey space; sublinear operator; commutator; BMO space; maximal operator; Calderón-Zygmund operator

MSC 2010: 42B20, 42B25, 42B35

1. Introduction

The classical Morrey spaces $M_{p,\lambda}$ were originally introduced by Morrey [29] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, the readers are referred to [29], [30], [31], [32].

Let $\mathbb{R}^{\ltimes}$ be the n-dimensional Euclidean space of points $x = (x_1, \ldots, x_n)$ with the norm $|x| = \left(\sum\limits_{i=1}^{n} x_i^2\right)^{1/2}$. For $x \in \mathbb{R}^{\ltimes}$ and $r > 0$, denote by $B(x,r)$ the open ball

The research of V. S. Guliyev was supported by the grant of Ahi Evran University Scientific Research Projects (PYO.FEN.4001.12.18).

centered at x of radius r. Let $\complement B(x,r)$ be the complement of the ball $B(x,r)$, and $|B(x,r)|$ the Lebesgue measure of $B(x,r)$.

A weight function is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0,\infty)$ almost everywhere. For a weight w and a measurable set E, we define $w(E)=\int_E w(x)\,\mathrm{d}x$, the Lebesgue measure of E by $|E|$, and the characteristic function of E by χ_E. Given a weight w, we say that w satisfies the doubling condition if there is a constant $D>0$ such that $w(2B)\leqslant Dw(B)$ for any ball B. When w satisfies the doubling condition, we write $w\in\Delta_2$, for short.

If w is a weight function, then we denote the weighted Lebesgue space by $L_p(w)\equiv L_p(\mathbb{R}^n,w)$ with the norm

$$\|f\|_{L_{p,w}}=\left(\int_{\mathbb{R}^n}|f(x)|^p w(x)\,\mathrm{d}x\right)^{1/p}<\infty\quad\text{when}\quad 1\leqslant p<\infty$$

and $\|f\|_{L_{\infty,w}}=\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}|f(x)|w(x)$ when $p=\infty$.

We recall that a weight function w is in Muckenhoupt's class A_p, $1<p<\infty$, if

$$\begin{aligned}[w]_{A_p}&:=\sup_B[w]_{A_p(B)}\\&=\sup_B\left(\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}\,\mathrm{d}x\right)^{p-1}<\infty,\end{aligned}$$

where the sup is taken with respect to all balls B and $1/p+1/p'=1$. Note that for all balls B we have

$$[w]_{A_p(B)}^{1/p}=|B|^{-1}\|w\|_{L_1(B)}^{1/p}\|w^{-1/p}\|_{L_{p'}(B)}\geqslant 1 \tag{1.1}$$

by Hölder's inequality. For $p=1$, the class A_1 is defined by the condition $Mw(x)\leqslant Cw(x)$ with $[w]_{A_1}=\sup_{x\in\mathbb{R}^n}Mw(x)/w(x)$, and for $p=\infty$ we define $A_\infty=\bigcup_{1\leqslant p<\infty}A_p$.

For $f\in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M and the sublinear commutator of the maximal operator are defined by

$$\begin{aligned}Mf(x)&=\sup_{t>0}|B(x,t)|^{-1}\int_{B(x,t)}|f(y)|\,\mathrm{d}y,\\M_b(f)(x)&=\sup_{t>0}|B(x,t)|^{-1}\int_{B(x,t)}|b(x)-b(y)|\,|f(y)|\,\mathrm{d}y,\end{aligned}$$

respectively. Let K be a Calderón-Zygmund singular integral operator, briefly a Calderón-Zygmund operator, i.e., a linear operator bounded from $L_2(\mathbb{R}^n)$ to

$L_2(\mathbb{R}^{\ltimes})$ for all bounded measurable functions f with a compact support, represented by

$$Kf(x) = \int_{\mathbb{R}^{\ltimes}} k(x,y)f(y)\,\mathrm{d}y, \quad x \notin \operatorname{supp} f.$$

Here, $k(x,y)$ is a continuous function away from the diagonal which satisfies the standard estimates: there exist $c_1 > 0$ and $0 < \varepsilon \leqslant 1$ such that

$$|k(x,y)| \leqslant c_1|x-y|^{-n}$$

for all $x, y \in \mathbb{R}^{\ltimes}$, $x \neq y$, and

$$|k(x,y) - k(x',y)| + |k(y,x) - k(y,x')| \leqslant c_1\left(\frac{|x-x'|}{|x-y|}\right)^{\varepsilon}|x-y|^{-n},$$

whenever $2|x-x'| \leqslant |x-y|$. Such operators were introduced in [6].

It is well known that the maximal operator and the Calderón-Zygmund operators play an important role in harmonic analysis (see [10]–[42]).

Let T represent a linear or a sublinear operator which satisfies that for any $f \in L_1(\mathbb{R}^{\ltimes})$ with compact support and $x \notin \operatorname{supp} f$

$$|Tf(x)| \leqslant c_0 \int_{\mathbb{R}^{\ltimes}} \frac{|f(y)|}{|x-y|^n}\,\mathrm{d}y, \tag{1.2}$$

where c_0 is independent of f and x.

For a function b, let T_b represent a linear or a sublinear operator which satisfies that for any $f \in L_1(\mathbb{R}^{\ltimes})$ with compact support and $x \notin \operatorname{supp} f$

$$|T_b f(x)| \leqslant c_0 \int_{\mathbb{R}^{\ltimes}} |b(x) - b(y)|\,|x-y|^{-n}|f(y)|\,\mathrm{d}y, \tag{1.3}$$

where c_0 is independent of f and x.

We point out that the condition (1.2) was first introduced by Soria and Weiss in [37]. The condition (1.2) is satisfied by many interesting operators in harmonic analysis, such as the Calderón-Zygmund operators, Carleson type maximal operators, Hardy-Littlewood maximal operators, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci-Stein's oscillatory singular integrals, and the Bochner-Riesz means (see [37], [36], [26] for details).

Definition 1.1. $\mathrm{BMO}(\mathbb{R}^{\ltimes})$ is the Banach space modulo constants with the norm $\|\cdot\|_*$ defined by

$$\|b\|_* = \sup_{x \in \mathbb{R}^{\ltimes}, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}|\,\mathrm{d}y < \infty,$$

where $b \in L_1^{\mathrm{loc}}(\mathbb{R}^{\ltimes})$ and

$$b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y)\,\mathrm{d}y.$$

Let K be a Calderón-Zygmund singular integral operator and $b \in \mathrm{BMO}(\mathbb{R}^{\ltimes})$. A well known result of Coifman, Rochberg and Weiss [7] states that if $b \in \mathrm{BMO}(\mathbb{R}^{\ltimes})$ and K is a Calderón-Zygmund operator, then the commutator operator $[b,K]f = K(bf) - bKf$ is bounded on $L_p(\mathbb{R}^{\ltimes})$ for $1 < p < \infty$. The commutators of a Calderón-Zygmund operator play an important role in studying the regularity of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order (see [4], [5], [8], [33]).

We define the weighted Morrey and generalized weighted Morrey spaces as follows.

Definition 1.2. Let $1 \leqslant p < \infty$, $0 < \kappa < 1$ and let w be a weight function. We denote by $L_{p,\kappa}(w) \equiv L_{p,\kappa}(\mathbb{R}^{\ltimes}, w)$ the weighted Morrey space of all classes of locally integrable functions f with the norm

$$\|f\|_{L_{p,\kappa}(w)} = \sup_{x \in \mathbb{R}^{\ltimes},\, r>0} w(B(x,r))^{-\kappa/p} \|f\|_{L_{p,w}(B(x,r))} < \infty.$$

Furthermore, by $WL_{p,\kappa}(w) \equiv WL_{p,\kappa}(\mathbb{R}^{\ltimes}, w)$ we denote the weak weighted Morrey space of all classes of locally integrable functions f with the norm

$$\|f\|_{WL_{p,\kappa}(w)} = \sup_{x \in \mathbb{R}^{\ltimes},\, r>0} w(B(x,r))^{-\kappa/p} \|f\|_{WL_{p,w}(B(x,r))} < \infty.$$

Definition 1.3. Let $1 \leqslant p < \infty$, let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^{\ltimes} \times (0,\infty)$ and w non-negative measurable function on $\mathbb{R}^{\ltimes}$. We denote by $M_{p,\varphi}(w) \equiv M_{p,\varphi}(\mathbb{R}^{\ltimes}, w)$ the generalized weighted Morrey space, the space of all classes of functions $f \in L_{p,w}^{\mathrm{loc}}(\mathbb{R}^{\ltimes})$ with finite norm

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x \in \mathbb{R}^{\ltimes},\, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-1/p} \|f\|_{L_{p,w}(B(x,r))}.$$

Furthermore, by $WM_{p,\varphi}(w) \equiv WM_{p,\varphi}(\mathbb{R}^{\ltimes}, w)$ we denote the weak generalized weighted Morrey space of all classes of functions $f \in WL_{p,w}^{\mathrm{loc}}(\mathbb{R}^{\ltimes})$ for which

$$\|f\|_{WM_{p,\varphi}(w)} = \sup_{x \in \mathbb{R}^{\ltimes},\, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-1/p} \|f\|_{WL_{p,w}(B(x,r))} < \infty.$$

In [12], [13], [14], [16], [20], [28] and [31], sufficient conditions on φ_1 and φ_2 for the boundedness of the maximal operator M and a Calderón-Zygmund operator K from the generalized Morrey spaces M_{p,φ_1} to M_{p,φ_2} for $1 < p < \infty$ and from M_{1,φ_1}

to WM_{1,φ_2} were obtained (see also [34], [2], [1]). In [9], the following condition was imposed on $\varphi(x,r)$:

$$c^{-1}\varphi(x,r) \leqslant \varphi(x,t) \leqslant c\varphi(x,r) \tag{1.4}$$

whenever $r \leqslant t \leqslant 2r$, where $c(\geqslant 1)$ does not depend on t, r and $x \in \mathbb{R}^{\ltimes}$, jointly with the condition

$$\int_r^\infty \varphi(x,t)^p \frac{\mathrm{d}t}{t} \leqslant C\varphi(x,r)^p, \tag{1.5}$$

for the sublinear operator T, satisfying condition (1.2), where $C(>0)$ does not depend on r and $x \in \mathbb{R}^{\ltimes}$.

The following statement was proved in [18].

Theorem 1.1. *Let $1 \leqslant p < \infty$, $w \in A_p$ and let (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w(B(x,s))^{1/p}}{w(B(x,t))^{1/p}} \frac{\mathrm{d}t}{t} \leqslant C\varphi_2(x,r), \tag{1.6}$$

where C does not depend on x and r. Let T be a sublinear operator satisfying the condition (1.2) *bounded on $L_p(w)$ for $p>1$, and bounded from $L_1(w)$ to $WL_1(w)$. Then the operator T is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p>1$ and from $M_{1,\varphi_1}(w)$ to $WM_{1,\varphi_2}(w)$.*

Remark 1.1. Note that Theorem 1.1 was proved in the case $w \equiv 1$ in [15] and in the case $w \equiv 1$ and $\varphi(x,r) = \varphi_1(x,r) = \varphi_2(x,r)$ satisfying conditions (1.4) and (1.5) in [9].

In this paper, we prove the boundedness of the sublinear commutator operators T_b satisfying condition (1.3) from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$ for $1 < p < \infty$ and $b \in \mathrm{BMO}(\mathbb{R}^{\ltimes})$. We apply this result to several particular operators such as the pseudo-differential operators, Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

By $A \lesssim B$ we mean that $A \leqslant CB$ with a positive constant C independent of the appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Main results

In the following, main results are given. First, we present some estimates which are the main tools for proving our theorems, for the boundedness of the operator T_b on the generalized weighted Morrey spaces.

Theorem 2.1. *Let $1 < p < \infty$, $w \in A_p$, $b \in \mathrm{BMO}(\mathbb{R}^\varkappa)$, and let T_b be a sublinear operator satisfying the condition* (1.3). *Let also T_b be bounded on $L_p(w)$. Then*

$$\|T_b f\|_{L_{p,w}(B)} \leqslant C w(B)^{1/p} \int_{2r}^{\infty} \ln\Big(\mathrm{e} + \frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-1/p} \frac{\mathrm{d}t}{t}$$

for all $f \in L^{\mathrm{loc}}_{p,w}(\mathbb{R}^\varkappa)$, where C does not depend on f, $x_0 \in \mathbb{R}^\varkappa$ and $r > 0$.

Now we give a theorem about the boundedness of the operator T_b on the generalized weighted Morrey spaces.

Theorem 2.2. *Let $1 < p < \infty$, $w \in A_p$, $b \in \mathrm{BMO}(\mathbb{R}^\varkappa)$ and let (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \ln\Big(\mathrm{e} + \frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w(B(x,s))^{1/p}}{w(B(x,t))^{1/p}} \frac{\mathrm{d}t}{t} \leqslant C\varphi_2(x,r), \tag{2.1}$$

where C does not depend on x and r. Let T_b be a sublinear operator satisfying the condition (1.3) *and bounded on $L_p(w)$. Then the operator T_b is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$. Moreover,*

$$\|T_b f\|_{M_{p,\varphi_2}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)}.$$

Note that for the sublinear commutator of the maximal operator M_b and for the linear commutator of the Calderón-Zygmund operator $[b, K]$, from Theorem 2.2 we get a new result. When $\varphi_1(x,r) = \varphi_2(x,r) \equiv w(B(x,r))^{(\kappa-1)/p}$, from Theorem 2.2 we also get the following new result.

Corollary 2.1. *Let $1 < p < \infty$, $0 < \kappa < 1$, $w \in A_p$, $b \in \mathrm{BMO}(\mathbb{R}^\varkappa)$ and let T_b be a sublinear operator satisfying the condition* (1.3). *Let also T_b be bounded on $L_p(w)$. Then the operator T_b is bounded on $L_{p,\kappa}(w)$.*

P r o o f. Let $1 < p < \infty$, $w \in A_p$, $0 < \kappa < 1$ and $b \in \mathrm{BMO}(\mathbb{R}^\varkappa)$. Then the pair $(w(B(x,r))^{(\kappa-1)/p}, w(B(x,r))^{(\kappa-1)/p})$ satisfies the condition (2.1). Indeed,

$$\begin{aligned}\int_r^{\infty} \ln\Big(\mathrm{e} + \frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} w(B(x,s))^{\kappa/p}}{w(B(x,t))^{1/p}} \frac{\mathrm{d}t}{t} &= \int_r^{\infty} \ln\Big(\mathrm{e} + \frac{t}{r}\Big) w(B(x,t))^{(\kappa-1)/p} \frac{\mathrm{d}t}{t} \\ &\leqslant C w(B(x,r))^{(\kappa-1)/p},\end{aligned}$$

where the last inequality follows from Lemma 13 in [3]. □

Note that from Corollary 2.1, for the operator $[b, K]$ we get results which are proved in [19].

3. Some lemmas

Lemma 3.1 ([11]).

(1) *If* $w \in A_p$ *for some* $1 \leqslant p < \infty$, *then* $w \in \Delta_2$. *Moreover, for all* $\lambda > 1$ *we have*

$$w(\lambda B) \leqslant \lambda^{np}[w]_{A_p} w(B).$$

(2) *If* $w \in A_\infty$, *then* $w \in \Delta_2$. *Moreover, for all* $\lambda > 1$ *we have*

$$w(\lambda B) \leqslant 2^{\lambda^n}[w]_{A_\infty}^{\lambda^n} w(B).$$

(3) *If* $w \in A_p$ *for some* $1 \leqslant p \leqslant \infty$, *then there exist* $C > 0$ *and* $\delta > 0$ *such that for any ball* B *and a measurable set* $S \subset B$,

$$\frac{w(S)}{w(B)} \leqslant C\left(\frac{|S|}{|B|}\right)^{\delta}.$$

We need the following statement on the boundedness of the Hardy type operator:

$$(H_1 g)(t) := \frac{1}{t}\int_0^t \ln\Bigl(\mathrm{e} + \frac{t}{r}\Bigr) g(r)\,\mathrm{d}\mu(r), \quad 0 < t < \infty,$$

where μ is a non-negative Borel measure on $(0,\infty)$.

Theorem 3.1. *The inequality*

$$\operatorname*{ess\,sup}_{t>0} w(t) H_1 g(t) \leqslant c \operatorname*{ess\,sup}_{t>0} v(t) g(t)$$

holds for all non-negative and non-increasing g *on* $(0,\infty)$ *if and only if*

$$A_1 := \sup_{t>0} \frac{w(t)}{t}\int_0^t \ln\Bigl(\mathrm{e} + \frac{t}{r}\Bigr)\frac{\mathrm{d}\mu(r)}{\operatorname*{ess\,sup}_{0<s<r} v(s)} < \infty,$$

and $c \approx A_1$.

Note that Theorem 3.1 is proved analogously to Theorem 4.3 in [15].

Lemma 3.2 ([30], Theorem 5, page 236). *Let $w \in A_\infty$. Then the norm of* $\mathrm{BMO}(\mathbb{R}^{\ltimes})$ *is equivalent to the norm of* $\mathrm{BMO}(w)$, *where*

$$\mathrm{BMO}(w) = \left\{b\colon\ \|b\|_{*,w} = \sup_{x\in\mathbb{R}^{\ltimes},\, r>0} \frac{1}{w(B(x,r))}\int_{B(x,r)} |b(y) - b_{B(x,r),w}| w(y)\,\mathrm{d}y < \infty\right\}$$

and

$$b_{B(x,r),w} = \frac{1}{w(B(x,r))}\int_{B(x,r)} b(y)w(y)\,\mathrm{d}y.$$

Remark 3.1.

(1) The John-Nirenberg inequality: there are constants $C_1, C_2 > 0$ such that for all $b \in \mathrm{BMO}(\mathbb{R}^{\ltimes})$ and $\beta > 0$

$$|\{x \in B\colon\ |b(x) - b_B| > \beta\}| \leqslant C_1|B|\mathrm{e}^{-C_2\beta/\|b\|_*}, \quad \forall B \subset \mathbb{R}^{\ltimes}.$$

(2) For $1 < p < \infty$ the John-Nirenberg inequality implies that

$$\|b\|_* \approx \sup_B \left(\frac{1}{|B|}\int_B |b(y) - b_B|^p\,\mathrm{d}y\right)^{1/p} \tag{3.1}$$

and for $1 \leqslant p < \infty$ and $w \in A_\infty$

$$\|b\|_* \approx \sup_B \left(\frac{1}{w(B)}\int_B |b(y) - b_B|^p w(y)\,\mathrm{d}y\right)^{1/p}. \tag{3.2}$$

Indeed, from the John-Nirenberg inequality and using Lemma 3.1 (3), we get

$$w(\{x \in B\colon\ |b(x) - b_B| > \beta\}) \leqslant Cw(B)\mathrm{e}^{-C_2\beta\delta/\|b\|_*}$$

for some $\delta > 0$. Hence, this inequality implies that

$$\begin{aligned}\int_B |b(y) - b_B|^p w(y)\,\mathrm{d}y &= p\int_0^\infty \beta^{p-1} w(\{x \in B\colon\ |b(x) - b_B| > \beta\})\,\mathrm{d}\beta \\ &\leqslant Cw(B)\int_0^\infty \beta^{p-1}\mathrm{e}^{-C_2\beta\delta/\|b\|_*}\,\mathrm{d}\beta \\ &= Cw(B)\|b\|_*^p.\end{aligned}$$

To prove the required equivalence we also need to have the right hand inequality, which is easily obtained using the Hölder inequality, and then we get (3.2). Note that (3.1) follows from (3.2) in the case $w \equiv 1$.

The following lemma was proved in [21].

Lemma 3.3. *Let b be a function in* $\mathrm{BMO}(\mathbb{R}^{\ltimes})$. *Let also* $1 \leqslant p < \infty$, $x \in \mathbb{R}^{\ltimes}$, *and* $r_1, r_2 > 0$. *Then*

$$\left(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)} |b(y) - b_{B(x,r_2)}|^p \,\mathrm{d}y\right)^{1/p} \leqslant C\Big(1 + \Big|\ln \frac{r_1}{r_2}\Big|\Big)\|b\|_*,$$

where $C > 0$ *is independent of* f, x, r_1 *and* r_2.

The following lemma is valid.

Lemma 3.4.

(i) *Let* $w \in A_\infty$ *and let* b *be a function in* $\mathrm{BMO}(\mathbb{R}^{\ltimes})$. *Let also* $1 \leqslant p < \infty$, $x \in \mathbb{R}^{\ltimes}$, *and* $r_1, r_2 > 0$. *Then*

$$\left(\frac{1}{w(B(x,r_1))}\int_{B(x,r_1)} |b(y) - b_{B(x,r_2),w}|^p w(y) \,\mathrm{d}y\right)^{1/p} \leqslant C\Big(1 + \Big|\ln \frac{r_1}{r_2}\Big|\Big)\|b\|_*,$$

where $C > 0$ *is independent of* f, x, r_1 *and* r_2.

(ii) *Let* $w \in A_p$ *and let* b *be a function in* $\mathrm{BMO}(\mathbb{R}^{\ltimes})$. *Let also* $1 < p < \infty$, $x \in \mathbb{R}^{\ltimes}$, *and* $r_1, r_2 > 0$. *Then*

$$\begin{aligned}\left(\frac{1}{w^{1-p'}(B(x,r_1))}\int_{B(x,r_1)} |b(y) - b_{B(x,r_2),w}|^{p'} w(y)^{1-p'} \,\mathrm{d}y\right)^{1/p'}& \\ \leqslant C\Big(1 + \Big|\ln \frac{r_1}{r_2}\Big|\Big)\|b\|_*,&\end{aligned}$$

where $C > 0$ *is independent of* f, x, r_1 *and* r_2.

Proof. We only consider the case $0 < r_1 \leqslant r_2$. Actually, the similar procedure works for the other case $0 < r_2 < r_1$.

For $0 < r_1 \leqslant r_2$, there are $k_1, k_2 \in \mathbb{Z}$ such that $2^{k_1-1} < r_1 \leqslant 2^{k_1}$ and $2^{k_2-1} < r_2 \leqslant 2^{k_2}$. Then $k_1 \leqslant k_2$ and $(k_2 - k_1 - 1)\ln 2 < \ln(r_2/r_1) < (k_2 - k_1 + 1)\ln 2$.

(i) From (3.2), Lemmas 3.1 (2) and 3.2 we have

$$
\begin{aligned}
&\left(\frac{1}{w(B(x,r_1))}\int_{B(x,r_1)}|b(y)-b_{B(x,r_2),w}|^p w(y)\,\mathrm{d}y\right)^{1/p}\\
&\qquad\leqslant \left(\frac{1}{w(B(x,r_1))}\int_{B(x,r_1)}|b(y)-b_{B(x,2^{k_1})}|^p w(y)\,\mathrm{d}y\right)^{1/p}\\
&\qquad\quad + |b_{B(x,2^{k_1}),w}-b_{B(x,r_2),w}| + |b_{B(x,2^{k_1})}-b_{B(x,2^{k_1}),w}|\\
&\qquad\leqslant \left(\frac{1}{w(B(x,r_1))}\int_{B(x,r_1)}|b(y)-b_{B(x,2^{k_1})}|^p w(y)\,\mathrm{d}y\right)^{1/p}\\
&\qquad\quad + |b_{B(x,r_2),w}-b_{B(x,2^{k_2}),w}| + \sum_{j=k_1}^{k_2-1}|b_{B(x,2^{j+1}),w}-b_{B(x,2^j),w}|\\
&\qquad\quad + |b_{B(x,2^{k_1})}-b_{B(x,2^{k_1}),w}|\\
&\qquad\leqslant \left(\frac{1}{w(B(x,r_1))}\int_{B(x,r_1)}|b(y)-b_{B(x,2^{k_1})}|^p w(y)\,\mathrm{d}y\right)^{1/p}\\
&\qquad\quad + \frac{1}{w(B(x,r_2))}\int_{B(x,r_2)}|b(y)-b_{B(x,2^{k_2}),w}|w(y)\,\mathrm{d}y\\
&\qquad\quad + \sum_{j=k_1}^{k_2-1}\frac{1}{w(B(x,2^j))}\int_{B(x,2^j)}|b(y)-b_{B(x,2^{j+1}),w}|w(y)\,\mathrm{d}y\\
&\qquad\quad + \frac{1}{w(B(x,2^{k_1}))}\int_{B(x,2^{k_1})}|b(y)-b_{B(x,2^{k_1})}|w(y)\,\mathrm{d}y\\
&\qquad\lesssim \left(\frac{1}{w(B(x,2^{k_1}))}\int_{B(x,2^{k_1})}|b(y)-b_{B(x,2^{k_1})}|^p w(y)\,\mathrm{d}y\right)^{1/p}\\
&\qquad\quad + \frac{1}{w(B(x,2^{k_2}))}\int_{B(x,2^{k_2})}|b(y)-b_{B(x,2^{k_2}),w}|w(y)\,\mathrm{d}y\\
&\qquad\quad + \sum_{j=k_1}^{k_2-1}\frac{1}{w(B(x,2^{j+1}))}\int_{B(x,2^{j+1})}|b(y)-b_{B(x,2^{j+1}),w}|w(y)\,\mathrm{d}y\\
&\qquad\quad + \frac{1}{w(B(x,2^{k_1}))}\int_{B(x,2^{k_1})}|b(y)-b_{B(x,2^{k_1})}|w(y)\,\mathrm{d}y\\
&\qquad\lesssim (1+k_2-k_1)\|b\|_* \lesssim \left(1+\ln\frac{r_2}{r_1}\right)\|b\|_*.
\end{aligned}
$$

This completes the proof of the first part of the lemma.
(ii) We have

$$
\left(\frac{1}{w^{1-p'}(B(x,r_1))}\int_{B(x,r_1)}|b(y)-b_{B(x,r_2),w}|^{p'} w(y)^{1-p'}\,\mathrm{d}y\right)^{1/p'}
$$

$$\leqslant \left(\frac{1}{w^{1-p'}(B(x,r_1))}\int_{B(x,r_1)}\{|b(y)-b_{B(x,2^{k_1}),w^{1-p'}}|\right.$$
$$\left.+|b_{B(x,2^{k_1}),w^{1-p'}}-b_{B(x,r_2),w}|\}^{p'}w(y)^{1-p'}\,\mathrm{d}y\right)^{1/p'}$$
$$\leqslant \left(\frac{1}{w^{1-p'}(B(x,r_1))}\int_{B(x,r_1)}|b(y)-b_{B(x,2^{k_1}),w^{1-p'}}|^{p'}w(y)^{1-p'}\,\mathrm{d}y\right)^{1/p'}$$
$$+|b_{B(x,2^{k_1}),w^{1-p'}}-b_{B(x,r_2),w}|=J_1+J_2.$$

It is known that, if $w\in A_p$ for $1\leqslant p<\infty$, then $w^{1-p'}\in A_{p'}\subset A_\infty$ and from Lemma 3.1 (1) and Lemma 3.4 we get

$$J_1\lesssim\left(\frac{1}{w^{1-p'}(B(x,2^{k_1}))}\int_{B(x,2^{k_1})}|b(y)-b_{B(x,2^{k_1}),w^{1-p'}}|^{p'}w(y)^{1-p'}\,\mathrm{d}y\right)^{1/p'}\lesssim\|b\|_*.$$

Now we estimate J_2:

$$\begin{aligned}J_2&=|b_{B(x,2^{k_1}),w^{1-p'}}-b_{B(x,r_2),w}|\\&\leqslant|b_{B(x,2^{k_1}),w^{1-p'}}-b_{B(x,2^{k_1})}|+|b_{B(x,2^{k_1})}-b_{B(x,r_2)}|+|b_{B(x,r_2)}-b_{B(x,r_2),w}|\\&=J_{21}+J_{22}+J_{23}.\end{aligned}$$

From (3.2) we have

$$\begin{aligned}J_{21}&=|b_{B(x,2^{k_1}),w^{1-p'}}-b_{B(x,2^{k_1})}|\\&\leqslant\frac{1}{w^{1-p'}(B(x,2^{k_1}))}\int_{B(x,2^{k_1})}|b(y)-b_{B(x,2^{k_1})}|w(y)^{1-p'}\,\mathrm{d}y\lesssim\|b\|_*.\end{aligned}$$

From Lemma 3.3 we get

$$\begin{aligned}J_{22}=|b_{B(x,2^{k_1})}-b_{B(x,r_2)}|&\leqslant\frac{1}{|B(x,2^{k_1})|}\int_{B(x,2^{k_1})}|b(y)-b_{B(x,r_2)}|\,\mathrm{d}y\\&\lesssim\left(1+\left|\ln\frac{2^{k_1}}{r_2}\right|\right)\|b\|_*\lesssim\left(1+\left|\ln\frac{r_1}{r_2}\right|\right)\|b\|_*.\end{aligned}$$

From (3.2) we have

$$\begin{aligned}J_{23}&=|b_{B(x,r_2)}-b_{B(x,r_2),w}|\\&\leqslant\frac{1}{w(B(x,r_2))}\int_{B(x,r_2)}|b(y)-b_{B(x,r_2)}|w(y)\,\mathrm{d}y\lesssim\|b\|_*.\end{aligned}$$

Then

$$J_1+J_2\lesssim\left(1+\ln\frac{r_2}{r_1}\right)\|b\|_*.$$

This completes the proof of the second part of the lemma. □

4. Proof of the theorems

Proof of Theorem 2.1. Let $p \in (1,\infty)$. For arbitrary $x_0 \in \mathbb{R}^n$ and $r > 0$, set $B = B(x_0,r)$. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement(2B)}$. Hence

$$\|T_b f\|_{L_{p,w}(B)} \leqslant \|T_b f_1\|_{L_{p,w}(B)} + \|T_b f_2\|_{L_{p,w}(B)}.$$

From the boundedness of T_b in $L_p(w)$ it follows that:

$$\|T_b f_1\|_{L_{p,w}(B)} \leqslant \|T_b f_1\|_{L_{p,w}} \lesssim \|f_1\|_{L_{p,w}} = \|f\|_{L_{p,w}(2B)}.$$

For $x \in B$ we have

$$|T_b f_2(x)| \lesssim \int_{\mathbb{R}^n} \frac{|b(y)-b(x)|}{|x-y|^n} |f_2(y)| \,\mathrm{d}y \approx \int_{\complement(2B)} \frac{|b(y)-b(x)|}{|x_0-y|^n} |f(y)| \,\mathrm{d}y.$$

Then

$$\begin{aligned}
\|T_b f_2\|_{L_{p,w}(B)} &\lesssim \biggl(\int_B \biggl(\int_{\complement(2B)} \frac{|b(y)-b(x)|}{|x_0-y|^n} |f(y)| \,\mathrm{d}y\biggr)^p w(x) \,\mathrm{d}x\biggr)^{1/p} \\
&\lesssim \biggl(\int_B \biggl(\int_{\complement(2B)} \frac{|b(y)-b_{B,w}|}{|x_0-y|^n} |f(y)| \,\mathrm{d}y\biggr)^p w(x) \,\mathrm{d}x\biggr)^{1/p} \\
&\quad + \biggl(\int_B \biggl(\int_{\complement(2B)} \frac{|b(x)-b_{B,w}|}{|x_0-y|^n} |f(y)| \,\mathrm{d}y\biggr)^p w(x) \,\mathrm{d}x\biggr)^{1/p} = I_1 + I_2.
\end{aligned}$$

Let us estimate I_1:

$$\begin{aligned}
I_1 &= w(B)^{1/p} \int_{\complement(2B)} \frac{|b(y)-b_{B,w}|}{|x_0-y|^n} |f(y)| \,\mathrm{d}y \\
&\approx w(B)^{1/p} \int_{\complement(2B)} |b(y)-b_{B,w}|\,|f(y)| \int_{|x_0-y|}^{\infty} \frac{\mathrm{d}t}{t^{n+1}} \,\mathrm{d}y \\
&\approx w(B)^{1/p} \int_{2r}^{\infty}\int_{2r\leqslant|x_0-y|\leqslant t} |b(y)-b_{B,w}|\,|f(y)| \,\mathrm{d}y \frac{\mathrm{d}t}{t^{n+1}} \\
&\lesssim w(B)^{1/p} \int_{2r}^{\infty}\int_{B(x_0,t)} |b(y)-b_{B,w}|\,|f(y)| \,\mathrm{d}y \frac{\mathrm{d}t}{t^{n+1}}.
\end{aligned}$$

Applying Hölder's inequality and by Lemma 3.4, we get

$$\begin{aligned}
I_1 &\lesssim w(B)^{1/p} \int_{2r}^{\infty} \biggl(\int_{B(x_0,t)} |b(y)-b_{B(x_0,r),w}|^{p'} w(y)^{1-p'} \,\mathrm{d}y\biggr)^{1/p'} \|f\|_{L_{p,w}(B(x_0,t))} \frac{\mathrm{d}t}{t^{n+1}} \\
&\lesssim [w]_{A_p}^{1/p} \|b\|_* w(B)^{1/p} \int_{2r}^{\infty} \Bigl(1+\ln\frac{t}{r}\Bigr) \|w^{-1/p}\|_{L_{p'}(B(x_0,t))} \|f\|_{L_{p,w}(B(x_0,t))} \frac{\mathrm{d}t}{t^{n+1}} \\
&\lesssim [w]_{A_p}^{1/p} \|b\|_* w(B)^{1/p} \int_{2r}^{\infty} \ln\Bigl(\mathrm{e}+\frac{t}{r}\Bigr) \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-1/p} \frac{\mathrm{d}t}{t}.
\end{aligned}$$

In order to estimate I_2 note that

$$I_2 = \left(\int_B |b(x) - b_{B,w}|^p w(x)\,\mathrm{d}x \right)^{1/p} \int_{\complement(2B)} \frac{|f(y)|}{|x_0 - y|^n}\,\mathrm{d}y.$$

By Lemma 3.4, we get

$$I_2 \lesssim w(B)^{1/p} \int_{\complement(2B)} \frac{|f(y)|}{|x_0 - y|^n}\,\mathrm{d}y.$$

Applying Hölder's inequality, we get

$$\begin{aligned} \int_{\complement(2B)} \frac{|f(y)|}{|x_0 - y|^n}\,\mathrm{d}y &\lesssim \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w^{-1/p}\|_{L_{p'}(B(x_0,t))} \frac{\mathrm{d}t}{t^{n+1}} \\ &\leqslant [w]_{A_p}^{1/p} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-1/p} \frac{\mathrm{d}t}{t}. \end{aligned} \tag{4.1}$$

Thus, by (4.1)

$$I_2 \lesssim w(B)^{1/p} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-1/p} \frac{\mathrm{d}t}{t}.$$

Summing up I_1 and I_2, for all $p \in [1,\infty)$ we get

$$\|T_b f_2\|_{L_{p,w}(B)} \lesssim w(B)^{1/p} \int_{2r}^{\infty} \ln\left(\mathrm{e} + \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-1/p} \frac{\mathrm{d}t}{t}. \tag{4.2}$$

On the other hand,

$$\begin{aligned} \|f\|_{L_{p,w}(2B)} &\approx |B| \|f\|_{L_{p,w}(2B)} \int_{2r}^{\infty} \frac{\mathrm{d}t}{t^{n+1}} \lesssim |B| \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \frac{\mathrm{d}t}{t^{n+1}} \\ &\leqslant w(B)^{1/p} \|w^{-1/p}\|_{L_{p'}(B)} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \frac{\mathrm{d}t}{t^{n+1}} \\ &\leqslant w(B)^{1/p} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w^{-1/p}\|_{L_{p'}(B(x_0,t))} \frac{\mathrm{d}t}{t^{n+1}} \\ &\leqslant [w]_{A_p}^{1/p} w(B)^{1/p} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-1/p} \frac{\mathrm{d}t}{t}. \end{aligned} \tag{4.3}$$

Finally,

$$\begin{aligned} \|T_b f\|_{L_{p,w}(B)} &\lesssim \|f\|_{L_{p,w}(2B)} \\ &\quad + w(B)^{1/p} \int_{2r}^{\infty} \ln\left(\mathrm{e} + \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-1/p} \frac{\mathrm{d}t}{t}, \end{aligned}$$

and the statement of Theorem 2.1 follows by (4.3). $\square$

P r o o f of Theorem 2.2. By Theorem 2.1 and Theorem 3.1 we have for $p > 1$

$$
\begin{aligned}
&\|T_b f\|_{M_{p,\varphi_2}(w)} \\
&\quad \lesssim \sup_{x\in\mathbb{R}^\varkappa,\, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \ln\Bigl(\mathrm{e}+\frac{t}{r}\Bigr) \|f\|_{L_{p,w}(B(x,t))} w(B(x,t))^{-1/p} \frac{\mathrm{d}t}{t} \\
&\quad = \sup_{x\in\mathbb{R}^\varkappa,\, r>0} \varphi_2(x,r)^{-1} \int_0^{r^{-1}} \ln\Bigl(\mathrm{e}+\frac{1}{tr}\Bigr) \|f\|_{L_{p,w}(B(x,t^{-1}))} w(B(x,t^{-1}))^{-1/p} \frac{\mathrm{d}t}{t} \\
&\quad = \sup_{x\in\mathbb{R}^\varkappa,\, r>0} \varphi_2(x,r^{-1})^{-1} r\frac{1}{r} \int_0^{r} \ln\Bigl(\mathrm{e}+\frac{r}{t}\Bigr) \|f\|_{L_{p,w}(B(x,t^{-1}))} w(B(x,t^{-1}))^{-1/p} \frac{\mathrm{d}t}{t} \\
&\quad \lesssim \sup_{x\in\mathbb{R}^\varkappa,\, r>0} \varphi_1(x,r^{-1})^{-1} w(B(x,r^{-1}))^{-1/p} \|f\|_{L_{p,w}(B(x,r^{-1}))} \\
&\quad = \sup_{x\in\mathbb{R}^\varkappa,\, r>0} \varphi_1(x,r)^{-1} w(B(x,r))^{-1/p} \|f\|_{L_{p,w}(B(x,r))} = \|f\|_{M_{p,\varphi_1}(w)}.
\end{aligned}
$$

□

5. Some applications

In this section we will apply Theorem 2.2 to several particular operators such as the pseudo-differential operators, Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

5.1. Pseudo-differential operators. Pseudo-differential operators are generalizations of differential operators and singular integrals. Let m be a real number, $0 \leqslant \delta < 1$ and $0 \leqslant \varrho < 1$. Following [17], [41], the symbol $S^m_{\varrho,\delta}$ stands for the set of smooth functions $\sigma(x,\xi)$ defined on $\mathbb{R}^\varkappa \times \mathbb{R}^\varkappa$ such that for all multi-indices α and β the following estimate holds:

$$
|D_x^\alpha D_\xi^\beta \sigma(x,\xi)| \leqslant C_{\alpha\beta}(1+|\xi|)^{m-\varrho|\beta|+\delta|\alpha|},
$$

where $C_{\alpha\beta} > 0$ is independent of x and ξ. The symbol $S^{-\infty}_{\varrho,\delta}$ stands for the set of functions which satisfy the above estimates for each real number m.

The operator A given by

$$
Af(x) = \int_{\mathbb{R}^\varkappa} \sigma(x,\xi) \mathrm{e}^{2\pi \mathrm{i} x\cdot\xi} \hat{f}(\xi)\,\mathrm{d}\xi
$$

is called a pseudo-differential operator with $\sigma(x,\xi) \in S^m_{\varrho,\delta}$, where f is a Schwartz function and $\hat{f}$ denotes the Fourier transform of f. As usual, $L^m_{\varrho,\delta}$ will denote the class of pseudo-differential operators with symbols in $S^m_{\varrho,\delta}$.

Miller [27] showed the boundedness of $L^0_{1,0}$ pseudo-differential operators on weighted L_p $(1 < p < \infty)$ spaces whenever the weight function belongs to Muckenhoupt's class A_p. In [6] it is shown that pseudo-differential operators in $L^0_{1,0}$ are Calderón-Zygmund operators. From Theorem 2.2, we get the following corollary.

Corollary 5.1. *Let $1 < p < \infty$, $w \in A_p$. Suppose that (φ_1, φ_2) satisfies the condition* (2.1) *and $b \in \mathrm{BMO}(\mathbb{R}^n)$. If A is a pseudo-differential operator of the Hörmander class $L^0_{1,0}$, then the operator $[b, A]$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$.*

From Corollary 2.1 we get

Corollary 5.2. *Let $1 < p < \infty$, $0 < \kappa < 1$, $w \in A_p$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. If A is a pseudo-differential operator of the Hörmander class $L^0_{1,0}$, then the operator $[b, A]$ is bounded on $L_{p,\kappa}(w)$.*

5.2. Littlewood-Paley operator. The Littlewood-Paley functions play an important role in classical harmonic analysis, for example in the study of non-tangential convergence of Fatou type and boundedness of Riesz transforms and multipliers [40], [38], [42], [39]. The Littlewood-Paley operator (see [42], [22]) is defined as follows.

Definition 5.1. Suppose that $\psi \in L_1(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} \psi(x)\,\mathrm{d}x = 0. \tag{5.1}$$

Then the generalized Littlewood-Paley g function g_ψ is defined by

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{\mathrm{d}t}{t}\right)^{1/2},$$

where $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$ and $F_t(f) = \psi_t * f$.

The sublinear commutator of the operator g_ψ is defined by

$$[b, g_\psi](f)(x) = \left(\int_0^\infty |F^b_t(f)(x)|^2 \frac{\mathrm{d}t}{t}\right)^{1/2},$$

where

$$F^b_t(f)(x) = \int_{\mathbb{R}^n} [b(x) - b(y)]\psi_t(x-y)f(y)\,\mathrm{d}y.$$

The following theorem is valid (see [25], Theorem 5.2.2).

Theorem 5.1. *Suppose that $\psi \in L_1(\mathbb{R}^n)$ satisfies* (5.1) *and the following conditions:*

$$|\psi(x)| \leqslant \frac{C}{(1+|x|)^{n+1}}, \tag{5.2}$$

$$|\nabla\psi(x)| \leqslant \frac{C}{(1+|x|)^{n+2}} \tag{5.3}$$

where $C > 0$ is independent of x. Then g_ψ is bounded on $L_p(w)$ for $1 < p < \infty$ and $w \in A_p$.

Let H be the space $H = \bigl\{h\colon \|h\| = \bigl(\int_0^\infty |h(t)|^2\,\mathrm{d}t/t\bigr)^{1/2} < \infty\bigr\}$, then, for each fixed $x \in \mathbb{R}^n$, $F_t(f)(x)$ may be viewed as a mapping from $[0,\infty)$ to H, and it is clear that $g_\psi(f)(x) = \|F_t(f)(x)\|$.

In fact, by the Minkowski inequality and the conditions on ψ we get

$$\begin{aligned} g_\psi(f)(x) &\leqslant \int_{\mathbb{R}^n} |f(y)| \biggl(\int_0^\infty |\psi_t(x-y)|^2 \frac{\mathrm{d}t}{t}\biggr)^{1/2} \mathrm{d}y \\ &\lesssim \int_{\mathbb{R}^n} |f(y)| \biggl(\int_0^\infty \frac{t^{-2n}}{(1+|x-y|/t)^{2(n+1)}} \frac{\mathrm{d}t}{t}\biggr)^{1/2} \mathrm{d}y \\ &= \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}\,\mathrm{d}y. \end{aligned}$$

Thus, we get

Corollary 5.3. *Let $1 < p < \infty$, $w \in A_p$. Suppose that (φ_1, φ_2) satisfies the condition* (2.1), *$b \in \mathrm{BMO}(\mathbb{R}^n)$ and $\psi \in L_1(\mathbb{R}^n)$ satisfies* (5.1)–(5.3). *Then the commutator of the Littlewood-Paley operator $[b, g_\psi]$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$.*

From Corollary 2.1 we get

Corollary 5.4. *Let $1 < p < \infty$, $0 < \kappa < 1$, $w \in A_p$, $b \in \mathrm{BMO}(\mathbb{R}^n)$. Suppose that $\psi \in L_1(\mathbb{R}^n)$ satisfies* (5.1)–(5.3). *Then the operator $[b, g_\psi]$ is bounded on $L_{p,\kappa}(w)$.*

5.3. Marcinkiewicz operator. Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue measure $d\sigma$. Suppose that Ω satisfies the following conditions:

(a) Ω is a homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, that is,

$$\Omega(tx) = \Omega(x) \quad \text{for any} \quad t > 0,\ x \in \mathbb{R}^n \setminus \{0\}.$$

(b) Ω has mean zero on S^{n-1}, that is,

$$\int_{S^{n-1}} \Omega(x')\,\mathrm{d}\sigma(x') = 0.$$

(c) $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$, $0 < \gamma \leqslant 1$, that is, there exists a constant $C > 0$ such that

$$|\Omega(x') - \Omega(y')| \leqslant C|x' - y'|^\gamma \quad \text{for any} \quad x', y' \in S^{n-1}.$$

In 1958, Stein [39] defined the Marcinkiewicz integral of higher dimension μ_Ω as

$$\mu_\Omega(f)(x) = \biggl(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{\mathrm{d}t}{t^3}\biggr)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\,\mathrm{d}y.$$

The continuity of the Marcinkiewicz operator μ_Ω has been extensively studied in [25], [40], [38], [43].

The sublinear commutator of the operator μ_Ω is defined by

$$[a, \mu_\Omega](f)(x) = \biggl(\int_0^\infty |F_{\Omega,t,a}(f)(x)|^2 \frac{\mathrm{d}t}{t^3}\biggr)^{1/2},$$

where

$$F_{\Omega,t,a}(f)(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [a(x) - a(y)] f(y)\,\mathrm{d}y.$$

Let H be the space

$$H = \biggl\{h\colon\ \|h\| = \biggl(\int_0^\infty |h(t)|^2 \frac{\mathrm{d}t}{t^3}\biggr)^{1/2} < \infty\biggr\}.$$

Then it is clear that $\mu_\Omega(f)(x) = \|F_{\Omega,t}(f)(x)\|$.

By the Minkowski inequality and the conditions on Ω, we get

$$\mu_\Omega(f)(x) \leqslant \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| \biggl(\int_{|x-y|}^\infty \frac{\mathrm{d}t}{t^3}\biggr)^{1/2} \mathrm{d}y \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}\,\mathrm{d}y.$$

Thus, μ_Ω satisfies the condition (1.2). It is known that μ_Ω is bounded on $L_p(w)$ for $1 < p < \infty$ and $w \in A_p$ (see [43]). From Theorem 2.2 we get

Corollary 5.5. *Let $1 < p < \infty$, $w \in A_p$. Suppose that (φ_1, φ_2) satisfies the condition* (2.1), $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and Ω satisfies the conditions* (a)–(c). *Then $[b, \mu_\Omega]$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$.*

From Corollary 2.1 we get

Corollary 5.6. *Let $1 < p < \infty$, $0 < \kappa < 1$, $w \in A_p$, $b \in \mathrm{BMO}(\mathbb{R}^n)$. Suppose that Ω satisfies conditions* (a)–(c). *Then the operator $[b, \mu_\Omega]$ is bounded on $L_{p,\kappa}(w)$.*

5.4. Bochner-Riesz operator. Let $\delta > (n-1)/2$, $B_t^\delta(\hat{f})(\xi) = (1 - t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(x) = t^{-n} B^\delta(x/t)$ for $t > 0$. The maximal Bochner-Riesz operator is defined by (see [24], [23])

$$B_{\delta,*}(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)|.$$

Let H be the space $H = \{h \colon \|h\| = \sup\limits_{t>0} |h(t)| < \infty\}$, then it is clear that $B_{\delta,*}(f)(x) = \|B_t^\delta(f)(x)\|$.

By the condition on B_r^δ (see [10]), we have

$$\begin{aligned} |B_r^\delta(x-y)| &\lesssim r^{-n}(1 + |x-y|/r)^{-(\delta+(n+1)/2)} \\ &= \Big(\frac{r}{r + |x-y|}\Big)^{\delta-(n-1)/2} \frac{1}{(r + |x-y|)^n} \\ &\lesssim |x-y|^{-n}, \end{aligned}$$

and

$$B_{\delta,*}(f)(x) \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} \, \mathrm{d}y.$$

Thus, $B_{\delta,*}$ satisfies the condition (1.2). It is known that $B_{\delta,*}$ is bounded on $L_p(w)$ for $1 < p < \infty$ and $w \in A_p$, and bounded from $L_1(w)$ to $WL_1(w)$ for $w \in A_1$ (see [35], [44]). From Theorem 2.2 we get

Corollary 5.7. *Let $1 < p < \infty$, $w \in A_p$. Suppose that (φ_1, φ_2) satisfies the condition* (2.1), *$\delta > (n-1)/2$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then the operator $[b, B_{\delta,*}]$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$.*

Remark 5.1. Recall that, under the assumptions that $w = 1$ and $\varphi(x,r) = \varphi_1(x,r) = \varphi_2(x,r)$ satisfy conditions (1.4) and (1.5), Corollary 5.7 was proved in [24].

From Corollary 2.1 we get

Corollary 5.8. *Let $1 < p < \infty$, $0 < \kappa < 1$, $w \in A_p$, $b \in \mathrm{BMO}(\mathbb{R}^{\varkappa})$ and $\delta > (n-1)/2$. Then the operator $[b, B_{\delta,*}]$ is bounded on $L_{p,\kappa}(w)$.*

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