Slim Chaabane; Imed Feki

Pointwise inequalities of logarithmic type in Hardy-Hölder spaces

Czechoslovak Mathematical Journal, Vol. 64 (2014), No. 2, 351–363

Persistent URL: `http://dml.cz/dmlcz/144003`

Czechoslovak Mathematical Journal, 64 (139) (2014), 351–363

POINTWISE INEQUALITIES OF LOGARITHMIC TYPE IN HARDY-HÖLDER SPACES

Slim Chaabane, Imed Feki, Sfax

(Received January 3, 2013)

Abstract. We prove some optimal logarithmic estimates in the Hardy space $H^{\infty}(G)$ with Hölder regularity, where G is the open unit disk or an annular domain of $\mathbb{C}$. These estimates extend the results established by S. Chaabane and I. Feki in the Hardy-Sobolev space $H^{k,\infty}$ of the unit disk and those of I. Feki in the case of an annular domain. The proofs are based on a variant of Hardy-Landau-Littlewood inequality for Hölder functions. As an application of these estimates, we study the stability of both the Cauchy problem for the Laplace operator and the Robin inverse problem.

Keywords: Hardy-Sobolev space; Hardy-Landau-Littlewood inequality; Hölder regularity; Cauchy problem; inverse problem; logarithmic estimate

MSC 2010: 30H05, 30H10, 30C40

1. Introduction

Let $\mathbb{D}$ be the open unit disk of $\mathbb{C}$ with boundary $\mathbb{T}$ and let $H^{\infty}(\mathbb{D})$ be the space of bounded analytic functions on $\mathbb{D}$. For $s \in\,]0,1[$, we denote by $G_s = \mathbb{D}\backslash s\overline{\mathbb{D}}$ the annulus with inner boundary $s\mathbb{T}$ and outer boundary $\mathbb{T}$ and by $H^{\infty}(G_s)$ the Hardy space of bounded analytic functions on G_s. For more details concerning the definitions and properties of the Hardy spaces $H^{\infty}(\mathbb{D})$ and $H^{\infty}(G_s)$, we can refer the reader to [1], [2], [10], [11], [12], [26], [24], [25].

In the sequel, we denote by G the open unit disk $\mathbb{D}$ or the annulus $G_s; s \in\,]0,1[$. For $k \in \mathbb{N}$ and $\alpha \in\,]0,1[$, we designate by $H^{k,\infty}(G)$ the Hardy-Sobolev space of G:

$$H^{k,\infty}(G) = \{f \in H^{\infty}(G);\ f^{(j)} \in H^{\infty}(G),\ j = 0,\dots,k\},$$

This research has been supported by the Laboratory of Applied Mathematics and Harmonic Analysis: L. A. M. H. A. LR 11ES52

where $f^{(j)}$ denotes the j^{th} complex derivative of f, and by $\mathcal{H}^{k,\alpha}(G)$ the Hölder-Hardy space:

$$\mathcal{H}^{k,\alpha}(G) = \Big\{ g \in H^{k,\infty}(G);\ \sup_{z_1 \neq z_2 \in G} \frac{|g^{(k)}(z_1) - g^{(k)}(z_2)|}{|z_1 - z_2|^{\alpha}} < \infty \Big\}.$$

We endow $H^{k,\infty}(G)$ with the usual norm

$$\|f\|_{H^{k,\infty}} = \max_{0 \leqslant j \leqslant k} (\|f^{(j)}\|_{L^{\infty}(\partial G)}).$$

Let $\mathcal{B}^{k,\alpha}(G)$ denote the unit ball of $\mathcal{H}^{k,\alpha}(G)$:

$$\mathcal{B}^{k,\alpha}(G) = \{ g \in \mathcal{H}^{k,\alpha}(G);\ [g]_{k,\alpha} \leqslant 1 \},$$

where $[g]_{k,\alpha}$ is the k^{th} Hölder quotient defined by

$$[g]_{k,\alpha} = \sup_{z_1 \neq z_2 \in G} \frac{|g^{(k)}(z_1) - g^{(k)}(z_2)|}{|z_1 - z_2|^{\alpha}}.$$

For any connected open subset I of ∂G with length $2\pi\lambda$; $\lambda \in]0, 1[$, the L^1 norm of f on I is given by

$$\|f\|_{L^1(I)} = \frac{1}{2\pi\lambda} \int_I |f(r\mathrm{e}^{\mathrm{i}\theta})| \,\mathrm{d}\theta,$$

where $r = s$ if $I \subset s\mathbb{T}$ and $r = 1$ if $I \subset \mathbb{T}$.

Control problems in Hardy spaces have been first considered by L. Baratchart and M. Zerner [3] in the class of Hardy-Sobolev spaces $H^{1,2}$ of the unit disk $\mathbb{D}$. They proved a $((\log - \log)/\log)$-type inequality with respect to the L^2 norm. More recently, a $(1/\log^k)$-type optimal estimates with respect to the L^{∞} norm have been established by the authors in $H^{k,\infty}(\mathbb{D})$ ([5]).

For the annulus G_s, a $(1/\log)$-type estimate with respect to the L^2 norm has been proved by J. Leblond and al. in [18]. Their method is based on the Hilbertian properties of the Hardy space H^2 and provides a control of the behavior of a function on the inner boundary $s\mathbb{T}$ from its behavior on $\mathbb{T}$. In the same situation, H. Meftahi and F. Wielonsky [20] gave a similar and explicit estimate in the Hardy-Sobolev spaces $H^{k,2}(G_s)$. Recently, the second author [13] proved a $(1/\log^k)$-type estimate in $H^{k,\infty}(G_s)$. His estimates control the behavior on the whole boundary ∂G_s with respect to the L^{∞} norm starting from its behavior on any open connected subset I of ∂G_s.

In the present paper we establish some optimal logarithmic estimates in the Hölder-Hardy spaces $\mathcal{H}^{k,\alpha}(G)$, extending thus the earlier cases [3], [5], [18], [20]. Our main result is the following:

Theorem 1.1. *Let $k \in \mathbb{N}$ and let I be a subarc of ∂G of length $2\pi\lambda$; $\lambda \in]0,1[$. There exist two non-negative constants C and γ, depending only on k, α, s and λ, such that for every $f \in \mathcal{B}^{k,\alpha}(G)$ satisfying $\|f\|_{L^1(I)} < \gamma$, we have*

$$\|f\|_{L^\infty(\partial G)} \leqslant \frac{C}{|\log \|f\|_{L^1(I)}|^{\alpha+k}}. \tag{1.1}$$

The following proposition clarifies the optimality of Theorem 1.1.

Proposition 1.2. *Let $k \in \mathbb{N}$, $\alpha \in]0,1[$ and let I be the semicircle $I = \{e^{i\theta}, -\pi/2 \leqslant \theta \leqslant \pi/2\}$. Then there exists a sequence of functions $g_n \in \mathcal{B}^{k,\alpha}(G)$ such that*

$$\lim_{n\to\infty} \|g_n\|_{L^\infty(\partial G)} |\log \|g_n\|_{L^1(I)}|^{\alpha+k} \geqslant \frac{(\log 2)^{\alpha+k}}{2^{1-\alpha}}. \tag{1.2}$$

The optimality of the equation (1.1) is clear in light of Proposition 1.2: we cannot replace in (1.1) the constant C with any function ε which tends to zero at zero such that for every function $f \in \mathcal{B}^{k,\alpha}(G)$ we would have

$$\|f\|_{L^\infty(\partial G)} \leqslant \frac{\varepsilon(\|f\|_{L^1(I)})}{|\log \|f\|_{L^1(I)}|^{\alpha+k}}.$$

In Section 2, we present some notation and preliminary results that will be useful in the sequel. We also prove a variant of the Hardy-Landau-Littlewood inequality amid the class of Hölder functions on a finite interval. Section 3 is devoted to the proof of our main results and to recovering the results of [5], [13] as a consequence of Theorem 1.1. As applications, in the last part of the paper we study the stability of Cauchy's problem for the Laplace operator in the class of Hölder solutions, and the stability of the Robin inverse problem in the case of a flux $\varphi \in W_0^{1,2}(I)$.

2. Notation and preliminary results

In this section we give some preliminary results which will be useful for the proof of our main estimates. For every $t \in [-\pi, \pi]$, we introduce the radial primitive of a function $f \in H^\infty(G)$ as

$$F_t(r) = \int_{x_0}^{r} f(xe^{it})\,dx \quad \text{for all} \quad r \in]x_0, 1[, \tag{2.1}$$

where $x_0 = 0$ if $G = \mathbb{D}$ and $x_0 = s$ if $G = G_s$.

Then, referring to [3], Lemma 4.1, [14], Lemma 2.4, and [5], Lemma 2.3, we get the following two estimates of the function f and its radial primitive F_t in the case of the unit disk $\mathbb{D}$.

Lemma 2.1. *Let I be a subarc of $\mathbb{T}$ of length $2\pi\lambda$; $0<\lambda<1$ and let $f\in H^\infty(\mathbb{D})$. For every constant $m\geqslant \|f\|_{L^\infty(\mathbb{T})}$ and $z\in\overline{\mathbb{D}}$ we have*

$$|f(z)|\leqslant m^{1-(\lambda/2)(1-|z|)}\|f\|_{L^1(I)}^{(\lambda/2)(1-|z|)} \quad \text{for every} \quad z\in\overline{\mathbb{D}}. \tag{2.2}$$

Lemma 2.2. *Let I be a subarc of $\mathbb{T}$ of length $2\pi\lambda$; $0<\lambda<1$ and let $f\in H^\infty(\mathbb{D})$, not identically zero. For every constant $m\geqslant \|f\|_{L^\infty(\mathbb{T})}$ and $(t,r)\in[-\pi,\pi]\times[0,1]$ we have*

$$|F_t(r)|\leqslant \frac{m}{\left|\frac{1}{2}\lambda\log\left(\|f/m\|_{L^1(I)}\right)\right|}. \tag{2.3}$$

In the case of an annular domain, similar estimates have been established in [13], Lemma 3.3, Lemma 3.4.

Lemma 2.3. *Let I be a subarc of ∂G_s of length $2\pi\lambda$; $0<\lambda<1$ and let $f\in H^\infty(G_s)$. For every constant $m\geqslant \|f\|_{L^\infty(\partial G_s)}$ there exists a non-negative constant C_s depending only on s such that for every $z\in\overline{G}_s$ we have*

$$|f(z)|\leqslant m\left\|\frac{f}{m}\right\|_{L^1(I)}^{2\lambda C_s(\log s-\log|z|)/\log s} \quad \text{if} \quad s<|z|\leqslant\sqrt{s},$$
$$|f(z)|\leqslant m\left\|\frac{f}{m}\right\|_{L^1(I)}^{2\lambda C_s\log|z|/\log s} \quad \text{if} \quad \sqrt{s}\leqslant|z|<1.$$

Lemma 2.4. *Let I be a subarc of ∂G_s of length $2\pi\lambda$; $0<\lambda<1$ and $q_0=-\log s$. Let $f\in H^\infty(G_s)$ not identically zero satisfy $\|f\|_{L^1(I)}<\mathrm{e}^{-q_0/(\lambda C_s)}$. Then for every constant $m\geqslant\|g\|_{L^\infty(\partial G_s)}$ and $(t,r)\in[-\pi,\pi]\times]s,1[$ we have*

$$|F_t(r)|\leqslant\frac{(2s+1)q_0m}{\left|2\lambda C_s\log\|f/m\|_{L^1(I)}\right|}. \tag{2.4}$$

In the following, we establish a variant of Hardy-Landau-Littlewood inequalities amid the class of Hölder functions defined on a finite interval. We can refer the reader to [4], [22], [15], [16], [17], [21], [23] for more details concerning Hardy-Landau-Littlewood inequalities.

Let $a<b$ and $H>0$. A function $f\colon [a,b]\to\mathbb{R}$ is said to be a $\mathcal{C}_H^{k,\alpha}$ function if

$$|f^{(k)}(x)-f^{(k)}(y)|\leqslant H|x-y|^\alpha \quad \text{for all} \quad x,y\in[a,b]. \tag{2.5}$$

Using arguments similar to [22], [19] for Lipschitz functions, we prove the following inequality.

Lemma 2.5. *Let $k \in \mathbb{N}^*$ and let $f\colon [a,b] \to \mathbb{R}$ be a $\mathcal{C}_H^{k,\alpha}$ function such that $\|f\|_{L^\infty[a,b]} \leqslant 1$.*

If $H > (2(1+\alpha))/(\alpha(b-a)^{1+\alpha})$, then we have

$$\|f'\|_{L^\infty_{([a,b])}} \leqslant C\|f\|^{(\alpha+k-1)/(\alpha+k)}_{L^\infty_{([a,b])}}, \tag{2.6}$$

where C is a non-negative constant depending only on k, α and H.

Proof. We prove the lemma by induction on k.

Let $f \in \mathcal{C}_H^{1,\alpha}$ and let $x_0 \in [a,b]$ be such that

$$|f'(x_0)| = \sup_{x\in[a,b]} |f'(x)|.$$

An asymptotic expansion of f at x_0 gives

$$|f(x) - f(x_0) - (x-x_0)f'(x_0)| \leqslant \frac{H}{1+\alpha}|x-x_0|^{1+\alpha} \quad \text{for all } \ x \in [a,b].$$

Moreover, if $x \in [a,b]$; $x \neq x_0$, we get

$$|f'(x_0)| \leqslant \Bigl|f'(x_0) - \frac{f(x)-f(x_0)}{x-x_0}\Bigr| + \Bigl|\frac{f(x)-f(x_0)}{x-x_0}\Bigr|.$$

Consequently,

$$\|f'\|_{L^\infty[a,b]} \leqslant g(|x-x_0|), \tag{2.7}$$

where g is the function defined on $[0, b-a]$ by

$$g(t) = \frac{H}{1+\alpha}t^\alpha + \frac{2M}{t}; \quad M = \sup_{x\in[a,b]} |f(x)|.$$

Since $H > 2(1+\alpha)/(\alpha(b-a)^{1+\alpha})$, one may set $t = (2M(1+\alpha)/(\alpha H))^{1/(1+\alpha)}$, which is then the optimal choice in (2.7). Then we obtain the estimate

$$\|f'\|_{L^\infty[a,b]} \leqslant \Bigl(\frac{2(1+\alpha)}{\alpha}\Bigr)^{\alpha/(1+\alpha)} H^{1/(1+\alpha)} M^{\alpha/(1+\alpha)}.$$

For $k \geqslant 2$, we suppose that inequality (2.6) holds for every function in $\mathcal{C}_H^{j,\alpha}$; $j = 1, \ldots, k-1$.

Let $f \in \mathcal{C}_H^{k,\alpha}$. Then by virtue of the Hardy-Landau-Littlewood inequality [4], Chapter VIII, page 147, there exists a non-negative constant C depending only on k, a, b such that

$$\|f'\|_{L^\infty_{([a,b])}} \leqslant C\|f\|^{1/k}_{W^{k,\infty}_{([a,b])}} \|f\|^{1-1/k}_{L^\infty_{([a,b])}}. \tag{2.8}$$

Let $s \in \{0, \dots, k\}$ be such that $\|f\|_{W^{k,\infty}([a,b])} = \|f^{(s)}\|_{L^\infty_{([a,b])}}$. Then we consider the following two cases:

First case: $s \in \{0, 1\}$.

From (2.8) we obtain

$$\|f'\|_{L^\infty_{([a,b])}} \leqslant C\|f\|_{L^\infty_{([a,b])}}, \tag{2.9}$$

which is better than (2.6).

Second case: $s \in \{2, \dots, k\}$.

For $j \in \{2, \dots, s\}$ we obtain from the induction hypothesis that

$$\|f^{(j)}\|_{L^\infty_{([a,b])}} \leqslant C\|f^{(j-1)}\|_{L^\infty_{([a,b])}}^{(\alpha+k-j)/(\alpha+k-j+1)}. \tag{2.10}$$

Making use of (2.10) for $j \in \{2, \dots, s\}$, we obtain

$$\|f^{(s)}\|_{L^\infty_{([a,b])}} \leqslant C\|f'\|_{L^\infty_{([a,b])}}^{(\alpha+k-s)/(\alpha+k-1)}.$$

From (2.8) and the fact that $\|f\|_{W^{k,\infty}([a,b])} = \|f^{(s)}\|_{L^\infty_{([a,b])}}$ we deduce that

$$\|f'\|_{L^\infty_{([a,b])}} \leqslant C\|f\|_{L^\infty_{([a,b])}}^{(\alpha+k-1)/(\alpha+k-(k-s)/(k-1))},$$

which is better than (2.6). $\square$

3. Optimal logarithmic estimates in $\mathcal{H}^{k,\alpha}(G)$

This section is devoted to proving Theorem 1.1 and Proposition 1.2. As a consequence, we establish some corollaries.

3.1. Proof of Theorem 1.1. Let $m \geqslant \max(\|f\|_{L^\infty(\partial G)}, 1)$. For every $t \in \,]-\pi, \pi[$, we denote by F_t the radial primitive of f defined by equation (2.1). According to Lemmas 2.2 and 2.4, there exists a non-negative constant λ_0 such that

$$|F_t(r)| \leqslant \frac{m}{|\lambda_0 \log \|f/m\|_{L^1(I)}|}, \tag{3.1}$$

where $\lambda_0 = \lambda/2$ if $G = \mathbb{D}$ and $\lambda_0 = \min(1, 2\lambda C_s/(1+2s)q_0)$ if $G = G_s$.

Applying Lemma 2.5 to the radial primitive F_t, we obtain

$$\|f\|_{L^\infty(\partial G)} \leqslant C\|F_t\|_{L^\infty(\partial G)}^{(\alpha+k)/(\alpha+k+1)},$$

consequently,

$$\|f\|_{L^\infty(\partial G)} \leqslant m_1 := C\biggl(\frac{m}{\bigl|\lambda_0 \log \|f/m\|_{L^1(I)}\bigr|}\biggr)^{(\alpha+k)/(\alpha+k+1)}. \tag{3.2}$$

Since (3.2) holds for every upper bound $m \geqslant \|f\|_{L^\infty(\partial G)}$, one can deduce

$$\|f\|_{L^\infty(\partial G)} \leqslant C\biggl(\frac{m_1}{\bigl|\lambda_0 \log \|f/m_1\|_{L^1(I)}\bigr|}\biggr)^{(\alpha+k)/(\alpha+k+1)}. \tag{3.3}$$

Let η be the function defined on $]0,1]$ by $\eta(x) = x|\log x|^{(\alpha+k)/(\alpha+k+1)}$. Then

$$\eta(x) \leqslant x^\beta \quad \text{for} \quad x \in \,]0,1], \ \text{where} \ \ \beta = 1 - \frac{1}{\mathrm{e}}\frac{\alpha+k}{\alpha+k+1}. \tag{3.4}$$

From (3.2) we get

$$\Bigl\|\frac{f}{m_1}\Bigr\|_{L^1(I)} = \frac{1}{C} m^{1/(\alpha+k+1)}(\lambda_0)^{(\alpha+k)/(\alpha+k+1)}\eta\Bigl(\Bigl\|\frac{f}{m}\Bigr\|_{L^1(I)}\Bigr). \tag{3.5}$$

Since we can choose $C \geqslant 1$, $m \geqslant 1$ and $\lambda_0 \in \,]0,1[$, we derive from (3.5) and (3.4) that

$$\Bigl\|\frac{f}{m_1}\Bigr\|_{L^1(I)} \leqslant \|f\|^\beta_{L^1(I)}. \tag{3.6}$$

From (3.3) and the monotonicity of the mapping $\varepsilon(x) = 1/|\log x|$, we obtain

$$\|f\|_{L^\infty(\partial G)} \leqslant C^{1+(\alpha+k)/(\alpha+k+1)} \frac{m^{((\alpha+k)/(\alpha+k+1))^2}(1/\beta)^{(\alpha+k)/(\alpha+k+1)}}{|\lambda_0 \log \|f\|_{L^1(I)}|^{(\alpha+k)[1+(\alpha+k)/(\alpha+k+1)]/(\alpha+k+1)}}.$$

Proceeding in this manner, we obtain for every $n \geqslant 1$

$$\|f\|_{L^\infty(\partial G)} \leqslant C^{b_n} \frac{m^{((\alpha+k)/(\alpha+k+1))^{n+1}}(1/\beta)^{c_n}}{|\lambda_0 \log \|f\|_{L^1(I)}|^{a_n}},$$

where a_n, b_n and c_n are three recurrent sequences satisfying

$$a_1 = \frac{\alpha+k}{\alpha+k+1}\Bigl(1 + \frac{\alpha+k}{\alpha+k+1}\Bigr), \quad b_1 = 1 + \frac{\alpha+k}{\alpha+k+1}, \quad c_1 = \frac{\alpha+k}{\alpha+k+1}$$
$$a_{n+1} = \frac{\alpha+k}{\alpha+k+1}(1+a_n), \quad b_{n+1} = 1 + \frac{\alpha+k}{\alpha+k+1}b_n, \quad c_{n+1} = \frac{\alpha+k}{\alpha+k+1}(1+c_n).$$

The proof of inequality (1.1) is completed by letting $n \to \infty$. □

3.2. Proof of Proposition 1.2. Let $u_n(z) = (z-1)^n$. Then we have

$$\|u_n\|_{L^\infty(\partial G)} = 2^n \quad \text{and} \quad \|u_n^{(k)}\|_{L^\infty(\partial G)} = \frac{n!}{(n-k)!}2^{n-k}. \tag{3.7}$$

Furthermore, we have

$$|u_n^{(k)}(z_1) - u_n^{(k)}(z_2)|^{(1/\alpha)-1} \leqslant \Big(\frac{n!}{(n-k)!}2^{n-k+1}\Big)^{(1/\alpha)-1} \tag{3.8}$$

and using the Taylor expansion, we obtain the asymptotic estimate

$$|u_n^{(k)}(z_1) - u_n^{(k)}(z_2)| \leqslant \frac{n!}{(n-k-1)!}2^{n-k-1}|z_1 - z_2|. \tag{3.9}$$

By multiplying the above inequalities (3.8) and (3.9), we get

$$|u_n^{(k)}(z_1) - u_n^{(k)}(z_2)| \leqslant \frac{n!(n-k)^{\alpha-1}}{(n-k-1)!}2^{n-k+1-2\alpha}|z_1 - z_2|^{\alpha}. \tag{3.10}$$

Let $f_n(z) = ((n-k-1)!u_n(z))/(n!(n-k)^{\alpha-1}2^{n-k+1-2\alpha})$.

Then we have

$$\|f_n\|_{L^\infty(\partial G)} = \frac{(n-k-1)!2^{k-1+2\alpha}}{n!(n-k)^{\alpha-1}}$$

and

$$\|f_n\|_{L^\infty(I)} = \frac{(n-k-1)!}{n!(n-k)^{\alpha-1}2^{(n/2)-k+1-2\alpha}}.$$

Consequently,

$$\|f_n\|_{L^\infty(\partial G)}|\log(\|f_n\|_{L^\infty(I)})|^{\alpha+k} = \frac{(\log(2))^{\alpha+k}}{2^{1-\alpha}}(1+o(1)) \quad \text{as} \quad n \to \infty,$$

from which we conclude the inequality (1.2). □

Using (2.3) and Theorem 1.1, we prove as in [14], Corollary 3.1, the following result:

Corollary 3.1. *Let m and k be two integers with $0 \leqslant m < k$ and let I be a subarc of G of length $2\pi\lambda$; $\lambda \in \,]0,1[$. There exists non-negative constants C and γ, depending only on m, k, α, s and λ, such that for every $f \in \mathcal{B}^{k,\alpha}(G)$ with $\|f\|_{W^{m,1}(I)} \leqslant \gamma$ we have*

$$\|f\|_{H^{m,\infty}} \leqslant \frac{C}{|\log(\|f\|_{W^{m,1}(I)})|^{\alpha+k-m}}. \tag{3.11}$$

3.3. Further remarks concerning Theorem 1.1. As a consequence of Theorem 1.1, we find again for $\alpha = 1$ the results of the authors [5], Theorem 2.6, when $G = \mathbb{D}$ and those of the second author [13], Theorem 3.7, when $G = G_s$. The following consequences are immediate.

Corollary 3.2. *Let $k \in \mathbb{N}$ and let I be a subarc of ∂G of length $2\pi\lambda$; $\lambda \in \,]0,1[$. There exist two non-negative constants C and γ, depending only on k, s and λ, such that every $f \in H^{k,\infty}(G)$ satisfying $\|f\|_{H^{k,\infty}} \leqslant 1$ and $\|f\|_{L^1(I)} < \gamma$, also satisfies*

$$\|f\|_{L^\infty(\partial G)} \leqslant \frac{C}{|\log \|f\|_{L^1(I)}|^k}. \tag{3.12}$$

In the particular case where $I = \mathbb{T}$, we find again the result given by the second author in [13], Corollary 3.10. A similar estimate with respect to the L^2-norm have been established by Leblond et al. in [18], Corollary 8.

Corollary 3.3. *Let $I = \mathbb{T}$, k and m be integers with $0 \leqslant m < k$. There exist non-negative constants C and γ depending only on k, m, s and λ, such that every $f \in H^{k,\infty}(G_s)$ satisfying $\|f\|_{H^{k,\infty}} \leqslant 1$ and $\|f\|_{H^{m,1}(\mathbb{T})} < \gamma$, also satisfies*

$$\|f\|_{H^{m,\infty}(s\mathbb{T})} \leqslant \frac{C}{|\log \|f\|_{W^{m,1}(\mathbb{T})}|^{k-m}}.$$

4. Applications

In this section we present two applications using the previous estimates: The first deals with the stability of the Cauchy problem for the Laplace operator in the class of Hardy-Hölder solutions. The other consists in establishing a logarithmic stability estimate for the inverse problem of the identification of an unknown Robin parameter in the case of a $W_0^{1,2}$-current flux.

Let us consider first the Cauchy problem

$$(\mathrm{CP}) \begin{cases} -\Delta u = 0 & \text{in } \mathbb{D}, \\ \partial_n u = \varphi & \text{on } I, \\ u = f & \text{on } I, \end{cases}$$

where $\partial_n u$ denotes the outer normal derivative of u, φ the imposed current flux and f the potential measurement.

Let c, $\alpha > 0$. We denote by $\mathcal{C}_c^{0,\alpha}$ the set

$$\mathcal{C}_c^{0,\alpha} = \Big\{ v \in \mathcal{C}^0(\overline{\mathbb{D}});\ \|v\|_{L^\infty(\mathbb{D})} + \sup_{x \neq y \in \mathbb{D}} \frac{|v(x) - v(y)|}{|x - y|^\alpha} \leqslant c \Big\}.$$

Then we establish the following theorem:

Theorem 4.1. *Let $\varphi \in \mathcal{C}^0(\overline{I})$ and let $u_i \in \mathcal{C}_c^{0,\alpha}$, $i = 1, 2$ be the respective solutions of* (CP) *when $f = f_i$; $i = 1, 2$. If $\|f_1 - f_2\|_{L^1(I)} < 1$, then*

$$\|u_1 - u_2\|_{L^\infty(\mathbb{T})} \leqslant \frac{\beta}{|\log \|f_1 - f_2\|_{L^1(I)}|^\alpha},$$

where $\beta > 0$ is a constant depending only upon φ, I, α and c.

Proof. Let $\theta_0 \in [0, 2\pi]$ be such that $\mathrm{e}^{\mathrm{i}\theta_0} \in I$. For $j = 1, 2$ we denote by v_j the harmonic conjugate function of u_j such that $v_j(\mathrm{e}^{\mathrm{i}\theta_0}) = 0$ and $g_j = u_j + \mathrm{i}v_j$. According to Privalov's theorem, see [12], Theorem 5.8, the function g_j is bounded on the Hardy-Hölder set $\mathcal{H}^{0,\alpha}$. Then, by using Theorem 1.1, we have

$$\|g_1 - g_2\|_{L^\infty(\mathbb{T})} \leqslant \frac{C}{|\log \|g_1 - g_2\|_{L^1(I)}|^\alpha}.$$

Using the Cauchy Riemann equations, we have $v_1(\mathrm{e}^{\mathrm{i}\theta}) = v_2(\mathrm{e}^{\mathrm{i}\theta}) = \int_{\theta_0}^{\theta} \varphi(\mathrm{e}^{\mathrm{i}s})\,\mathrm{d}s$ for every $\theta \in [0, 2\pi]$ such that $\mathrm{e}^{\mathrm{i}\theta} \in I$. Consequently,

$$\|u_1 - u_2\|_{L^\infty(\mathbb{T})} \leqslant \frac{C}{|\log \|f_1 - f_2\|_{L^1(I)}|^\alpha}.$$

$\square$

For the second application, we consider a prescribed flux $\varphi \not\equiv 0$ together with measurements f given on a subarc I of the unit circle $\mathbb{T}$, and we will find a function

q on $J = \mathbb{T} \setminus I$ such that the solution u of

$$\text{(RP)} \begin{cases} -\Delta u = 0 & \text{in } \mathbb{D}, \\ \partial_n u = \varphi & \text{on } I, \\ \partial_n u + qu = 0 & \text{on } \mathbb{T} \setminus I, \end{cases}$$

also satisfies $u|_I = f$.

Let c, $c' > 0$ and let K be a non-empty connected subset of J for which the boundary does not intersect that of I. We suppose that q belongs to the class of admissible Robin coefficients

$$Q_{\text{ad}} = \{q \in \mathcal{C}_0^1(\overline{J}),\ \|q\|_{W^{1,\infty}(J)} \leqslant c',\ \text{and } q \geqslant c\chi_K\},$$

where $\mathcal{C}_0^1$ denotes the set of differentiable functions such that the functions and their first derivatives vanish on the boundary. Let $W_0^{1,2}(I)$ denote the closure of $\mathcal{C}_0^1(I)$ in $W^{1,2}(I)$. Referring to [8], [9], [7], we have the following lemma:

Lemma 4.2 ([8], [9], [7]). *Let $\varphi \in W_0^{1,2}(I)$ with non-negative values satisfy $\varphi \not\equiv 0$ and assume that $q \in Q_{\text{ad}}$ for some constants $c, c' > 0$. Then the solution u_q of the Robin problem* (RP) *belongs to the set $\mathcal{C}^{1,1/2}(\overline{\mathbb{D}})$.*

Furthermore, there exist non-negative constants α, β and γ such that for every $q \in Q_{\text{ad}}$ we have

$$u_q \geqslant \alpha > 0, \quad \|u\|_{W^{1,\infty}(\mathbb{T})} \leqslant \beta$$

and

$$|u_q^{(k)}(x) - u_q^{(k)}(y)| \leqslant \gamma |x-y|^{1/2} \quad \forall x, y \in \overline{\mathbb{D}}\ \forall k = 0, 1.$$

Using again Privalov's theorem and equation (1.1), we prove as in [6] the following stability result:

Theorem 4.3. *Let $\varphi \in W_0^{1,2}(I)$ with non-negative values satisfy $\varphi \not\equiv 0$. Then there exists a non-negative constant C such that for any $q_1, q_2 \in Q_{\text{ad}}$ we have*

$$\|q_1 - q_2\|_{L^\infty(J)} \leqslant \frac{C}{|\log \|u_1 - u_2\|_{L^1(I)}|^{1/2}}$$

provided that $\|u_1 - u_2\|_{L^1(I)} < 1$, where u_i denotes the solution of (RP) *with $q = q_i$; $i = 1, 2$.*

Note that this result improves [6], Corollary 1, where the authors supposed that $\varphi \in W_0^{2,2}(I)$.

P r o o f. The proof is the same as the one of [6], Theorem 3, except that we use Privalov's theorem, see also [12], Theorem 5.8, instead of [6], Theorem 2, and our Theorem 1.1 instead of [6], Corollary 3. $\square$

Acknowledgment. The authors are grateful to the referee for having suggested many enriching and useful comments improving the form of the paper.

References

[1] *L. Baratchart, J. Leblond*: Harmonic identification and Hardy class trace on an arc of the circle. Optimisation et Contrôle. Proceedings of the Colloquium in Honor of Jean Céa's sixtieth birthday held in Sophia-Antipolis, 1992 (J.-A. Désidéri, eds.). Cépaduès Éditions, Toulouse, 1993, pp. 17–29. (In French.)

[2] *L. Baratchart, J. Leblond, J. R. Partington*: Hardy approximation to L^∞ functions on subsets of the circle. Constr. Approx. *12* (1996), 423–435.

[3] *L. Baratchart, M. Zerner*: On the recovery of functions from pointwise boundary values in a Hardy-Sobolev class of the disk. J. Comput. Appl. Math *46* (1993), 255–269.

[4] *H. Brézis*: Functional Analysis. Theory and Applications. Collection of Applied Mathematics for the Master's Degree, Masson, Paris, 1983. (In French.)

[5] *S. Chaabane, I. Feki*: Optimal logarithmic estimates in Hardy-Sobolev spaces $H^{k,\infty}$. C. R., Math., Acad. Sci. Paris *347* (2009), 1001–1006.

[6] *S. Chaabane, I. Fellah, M. Jaoua, J. Leblond*: Logarithmic stability estimates for a Robin coefficient in two-dimensional Laplace inverse problems. Inverse Probl. *20* (2004), 47–59.

[7] *S. Chaabane, J. Ferchichi, K. Kunisch*: Differentiability properties of the L^1-tracking functional and application to the Robin inverse problem. Inverse Probl. *20* (2004), 1083–1097.

[8] *S. Chaabane, M. Jaoua*: Identification of Robin coefficients by the means of boundary measurements. Inverse Probl. *15* (1999), 1425–1438.

[9] *S. Chaabane, M. Jaoua, J. Leblond*: Parameter identification for Laplace equation and approximation in Hardy classes. J. Inverse Ill-Posed Probl. *11* (2003), 33–57.

[10] *I. Chalendar, J. R. Partington*: Approximation problems and representations of Hardy spaces in circular domains. Stud. Math. *136* (1999), 255–269.

[11] *B. Chevreau, C. M. Pearcy, A. L. Shields*: Finitely connected domains G, representations of $H^\infty(G)$, and invariant subspaces. J. Oper. Theory *6* (1981), 375–405.

[12] *P. L. Duren*: Theory of H^p Spaces. Pure and Applied Mathematics 38, Academic Press, New York, 1970.

[13] *I. Feki*: Estimates in the Hardy-Sobolev space of the annulus and stability result. Czech. Math. J. *63* (2013), 481–495.

[14] *I. Feki, H. Nfata, F. Wielonsky*: Optimal logarithmic estimates in the Hardy-Sobolev space of the disk and stability results. J. Math. Anal. Appl. *395* (2012), 366–375.

[15] *E. Gagliardo*: Proprietà di alcune classi di funzioni più variabili. Ricerche Mat. *7* (1958), 102–137. (In Italian.)

[16] *G. H. Hardy, E. Landau, J. E. Littlewood*: Some inequalities satisfied by the integrals or derivatives of real or analytic functions. Math. Z. *39* (1935), 677–695.

[17] *M. K. Kwong, A. Zettl*: Norm Inequalities for Derivatives and Differences. Lecture Notes in Mathematics 1536, Springer, Berlin, 1992.

[18] *L. Leblond, M. Mahjoub, J. R. Partington*: Analytic extensions and Cauchy-type inverse problems on annular domains: stability results. J. Inverse Ill-Posed Probl. *14* (2006), 189–204.

[19] *L. J. Marangunić, J. Pečarić*: On Landau type inequalities for functions with Hölder continuous derivatives. JIPAM, J. Inequal. Pure Appl. Math. (electronic only) *5* (2004), Paper No. 72, 5 pages.

[20] *H. Meftahi, F. Wielonsky*: Growth estimates in the Hardy-Sobolev space of an annular domain with applications. J. Math. Anal. Appl. *358* (2009), 98–109.
[21] *D. S. Mitrinović, J. E. Pečarić, A. M. Fink*: Inequalities Involving Functions and Their Integrals and Derivatives. Mathematics and Its Applications: East European Series 53, Kluwer Academic Publishers, Dordrecht, 1991.
[22] *C. P. Niculescu, C. Buşe*: The Hardy-Landau-Littlewood inequalities with less smoothness. JIPAM, J. Inequal. Pure Appl. Math. (electronic only) *4* (2003), Paper No. 51, 8 pages.
[23] *L. Nirenberg*: An extended interpolation inequality. Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser. *20* (1966), 733–737.
[24] *W. Rudin*: Analytic functions of class H^p. Trans. Am. Math. Soc. *78* (1955), 46–66.
[25] *D. Sarason*: The H^p Spaces of An Annulus. Memoirs of the American Mathematical Society 56, AMS, Providence, 1965.
[26] *H.-C. Wang*: Real Hardy spaces of an annulus. Bull. Aust. Math. Soc. *27* (1983), 91–105.

Authors' address: Slim Chaabane, Imed Feki, Department of Mathematics, Faculty of Sciences, Sfax University, B.P 1171, Sfax 3018, Tunisia, e-mail: `slim.chaabane@fsm.rnu.tn`, `imed.feki@fss.rnu.tn`.