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THE GROUP OF COMMUTATIVITY PRESERVING MAPS ON STRICTLY UPPER TRIANGULAR MATRICES

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Abstract. Let $\mathscr{N} = N_n(R)$ be the algebra of all $n \times n$ strictly upper triangular matrices over a unital commutative ring R. A map φ on $\mathscr{N}$ is called preserving commutativity in both directions if $xy = yx \Leftrightarrow \varphi(x)\varphi(y) = \varphi(y)\varphi(x)$. In this paper, we prove that each invertible linear map on $\mathscr{N}$ preserving commutativity in both directions is exactly a quasi-automorphism of $\mathscr{N}$, and a quasi-automorphism of $\mathscr{N}$ can be decomposed into the product of several standard maps, which extains the main result of Y. Cao, Z. Chen and C. Huang (2002) from fields to rings.

Keywords: commutativity preserving map; automorphism; commutative ring

MSC 2010: 17C30, 15A04, 15A27, 15A99, 13C10

1. Introduction

A lot of attention has been paid to the commutativity preserver problem on associative algebras, particularly on matrix algebras. The earliest paper on such problem dates back to 1976, when Watkins [8] studied commutativity preserving maps on the full matrix algebra $M_n(F)$ over a field F. If $n \geqslant 3$, then every invertible linear commutativity preserving map φ on $M_n(F)$ was shown to be one of the two standard forms: $\varphi(x) = ctxt^{-1} + f(x)e$, $x \in M_n(F)$, or $\varphi(x) = ctx't^{-1} + f(x)e$, $x \in M_n(F)$, where c is a nonzero element in F, t an invertible matrix, and f a linear function on $M_n(F)$. In 1999, Marcoux et al. [4] described commutativity preserving maps on $T_n(F)$ of all upper triangular matrices, and in 2002, Cao et al. [2] determined commutativity preserving maps on $N_n(F)$ of all strictly upper triangular matrices with F a field. It was Omladič who was the first to considered commutativity pre-

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serving maps on infinite-dimensional algebras. In [5], he considers such maps on $\mathfrak{B}(X)$ of all linear operators on an infinite-dimensional Banach space X. In 1993, Brešar [1] improved Omladič's result by using a ring theoretic approach called commuting mappings. Šemrl [6] turned to study nonlinear maps on $M_n(F)$ preserving commutativity, and he found that, without the linear condition, these types of maps can have wild behavior. Commutativity preserving linear maps on a finite dimensional simple Lie algebra was characterized in [7]. We find that the known results on the topic of characterizing commutativity preserving maps are all for algebras over fields, no result concerns commutativity preserving maps on algebras over rings. The reason why people did not study commutativity preserving maps on matrix algebras over rings is probably that the technique of dealing with matrices over fields cannot directly be transferred to matrices over rings.

In this paper, based on the characterization of automorphisms of $\mathscr{N}_n(R)$ due to Cao et al. [3], we characterize the invertible linear maps on $\mathscr{N} = \mathscr{N}_n(R)$ over a commutative ring R which preserve commutativity in both directions, thus extending the main result of [2] from fields to rings. Our main idea is to show that a commutativity preserving map on $\mathscr{N}$ is exactly a quasi-automorphism of $\mathscr{N}$, and a quasi-automorphism of $\mathscr{N}$ can be decomposed into the product of several standard maps.

2. Quasi-automorphisms of $\mathscr{N}$

Let R be a commutative ring with identity. We denote by R^* the set of all invertible elements in R. By e_{ij} we denote the matrix unit which has 1 in the (i,j) position and 0 elsewhere. The set of all $n \times n$ strictly upper triangular matrices over R is denoted by $\mathscr{N}$. The derived subalgebra $[\mathscr{N}, \mathscr{N}]$ of $\mathscr{N}$ is denoted by $\mathscr{N}_1$. If $n \geqslant 3$, the center of $\mathscr{N}$ is Re_{1n}, which is denoted by Z. Set

$$\begin{aligned}
\Phi &= \{(i,j);\ 1 \leqslant i < j \leqslant n\};\\
\Psi &= \{(i,j) \in \Phi;\ j - i \geqslant 2\};\\
\Omega &= \Psi - \{(1,n)\}.
\end{aligned}$$

One easily sees that all e_α, $\alpha \in \Phi$, form an R-basis of $\mathscr{N}$, and all $e_\beta, \beta \in \Psi$, form an R-basis of $\mathscr{N}_1$. The Lie product of $\mathscr{N}$ is defined as $[x,y] = xy - yx$.

Definition 2.1. Let $\varphi\colon \mathscr{N} \to \mathscr{N}$ be an invertible linear map.

(i) φ is called a (Lie) *automorphism* of $\mathscr{N}$ if $\varphi([x,y]) = [\varphi(x), \varphi(y)]$, for all maps $x, y \in \mathscr{N}$;

(ii) φ is called a (Lie) *quasi-automorphism* of $\mathscr{N}$ if there exists an invertible linear map $\bar\varphi$ on $\mathscr{N}_1$ such that $\bar\varphi([x,y]) = [\varphi(x), \varphi(y)]$, for all $x, y \in \mathscr{N}$;

(iii) φ is called *preserving commutativity in both directions* if

$$\varphi(x)\varphi(y) = \varphi(y)\varphi(x) \Leftrightarrow xy = yx,$$

or equivalently, $[\varphi(x), \varphi(y)] = 0 \Leftrightarrow [x, y] = 0$.

We denote by $\mathrm{Aut}(\mathscr{N})$; $Q\,\mathrm{Aut}(\mathscr{N})$; $\mathrm{Inv_c}(\mathscr{N})$; $\mathrm{GL}(\mathscr{N})$, respectively, the set of all automorphisms of $\mathscr{N}$; the set of all quasi-automorphisms of $\mathscr{N}$; the set of all invertible linear maps on $\mathscr{N}$ preserving commutativity in both directions; the set of all invertible linear maps on $\mathscr{N}$. Observe that for $\varphi, \varphi_1, \varphi_2 \in Q\,\mathrm{Aut}(\mathscr{N})$, $\varphi^{-1}, \varphi_2\varphi_1 \in Q\,\mathrm{Aut}(\mathscr{N})$ and $\overline{\varphi_2 \cdot \varphi_1} = \bar{\varphi}_2 \cdot \bar{\varphi}_1$, and $\overline{\varphi}^{-1} = \overline{\varphi^{-1}}$. So $Q\,\mathrm{Aut}(\mathscr{N})$ forms a group. Also, $\mathrm{Inv_c}(\mathscr{N})$ forms another group. We now construct some standard quasi-automorphisms for $\mathscr{N}$ as follows:

(A) *Automorphisms.* It is easy to check that every automorphism of $\mathscr{N}$ is a quasi-automorphism of $\mathscr{N}$, and a quasi-automorphism of $\mathscr{N}$ preserves commutativity in both directions. Then we obtain a series of subgroups for $\mathrm{GL}(\mathscr{N})$ as follows:

$$\mathrm{Aut}(\mathscr{N}) \leqslant Q\,\mathrm{Aut}(\mathscr{N}) \leqslant \mathrm{Inv_c}(\mathscr{N}) \leqslant \mathrm{GL}(\mathscr{N}).$$

(B) *Scalar multiplication maps.* Let $r \in R$ be invertible. Define

$$\eta_r\colon\ x \to rx, \quad \forall\, x \in \mathscr{N};$$
$$\overline{\eta_r}\colon\ x \to r^2 x, \quad \forall\, x \in \mathscr{N}_1.$$

Then η_r is a quasi-automorphism of $\mathscr{N}$ since

$$\overline{\eta_r}([x, y]) = r^2([x, y]) = [rx, ry] = [\eta_r(x), \eta_r(y)], \quad \forall\, x \in \mathscr{N}.$$

We call η_r a *scalar multiplication map* on $\mathscr{N}$. Obviously, η_r is an automorphism if and only if $r = 1$.

(C) *Central quasi-automorphisms.* Suggested by the concept of *central automorphisms* introduced by [3], we define a related concept, called central quasi-automorphisms of $\mathscr{N}$. Let $n \geqslant 4$ and let $f\colon\ \mathscr{N} \to R$ be a linear function such that $1 + f(e_{1n})$ is invertible. Define $\theta_f\colon\ x \mapsto x + f(x)e_{1n}$, for all $x \in \mathscr{N}$. Then θ_f is invertible, its inverse being the mapping defined as $x \mapsto x - f(x)(1 + f(e_{1n}))^{-1}e_{1n}$, for all $x \in \mathscr{N}$. Set $\overline{\theta_f}$ to be the identity map on $\mathscr{N}_1$. Observe that

$$\overline{\theta_f}([x, y]) = [x, y] = [\theta_f(x), \theta_f(y)], \quad \forall\, x, y \in \mathscr{N}.$$

Thus θ_f is a quasi-automorphism of $\mathscr{N}$, called a *central quasi-automorphism* of $\mathscr{N}$. Obviously, θ_f is an automorphism if and only if $f(x) = 0$ for all $x \in \mathscr{N}_1$ and $\theta_f \in \mathrm{Inv_c}(\mathscr{N})$.

(D) *Extremal quasi-automorphisms.* Let $n \geqslant 4$ and $a, b \in R$. Define

$$\begin{aligned}
\psi_a\colon\ & x = (x_{ij}) \in \mathscr{N} \mapsto x + a x_{12} e_{3n} + a x_{13} e_{2n};\\
\overline{\psi_a}\colon\ & x = (x_{ij}) \in \mathscr{N}_1 \mapsto x - a x_{13} e_{2n};\\
\chi_b\colon\ & x = (x_{ij}) \in \mathscr{N} \mapsto x + b x_{n-1,n} e_{1,n-2} + b x_{n-2,n} e_{1,n-1};\\
\overline{\chi_b}\colon\ & x = (x_{ij}) \in \mathscr{N}_1 \mapsto x - b x_{n-2,n} e_{1,n-1}.
\end{aligned}$$

One easily checks that

$$\begin{aligned}
\overline{\psi_a}([x,y]) &= [\psi_a(x), \psi_a(y)], \quad \forall\, x, y \in \mathscr{N},\\
\overline{\chi_b}([x,y]) &= [\chi_b(x), \chi_b(y)], \quad \forall\, x, y \in \mathscr{N},
\end{aligned}$$

showing that both ψ_a and χ_b are quasi-automorphisms of $\mathscr{N}$. We call ψ_a and χ_b extremal quasi-automorphisms of $\mathscr{N}$. They are automorphisms of $\mathscr{N}$ if and only if $\operatorname{char} F = 2$.

The maps introduced above are all quasi-automorphisms of $\mathscr{N}$. Next we will prove that each quasi-automorphism of $\mathscr{N}$ can conversely be decomposed into the product of these standard quasi-automorphisms when $n \geqslant 5$. Let $\varphi\colon \mathscr{N} \to \mathscr{N}$ be a quasi-automorphism of $\mathscr{N}$ with an invertible linear map $\bar{\varphi}\colon \mathscr{N}_1 \to \mathscr{N}_1$ such that $[\varphi(x), \varphi(y)] = \bar{\varphi}([x,y])$, $\forall\, x, y \in \mathscr{N}$. We now give some elementary results for φ.

Lemma 2.1. *Let $n \geqslant 3$. Then φ stabilizes the center Z of $\mathscr{N}$.*

Proof. For any $y \in \mathscr{N}$, assume that $y = \varphi(x)$. Since

$$[\varphi(e_{1n}), y] = \bar{\varphi}([e_{1n}, x]) = 0,$$

$\varphi(e_{1n})$ lies in the center Z of $\mathscr{N}$. This implies that φ stabilizes Z. □

We now consider the action of $\bar{\varphi}$ on e_β for $\beta \in \Psi$.

Lemma 2.2. *Let $\varphi \in Q\operatorname{Aut}(\mathscr{N})$ and $n \geqslant 3$. Then $\bar{\varphi}$ stabilizes Z.*

Proof. If $n = 3$, then by $\bar{\varphi}(e_{13}) = [\varphi(e_{12}), \varphi(e_{23})] \in \mathscr{N}_1 = Z$, we obtain the result. Now we consider the case that $n \geqslant 4$. Because all $\varphi(e_\alpha)$, $\alpha \in \Phi$ form an R-basis of $\mathscr{N}$, we may assume that

$$\bar{\varphi}(e_{1n}) = \sum_{1 \leqslant i < j \leqslant n} a_{ij} \varphi(e_{ij}).$$

For $2 \leqslant k < n$, we choose p such that $2 \leqslant p \leqslant n-1$ and $p \neq k$. Then by

$$\sum_{i=1}^{k-1} a_{ik} \bar{\varphi}(e_{in}) = \Bigl[\sum_{1 \leqslant i < j \leqslant n} a_{ij} \varphi(e_{ij}), \varphi(e_{kn})\Bigr]$$

$$
\begin{aligned}
&= [\bar{\varphi}(e_{1n}), \varphi(e_{kn})] \\
&= [[\varphi(e_{1p}), \varphi(e_{pn})], \varphi(e_{kn})] \\
&= [[\varphi(e_{1p}), \varphi(e_{kn})], \varphi(e_{pn})] + [\varphi(e_{1p}), [\varphi(e_{pn}), \varphi(e_{kn})]] \\
&= [\bar{\varphi}([e_{1p}, e_{kn}]), \varphi(e_{pn})] + [\varphi(e_{1p}), \bar{\varphi}([e_{pn}, e_{kn}])] = 0
\end{aligned}
$$

we have that $a_{ik} = 0$ for $1 \leqslant i < k < n$. Similarly, by considering $[\varphi(e_{1k}), \bar{\varphi}(e_{1n})]$ we can show that $a_{kj} = 0$ for $1 < k < j \leqslant n$. Thus $\bar{\varphi}(e_{1n}) = a_{1n}\varphi(e_{1n}) \in Z$ (using Lemma 2.1). So $\bar{\varphi}$ stabilizes Z. □

Lemma 2.3. *Let $n \geqslant 4$ and let $\varphi \in Q\,\mathrm{Aut}(\mathscr{N})$. Then*
(i) $\bar{\varphi}(e_{1n}) = a_{1n}\varphi(e_{1n})$;
(ii) $\bar{\varphi}(e_{1k}) \equiv a_{1k}\varphi(e_{1k}) \pmod Z$ *for* $k = 4, 5, \ldots, n-1$;
(iii) $\bar{\varphi}(e_{13}) \equiv a_{13}\varphi(e_{13}) + a\varphi(e_{2n}) \pmod Z$;
where $a_{1k}, a \in R$.

P r o o f. (i) has been shown in Lemma 2.2. Assume that

$$\bar{\varphi}(e_{1k}) = \sum_{1\leqslant i<j\leqslant n} a_{ij}^{(1k)}\varphi(e_{ij}) \quad \text{for } k = 3, 4, \ldots, n-1.$$

To complete the proof of (ii) and (iii) we need to verify several assertions.

Assertion 1. $a_{1l}^{(1k)} = 0$ if $l \neq k$ and $l \neq n$.

We consider $[\bar{\varphi}(e_{1k}), \varphi(e_{ln})]$. Applying Jacobi's identity and Lemma 2.2, we have that

$$
\begin{aligned}
[\bar{\varphi}(e_{1k}), \varphi(e_{ln})] &= [[\varphi(e_{12}), \varphi(e_{2k})], \varphi(e_{ln})] \\
&= [[\varphi(e_{12}), \varphi(e_{ln})], \varphi(e_{2k})] + [\varphi(e_{12}), [\varphi(e_{2k}), \varphi(e_{ln})]] \\
&= [\bar{\varphi}([e_{12}, e_{ln}]), \varphi(e_{2k})] + [\varphi(e_{12}), \bar{\varphi}([e_{2k}, e_{ln}])] \\
&= \delta_{2,l}[\bar{\varphi}(e_{1n}), \varphi(e_{2k})] = 0.
\end{aligned}
$$

Here $\delta_{2,l}$ denotes the Kronecker symbol, i.e., $\delta_{2,l} = 1$ if $l = 2$; $\delta_{2,l} = 0$ otherwise. Thus,

$$0 = [\bar{\varphi}(e_{1k}), \varphi(e_{ln})] = \Bigg[\sum_{1\leqslant i<j\leqslant n} a_{ij}^{(1k)}\varphi(e_{ij}), \varphi(e_{ln})\Bigg] = \sum_{i=1}^{l-1} a_{il}^{(1k)}\bar{\varphi}(e_{in}).$$

This equality implies that $a_{1l}^{(1k)} = 0$ provides that $l \neq k$ and $l \neq n$.

Assertion 2. $a_{ml}^{(1k)} = 0$ for $3 \leqslant m < l \leqslant n$.

For this assertion we consider $[\varphi(e_{1m}), \bar{\varphi}(e_{1k})]$. Applying Jacobi's identity we have

$$\begin{aligned}[\varphi(e_{1m}), \bar{\varphi}(e_{1k})] &= [\varphi(e_{1m}), [\varphi(e_{12}), \varphi(e_{2k})]] \\ &= [[\varphi(e_{1m}), \varphi(e_{12})], \varphi(e_{2k})] + [\varphi(e_{12}), [\varphi(e_{1m}), \varphi(e_{2k})]] \\ &= [\bar{\varphi}([e_{1m}, e_{12}]), \varphi(e_{2k})]] + [\varphi(e_{12}), \bar{\varphi}([e_{1m}, e_{2k}])] = 0.\end{aligned}$$

Thus we have

$$0 = [\varphi(e_{1m}), \bar{\varphi}(e_{1k})] = \left[\varphi(e_{1m}), \sum_{1 \leqslant i < j \leqslant n} a_{ij}^{(1k)} \varphi(e_{ij})\right] = \sum_{j=m+1}^{n} a_{mj}^{(1k)} \bar{\varphi}(e_{1j}).$$

This implies that $a_{ml}^{(1k)} = 0$ for $3 \leqslant m < l \leqslant n$.

Assertion 3. $a_{2l}^{(1k)} = 0$ for $3 \leqslant l \leqslant n-1$.

For this case we also consider $[\bar{\varphi}(e_{1k}), \varphi(e_{ln})]$. On the one hand,

$$\begin{aligned}[\bar{\varphi}(e_{1k}), \varphi(e_{ln})] &= [[\varphi(e_{12}), \varphi(e_{2k})], \varphi(e_{ln})] \\ &= [[\varphi(e_{12}), \varphi(e_{ln})], \varphi(e_{2k})] + [\varphi(e_{12}), [\varphi(e_{2k}), \varphi(e_{ln})]] \\ &= [\bar{\varphi}([e_{12}, e_{ln}]), \varphi(e_{2k})] + [\varphi(e_{12}), \bar{\varphi}([e_{2k}, e_{ln}])] \\ &= \delta_{k,l}[\varphi(e_{12}), \bar{\varphi}(e_{2n})].\end{aligned}$$

Assume $\bar{\varphi}(e_{2n}) = \sum_{1 \leqslant i < j \leqslant n} a_{ij}^{(2n)} \varphi(e_{ij})$. Then we further have

$$[\bar{\varphi}(e_{1k}), \varphi(e_{ln})] = \delta_{k,l}\left[\varphi(e_{12}), \sum_{1 \leqslant i < j \leqslant n} a_{ij}^{(2n)} \varphi(e_{ij})\right] = \delta_{k,l} \sum_{j=3}^{n} a_{2j}^{(2n)} \bar{\varphi}(e_{1j}).$$

On the other hand, we have

$$[\bar{\varphi}(e_{1k}), \varphi(e_{ln})] = \sum_{i=1}^{l-1} a_{il}^{(1k)} \bar{\varphi}(e_{in}).$$

By

$$\delta_{k,l} \sum_{j=3}^{n} a_{2j}^{(2n)} \bar{\varphi}(e_{1j}) = \sum_{i=1}^{l-1} a_{il}^{(1k)} \bar{\varphi}(e_{in}),$$

we have $a_{2l}^{(1k)} = 0$ for $3 \leqslant l \leqslant n-1$.

Assertion 4. $a_{2n}^{(1k)} = 0$ for $4 \leqslant k \leqslant n-1$.

For this case we consider $[\varphi(e_{12}), \bar{\varphi}(e_{1k})]$. Applying Jacobi's identity we have that

$$\begin{aligned}[\varphi(e_{12}), \bar{\varphi}(e_{1k})] &= [\varphi(e_{12}), [\varphi(e_{13}), \varphi(e_{3k})]] \\ &= [[\varphi(e_{12}), \varphi(e_{13})], \varphi(e_{3k})] + [\varphi(e_{13}), [\varphi(e_{12}), \varphi(e_{3k})]] \\ &= [\bar{\varphi}([e_{12}, e_{13}], \varphi(e_{3k}))] + [\varphi(e_{13}), \bar{\varphi}([e_{12}, e_{3k}])] = 0.\end{aligned}$$

Thus

$$\sum_{j=3}^{n} a_{2j}^{(1k)} \bar{\varphi}(e_{1j}) = \Bigg[\varphi(e_{12}), \sum_{1\leqslant i<j\leqslant n} a_{ij}^{(1k)} \varphi(e_{ij})\Bigg] = [\varphi(e_{12}), \bar{\varphi}(e_{1k})] = 0.$$

This implies that $a_{2n}^{(1k)} = 0$ for $4 \leqslant k \leqslant n-1$.

For brevity we denote $a_{1k}^{(1k)}$ by a_{1k}, and denote $a_{2n}^{(13)}$ by a, then the result follows.

$\square$

Lemma 2.4. *Let $n \geqslant 4$ and let $\varphi \in Q\operatorname{Aut}(\mathscr{N})$. Then*

(i) $\bar{\varphi}(e_{in}) \equiv a_{in}\varphi(e_{in}) \pmod{Z}$ *for $i = 2, 3, \ldots, n-3$;*

(ii) $\bar{\varphi}(e_{n-2,n}) \equiv a_{n-2,n}\varphi(e_{n-2,n}) + b\varphi(e_{1,n-1}) \pmod{Z}$,

where $a_{in}, b \in R$.

P r o o f. Let $w = e_{1n} + e_{2,n-1} + \ldots + e_{i,n-i+1} + \ldots + e_{n1}$ and define $\omega\colon \mathscr{N} \to \mathscr{N}$ by $x \mapsto -wx'w$. Then it is easy to check that ω is an automorphism of $\mathscr{N}$, and it sends e_{ij} to $-e_{n-j+1,n-i+1}$. Denote $\varphi \circ \omega$ by φ_1. Then we have $\overline{\varphi}(e_{in}) = -\overline{\varphi_1}(e_{1,n-i+1})$, and $\varphi(e_{in}) = -\varphi_1(e_{1,n-i+1})$. By Lemma 2.3, we may assume that

$$\begin{aligned}\overline{\varphi}_1(e_{1,n-i+1}) &\equiv a_{in}\varphi_1(e_{1,n-i+1}) \pmod{Z}, \quad i = 2, 3, \ldots, n-3; \\ \overline{\varphi}_1(e_{13}) &\equiv a_{n-2,n}\varphi_1(e_{13}) + b\varphi_1(e_{2n}) \pmod{Z}.\end{aligned}$$

Thus

$$\begin{aligned}\overline{\varphi}(e_{in}) &\equiv a_{in}\varphi(e_{in}) \pmod{Z}, \quad i = 2, 3, \ldots, n-3; \\ \overline{\varphi}(e_{n-2,n}) &\equiv a_{n-2,n}\varphi(e_{n-2,n}) + b\varphi(e_{1,n-1}) \pmod{Z}.\end{aligned}$$

$\square$

Lemma 2.5. *Let $n \geqslant 4$ and let $\varphi \in Q\operatorname{Aut}(\mathscr{N})$. If $2 \leqslant i < j-1 < j \leqslant n-1$, then $\bar{\varphi}(e_{ij}) \equiv a_{ij}\varphi(e_{ij}) \pmod{Z}$, where $a_{ij} \in R$.*

P r o o f. Assume that

$$\bar{\varphi}(e_{ij}) = \sum_{1\leqslant k<l\leqslant n} a_{kl}^{(ij)} \varphi(e_{kl}).$$

We need to show that $a_{kl}^{(ij)} = 0$ for $(kl) \notin \{(ij), (1n)\}$. First, we prove that $a_{kl}^{(ij)} = 0$ for the case that $2 \leqslant k < l \leqslant n$ and $(kl) \neq (ij)$. For this case we consider $[\varphi(e_{1k}), \bar{\varphi}(e_{ij})]$. On the one hand,

$$\begin{aligned}[\varphi(e_{1k}), \bar{\varphi}(e_{ij})] &= [\varphi(e_{1k}), [\varphi(e_{i,i+1}), \varphi(e_{i+1,j})]] \\ &= [\delta_{k,i}\bar{\varphi}(e_{1,i+1}), \varphi(e_{i+1,j})] + [\varphi(e_{i,i+1}), \delta_{k,i+1}\bar{\varphi}(e_{1j})].\end{aligned}$$

By Lemma 2.3, we may assume that $\bar{\varphi}(e_{1k}) \equiv a_{1k}\varphi(e_{1k}) \pmod Z$ for $4 \leqslant k \leqslant n-1$. Thus, we have $[\varphi(e_{i,i+1}), \delta_{k,i+1}\bar{\varphi}(e_{1j})] = 0$. Hence

$$[\varphi(e_{1k}), \bar{\varphi}(e_{ij})] = \delta_{k,i}a_{1,i+1}[\varphi(e_{1,i+1}), \varphi(e_{i+1,j})] = \delta_{k,i}a_{1,i+1}\bar{\varphi}(e_{1j}).$$

On the other hand,

$$[\varphi(e_{1k}), \bar{\varphi}(e_{ij})] = \Bigg[\varphi(e_{1k}), \sum_{1\leqslant k<l\leqslant n} a_{st}^{(ij)}\varphi(e_{st})\Bigg] = \sum_{t=k+1}^{n} a_{kt}^{(ij)}\bar{\varphi}(e_{1t}).$$

By

$$\delta_{k,i}a_{1,i+1}\bar{\varphi}(e_{1j}) = \sum_{t=k+1}^{n} a_{kt}^{(ij)}\bar{\varphi}(e_{1t}),$$

we have that $a_{kt}^{(ij)} = 0$ provides $(k,t) \neq (i,j)$.

Similarly, by considering $[\bar{\varphi}(e_{ij}), \varphi(e_{ln})]$ we can prove that $a_{kl}^{(ij)} = 0$ for the case that $1 \leqslant k < l \leqslant n-1$ and $(k,l) \neq (i,j)$. The verification is left to the reader.

Thus $a_{kl}^{(ij)} = 0$ for $(k,l) \notin \{(i,j), (1,n)\}$. □

For brevity, we denote $a_{ij}^{(ij)}$ by a_{ij}. Then the result follows.

Lemma 2.6. *All a_{ij} (as in Lemmas 2.3–2.5) for $(i,j) \in \Omega$ are consistent.*

P r o o f. For $3 \leqslant k \leqslant n-1$, by

$$\begin{aligned}a_{1k}\bar{\varphi}(e_{1n}) &= [a_{1k}\varphi(e_{1k}), \varphi(e_{kn})] \\ &= [\bar{\varphi}(e_{1k}), \varphi(e_{kn})] \\ &= [[\varphi(e_{12}), \varphi(e_{2k})], \varphi(e_{kn})] \\ &= [[\varphi(e_{12}), \varphi(e_{kn})], \varphi(e_{2k})] + [\varphi(e_{12}), [\varphi(e_{2k}), \varphi(e_{kn})]] \\ &= [\varphi(e_{12}), \bar{\varphi}(e_{2n})] \\ &= a_{2n}[\varphi(e_{12}), \varphi(e_{2n})] \\ &= a_{2n}\bar{\varphi}(e_{1n})\end{aligned}$$

we have that $a_{1k} = a_{2n}$ for $k = 3, 4, \ldots, n-1$. A similar discussion shows that $a_{in} = a_{1,n-1}$ for $i = 2, 3, \ldots, n-2$ (we omit the analogous argument). Now we consider a_{ij} for the case that $2 \leqslant i < j-1 < j \leqslant n-1$. By

$$\begin{aligned}
a_{ij}\bar{\varphi}(e_{in}) &= [a_{ij}\varphi(e_{ij}), \varphi(e_{jn})] \\
&= [\bar{\varphi}(e_{ij}), \varphi(e_{jn})] \\
&= [[\varphi(e_{i,i+1}), \varphi(e_{i+1,j})], \varphi(e_{jn})] \\
&= [[\varphi(e_{i,i+1}), \varphi(e_{jn})], \varphi(e_{i+1,j})] + [\varphi(e_{i,i+1}), [\varphi(e_{i+1,j}), \varphi(e_{jn})]] \\
&= [\varphi(e_{i,i+1}), \bar{\varphi}(e_{i+1,n})] \\
&= a_{i+1,n}[\varphi(e_{i,i+1}), \varphi(e_{i+1,n})] \\
&= a_{i+1,n}\bar{\varphi}(e_{in})
\end{aligned}$$

we have that $a_{ij} = a_{i+1,n}$. So all a_{ij} are consistent. □

By Lemma 2.6, we may assume that

$$\bar{\varphi}(e_{13}) \equiv r\varphi(e_{13}) + a\varphi(e_{2n}) \pmod{Z}; \tag{2.1}$$

$$\bar{\varphi}(e_{n-2,n}) \equiv r\varphi(e_{n-2,n}) + b\varphi(e_{1,n-1}) \pmod{Z}; \tag{2.2}$$

$$\bar{\varphi}(e_{ij}) \equiv r\varphi(e_{ij}) \pmod{Z} \quad \text{for } (i,j) \in \Omega - \{(1,3), (n-2,n)\}. \tag{2.3}$$

Lemma 2.7. *Let φ be a quasi-automorphism of $\mathscr{N}$. Then $\varphi(\mathscr{N}_1) = \mathscr{N}_1$.*

Proof. The equations (2.1)–(2.3), together with Lemmas 2.1 and 2.2, show that $(\varphi^{-1} \circ \bar{\varphi})(e_{ij}) \in \mathscr{N}_1$ for all $(i,j) \in \Psi$. Furthermore, we have $(\varphi^{-1} \circ \bar{\varphi})(\mathscr{N}_1) \subseteq \mathscr{N}_1$, and $\mathscr{N}_1 = \bar{\varphi}(\mathscr{N}_1) \subseteq \varphi(\mathscr{N}_1)$. A similar discussion on φ^{-1} leads to $\mathscr{N}_1 \subseteq \varphi^{-1}(\mathscr{N}_1)$. That is $\varphi(\mathscr{N}_1) \subseteq \mathscr{N}_1$. Thus the required result follows. □

Theorem 2.8.

(i) *If $n = 2$, then every invertible linear map on $\mathscr{N}$ is a quasi-automorphism of $\mathscr{N}$.*

(ii) *If $n = 3$, then each quasi-automorphism of $\mathscr{N}$ can be decomposed into the product of a scalar multiplication map and an automorphism.*

(iii) *If $n \geqslant 5$ and $2 \in R^*$, then each quasi-automorphism of $\mathscr{N}$ can be decomposed into the product of a scalar multiplication map, an automorphism, a central quasi-automorphism and two extremal quasi-automorphisms.*

Proof. Let $\varphi \in Q\operatorname{Aut}(\mathscr{N})$ with $\bar{\varphi}$ such that $[\varphi(x), \varphi(y)] = \bar{\varphi}([x,y])$, for all $x, y \in \mathscr{N}$. (i) is obvious. When $n = 3$, we may assume (by Lemmas 2.1–2.2) that

$$\bar{\varphi}(e_{13}) = r\varphi(e_{13}), \quad \overline{\varphi^{-1}}(e_{13}) = s\varphi^{-1}(e_{13}).$$

Thus $rs = 1$, showing that r is invertible. Let $\varphi = \eta_{r^{-1}} \circ \varphi$. Then we have

$$\bar{\varphi}(e_{13}) = \overline{\eta_{r^{-1}}} \circ \bar{\varphi}(e_{13}) = r^{-2} r \varphi(e_{13}) = r^{-1}\varphi(e_{13}) = \varphi(e_{13}).$$

For any $x = (x_{ij})$, $y = (y_{ij}) \in \mathscr{N}$, $[x, y] = (x_{12}y_{23} - y_{12}x_{23})e_{13}$. Then

$$\begin{aligned}
\varphi([x, y]) &= (x_{12}y_{23} - y_{12}x_{23})\varphi(e_{13}) \\
&= (x_{12}y_{23} - y_{12}x_{23})\overline{\varphi}(e_{13}) \\
&= (x_{12}y_{23} - y_{12}x_{23})[\varphi(e_{12}), \varphi(e_{23})] \\
&= [\varphi(x), \varphi(y)].
\end{aligned}$$

This implies that φ is an automorphism of $\mathscr{N}$. So $\varphi = \eta_r \circ \varphi$, as desired.

Now we consider the case that $n \geqslant 5$. Assume as before that $\overline{\varphi}(e_{13}) \equiv r\varphi(e_{13}) + a\varphi(e_{2n}) \pmod{Z}$. First, we prove that r is invertible. The equation (2.1) shows that $(\varphi^{-1} \circ \overline{\varphi})(e_{13}) \equiv re_{13} + ae_{2n} \pmod{Z}$. By Lemma 2.7, we have $(\varphi^{-1} \circ \overline{\varphi})(\mathscr{N}_1) = \varphi^{-1}(\mathscr{N}_1) = \mathscr{N}_1$. So there exists an element z in $\mathscr{N}_1$, written as $z = \sum_{j-i\geqslant 2} a_{ij}e_{ij}$, such that $(\varphi^{-1} \circ \overline{\varphi})(z) = e_{13}$. Applying (2.1)–(2.3), we have $rz + a_{13}ae_{2n} + a_{n-2,n}be_{1,n-1} \equiv e_{13} \pmod{Z}$. This implies that $ra_{13} = 1$, and r is invertible.

Denote $\eta_{r^{-1}} \circ \varphi$ by φ_1. Then for $(i, j) \in \Omega - \{(1, 3), (n-2, n)\}$,

$$\begin{aligned}
\overline{\varphi_1}(e_{ij}) &= \overline{\eta_{r^{-1}}} \circ \bar{\varphi}(e_{ij}) \\
&\equiv \overline{\eta_{r^{-1}}}(r\varphi(e_{ij})) \pmod{Z} \\
&\equiv r^{-2} r\varphi(e_{ij}) \pmod{Z} \\
&\equiv r^{-1}\varphi(e_{ij}) \pmod{Z} \\
&\equiv \varphi_1(e_{ij}) \pmod{Z}.
\end{aligned}$$

A similar discussion shows that

$$\begin{aligned}
\overline{\varphi_1}(e_{13}) &\equiv \varphi_1(e_{13}) + r^{-1}a\varphi_1(e_{2n}) \pmod{Z}; \\
\overline{\varphi_1}(e_{n-2,n}) &\equiv \varphi_1(e_{n-2,n}) + r^{-1}b\varphi_1(e_{1,n-1}) \pmod{Z}.
\end{aligned}$$

Denote $\varphi_1 \circ \psi_{a/(2r)} \circ \chi_{b/(2r)}$ by φ_2. Then we have

$$\begin{aligned}
\overline{\varphi_2}(e_{13}) &= \overline{\varphi_1}\Big(e_{13} - \frac{a}{2r}e_{2n}\Big) \\
&\equiv \varphi_1(e_{13}) + r^{-1}a\varphi_1(e_{2n}) - \frac{a}{2r}\varphi_1(e_{2n}) \pmod{Z} \\
&\equiv \varphi_1(e_{13}) + \frac{a}{2r}\varphi_1(e_{2n}) \pmod{Z} \\
&\equiv \varphi_2(e_{13}) \pmod{Z}.
\end{aligned}$$

Similarly, we have

$$\overline{\varphi_2}(e_{n-2,n}) \equiv \varphi_2(e_{n-2,n}) \pmod{Z}.$$

Now we have

$$\overline{\varphi_2}(e_{ij}) \equiv \varphi_2(e_{ij}) \pmod{Z} \quad \text{for } (i,j) \in \Omega.$$

Since Z can be spanned by $\varphi_2(e_{1n})$ (recalling Lemma 2.1), we may assume that

$$\overline{\varphi_2}(e_{ij}) = \varphi_2(e_{ij}) + s_{ij}\varphi_2(e_{1n}) \quad \text{for } (i,j) \in \Psi.$$

Now $(\varphi_2^{-1} \circ \overline{\varphi_2})(e_{1n}) = (1+s_{1n})e_{1n}$. Since $(\varphi_2^{-1} \circ \overline{\varphi_2})(Z) = Z$ we may suppose $(\varphi_2^{-1} \circ \overline{\varphi_2})(te_{1n}) = e_{1n}$. Then we obtain $t(1+s_{1n}) = 1$, showing that $1+s_{1n}$ is invertible. Let f be the linear function defined in the way $f\colon\ x = (x_{ij}) \in \mathscr{N} \mapsto \sum_{(i,j)\in\Psi} x_{ij}s_{ij}$. Then $1 + f(e_{1n}) = 1 + s_{1n}$ is invertible, and $\overline{\varphi_2}(x) = \varphi_2(x) + f(x)\varphi_2(e_{1n})$. Using f we define the central quasi-automorphism θ_f. Let $\varphi = \varphi_2 \circ \theta_f$. Then we have that

$$\overline{\varphi}(x) = \varphi_2(x) + f(x)\varphi_2(e_{1n}) = \varphi(x), \quad \forall\, x \in \mathscr{N}_1.$$

This implies that the restriction of φ to $\mathscr{N}_1$ is consistent to $\overline{\varphi}$. So

$$\varphi([x,y]) = \overline{\varphi}([x,y]) = [\varphi(x), \varphi(y)].$$

This means that φ is an automorphism of $\mathscr{N}$. Finally, we have that

$$\varphi = \eta_r \circ \varphi \circ \theta_{-f} \circ \chi_{-b/(2r)} \circ \psi_{-a/(2r)}.$$

□

3. Commutativity preserving maps on $\mathscr{N}$

Based on the characterization of quasi-automorphisms of $\mathscr{N}$, we can now describe the commutativity preserving maps on $\mathscr{N}$.

Lemma 3.1. *An invertible linear map σ on $\mathscr{N}$ preserves commutativity in both directions if and only if σ is a quasi-automorphism of $\mathscr{N}$, i.e.,* $\mathrm{Inv}_c(\mathscr{N}) = Q\,\mathrm{Aut}(\mathscr{N})$.

Proof. Obviously, $Q\,\mathrm{Aut}(\mathscr{N})$ is a subgroup of $\mathrm{Inv}_c(\mathscr{N})$. Let φ be an invertible linear map on $\mathscr{N}$ preserving commutativity in both directions. To show that φ is a quasi-automorphism it is necessary to find an invertible linear map $\bar{\varphi}\colon\ \mathscr{N}_1 \to \mathscr{N}_1$ such that

$$\bar{\varphi}([x,y]) = [\varphi(x), \varphi(y)], \quad \forall\, x, y \in \mathscr{N}.$$

If $n = 2$ we can take $\bar{\varphi}$ to be the identity map on $\mathscr{N}_1$. Now we consider the case when $n \geqslant 3$. Naturally, we define $\bar{\varphi}$ on e_{ij}, for $(i,j) \in \Psi$, by

$$\bar{\varphi}(e_{ij}) = [\varphi(e_{i,i+1}), \varphi(e_{i+1,j})] \quad \text{for } (i,j) \in \Psi,$$

and extend it linearly to the whole $\mathscr{N}_1$. More definitely,

$$\bar{\varphi}\Bigg(\sum_{(i,j)\in\Psi} a_{ij}e_{ij}\Bigg) = \sum_{(i,j)\in\Psi} a_{ij}[\varphi(e_{i,i+1}), \varphi(e_{i+1,j})].$$

We now verify the following assertions.

Assertion 1. $\bar{\varphi}([e_{ij}, e_{kl}]) = [\varphi(e_{ij}), \varphi(e_{kl})]$ for $(i,j), (k,l) \in \Phi$.

If $[e_{ij}, e_{kl}] = 0$, the assertion obviously holds. Otherwise, if $j = k$, then by the definition of $\bar{\varphi}$,

$$\bar{\varphi}([e_{ij}, e_{kl}]) = \bar{\varphi}(e_{il}) = [\varphi(e_{i,i+1}), \varphi(e_{i+1,l})].$$

As $[e_{i,i+1} - e_{ij}, e_{i+1,l} + e_{jl}] = 0$ we have

$$[\varphi(e_{i,i+1}) - \varphi(e_{ij}), \varphi(e_{i+1,l}) + \varphi(e_{jl})] = 0.$$

Since $[\varphi(e_{i,i+1}), \varphi(e_{jl})] = [\varphi(e_{ij}), \varphi(e_{i+1,l})] = 0$, we have

$$[\varphi(e_{i,i+1}), \varphi(e_{i+1,l})] = [\varphi(e_{ij}), \varphi(e_{jl})].$$

Thus the assertion follows. If $i = l$, by using the result just obtained, we have

$$\bar{\varphi}([e_{ij}, e_{ki}]) = -\bar{\varphi}([e_{ki}, e_{ij}] = -[\varphi(e_{ki}), \varphi(e_{ij})] = [\varphi(e_{ij}), \varphi(e_{ki})].$$

The assertion also holds.

Assertion 2. $\bar{\varphi}([x,y]) = [\varphi(x), \varphi(y)]$, for $x, y \in \mathscr{N}$.

Express $x, y \in \mathscr{N}$ as $x = \sum\limits_{1\leqslant i<j\leqslant n} x_{ij}e_{ij}$, $y = \sum\limits_{1\leqslant k<l\leqslant n} y_{kl}e_{kl}$. Then

$$\begin{aligned}\bar{\varphi}([x,y]) &= \sum_{1\leqslant i<j\leqslant n}\ \sum_{1\leqslant k<l\leqslant n} x_{ij}y_{kl}\bar{\varphi}([e_{ij}, e_{kl}]) \\ &= \sum_{1\leqslant i<j\leqslant n}\ \sum_{1\leqslant k<l\leqslant n} x_{ij}y_{kl}[\varphi(e_{ij}), \varphi(e_{kl})] \\ &= [\varphi(x), \varphi(y)].\end{aligned}$$

Hence, φ is a quasi-automorphism of $\mathscr{N}$. $\square$

Applying Lemma 3.1 and Theorem 2.7 we immediately obtain the main result of this paper as follows.

Theorem 3.2.

(i) *If $n = 2$, then every invertible linear map on $\mathcal{N}$ preserves commutativity in both directions.*

(ii) *If $n = 3$, then each invertible linear map on $\mathcal{N}$ preserving commutativity in both directions can be decomposed into the product of a scalar multiplication map and an automorphism.*

(iii) *If $n \geqslant 5$ and $2 \in R^*$, then each invertible linear map on $\mathcal{N}$ preserving commutativity in both directions can be decomposed into the product of a scalar multiplication map, an automorphism, a central quasi-automorphism and two extremal quasi-automorphisms.*

Remark. Note that the automorphisms of $\mathcal{N}$ have been completely characterized by Cao et al. in [3]. So the commutativity preserving maps on $\mathcal{N}$ are completely characterized provides that $n \neq 4$.

In Theorem 3.2 the case that $n = 4$ is left unsolved. For this special case, we shall directly characterize the mapping $\varphi \in \mathrm{Inv}_c(\mathcal{N})$, without using quasi-automorphisms. In the sequel, $n = 4$ is always assumed. First, we introduce some standard mappings on $\mathcal{N}$.

(i) Let x be an invertible upper triangular matrix over R. The map $\sigma_x\colon\ y \mapsto xyx^{-1}$ is verified to be an automorphism of $\mathcal{N}$. Thus $\sigma_x \in \mathrm{Inv}_c(\mathcal{N})$.

(ii) For $s, t \in R$, we define $\varrho_{s,t}$ on $\mathcal{N}$ by $\varrho_{s,t}\colon\ x = (x_{ij}) \mapsto x + sx_{12}e_{24} + tx_{34}e_{13}$. Then it is easy to verify that $\varrho_{s,t}$ is an automorphism of $\mathcal{N}$. Thus, $\varrho_{s,t} \in \mathrm{Inv}_c(\mathcal{N})$.

(iii) Let $r \in R^*$. We define τ_r on $\mathcal{N}$ by $\tau_r\colon\ x = (x_{ij}) \mapsto x + (r-1)(x_{13}e_{13} + x_{24}e_{24})$. It is verified that $\tau_r \in \mathrm{Inv}_c(\mathcal{N})$ with the inverse $\tau_{r^{-1}}$. Obviously, τ_r is an automorphism if and only if $r = 1$.

(iv) Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an invertible 2×2 matrix over R and let $\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$ be its inverse. Define $\lambda_{a,b,c,d}$ on $\mathcal{N}$ by

$$\begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ 0 & 0 & x_{23} & x_{24} \\ 0 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & ax_{12} + cx_{34} & ax_{13} + cx_{24} & x_{14} \\ 0 & 0 & x_{23} & dx_{24} + bx_{13} \\ 0 & 0 & 0 & dx_{34} + bx_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then it is not difficult to check that $\lambda_{a,b,c,d} \in \mathrm{Inv}_c(\mathcal{N})$ with the inverse $\lambda_{a_1,b_1,c_1,d_1}$.

Theorem 3.3. *Let $n = 4$, and let φ be an invertible linear map on $\mathcal{N}$ preserving commutativity in both directions. Then*

$$\varphi = \lambda_{a,b,c,d} \circ \sigma_z \circ \varrho_{s,t} \circ \tau_r \circ \theta_f,$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *is an invertible matrix over* R, $\lambda_{a,b,c,d}$, σ_z, $\varrho_{s,t}$ *and* τ_r *are maps defined in* (iv), (i), (ii) *and* (iii) *above, respectively, and* θ_f *is a central quasi-automorphism (as defined in Section* 2*).*

Proof. By Lemma 3.1, φ is a quasi-automorphism of $\mathscr{N}$. Lemma 2.7 says that $\varphi(\mathscr{N}_1) = \mathscr{N}_1$. Assume that

$$\begin{aligned} \varphi(e_{13}) &\equiv ae_{13} + be_{24} \pmod{Z}, \quad \varphi(e_{24}) \equiv de_{24} + ce_{13} \pmod{Z}; \\ \varphi^{-1}(e_{13}) &\equiv a_1e_{13} + b_1e_{24} \pmod{Z}, \quad \varphi^{-1}(e_{24}) \equiv d_1e_{24} + c_1e_{13} \pmod{Z}. \end{aligned}$$

It follows from $(\varphi^{-1} \circ \varphi)(e_{13}) = e_{13}$ and $(\varphi^{-1} \circ \varphi)(e_{24}) = e_{24}$ that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This implies that $\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$ is invertible. Using this matrix we define the mapping $\lambda_{a_1,b_1,c_1,d_1}$, and denote $\lambda_{a_1,b_1,c_1,d_1} \circ \varphi$ by φ_1. Then

$$\varphi_1(e_{13}) \equiv e_{13} \pmod{Z} \quad \text{and} \quad \varphi_1(e_{24}) \equiv e_{24} \pmod{Z}.$$

Now we consider the action of φ_1 on $e_{i,i+1}$ for $i = 1, 2, 3$. Suppose that

$$\varphi_1(e_{i,i+1}) \equiv \sum_{k=1}^{3} a_{k,k+1}^{(i)} e_{k,k+1} \pmod{\mathscr{N}_1}.$$

Since

$$[\varphi_1(e_{12}), \varphi_1(e_{13})] = [\varphi_1(e_{34}), \varphi_1(e_{24})] = [\varphi_1(e_{23}), \varphi_1(e_{13})] = [\varphi_1(e_{23}), \varphi_1(e_{24})] = 0,$$

we have

$$a_{34}^{(1)} = a_{12}^{(3)} = a_{34}^{(2)} = a_{12}^{(2)} = 0.$$

Now, since φ_1 is invertible, one can easily find that $a_{23}^{(2)}$ is invertible. Now suppose that $\varphi_1^{-1}(e_{12}) \equiv c_{12}^{(1)}e_{12} + c_{23}^{(1)}e_{23} \pmod{\mathscr{N}}$ and $\varphi_1^{-1}(e_{23}) \equiv c_{23}^{(2)}e_{23} \pmod{\mathscr{N}}$. By $(\varphi_1^{-1} \circ \varphi_1)(e_{12}) = e_{12}$ we have $a_{12}^{(1)}c_{12}^{(1)} = 1$, which implies that $a_{12}^{(1)}$ is invertible. Similarly, $a_{34}^{(3)}$ is invertible. Now following from $[\varphi_1(e_{12}), \varphi_1(e_{34})] = 0$, we have $a_{12}^{(1)}a_{23}^{(3)} = a_{34}^{(3)}a_{23}^{(1)} = 0$, which further leads to

$$a_{23}^{(3)} = a_{23}^{(1)} = 0.$$

Take $x = \mathrm{diag}(1, a_{12}^{(1)}, a_{12}^{(1)} a_{23}^{(2)}, a_{12}^{(1)} a_{23}^{(2)} a_{34}^{(3)})$, then

$$(\sigma_x \circ \varphi_1)(e_{i,i+1}) \equiv e_{i,i+1} \pmod{\mathscr{N}_1} \quad \text{for } i = 1, 2, 3.$$

Now one must note that

$$(\sigma_x \circ \varphi_1)(e_{13}) \equiv (a_{12}^{(1)} a_{23}^{(2)})^{-1} e_{13} \pmod{Z}$$

and

$$(\sigma_x \circ \varphi_1)(e_{24}) \equiv (a_{23}^{(2)} a_{34}^{(3)})^{-1} e_{24} \pmod{Z}.$$

Denote $\sigma_x \circ \varphi_1$ by φ_2, and assume

$$\varphi_2(e_{i,i+1}) \equiv e_{i,i+1} + b_{13}^{(i)} e_{13} + b_{24}^{(i)} e_{24} \pmod{Z} \quad \text{for } i = 1, 2, 3, \dots .$$

By $[\varphi_2(e_{12}), \varphi_2(e_{34})] = 0$ we obtain $b_{24}^{(3)} = -b_{13}^{(1)}$. Taking

$$y = \begin{pmatrix} 1 & -b_{13}^{(2)} & 0 & 0 \\ 0 & 1 & b_{13}^{(1)} & 0 \\ 0 & 0 & 1 & b_{24}^{(2)} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we have that

$$(\sigma_y \circ \varphi_2)(e_{12}) \equiv e_{12} + b_{24}^{(1)} e_{24} \pmod{Z}; \quad (\sigma_y \circ \varphi_2)(e_{23}) \equiv e_{23} \pmod{Z};$$
$$(\sigma_y \circ \varphi_2)(e_{34}) \equiv e_{34} + b_{13}^{(3)} e_{13} \pmod{Z}.$$

Further, we find that

$$(\varrho_{-s,-t} \circ \sigma_y \circ \varphi_2) \equiv e_{i,i+1} \pmod{Z} \quad \text{for } i = 1, 2, 3, \dots,$$

where $s = b_{24}^{(1)}$, $t = b_{13}^{(3)}$. Denote $\varrho_{-s,-t} \circ \sigma_y \circ \varphi_2$ by φ_3. Now, it has been proved that

$$\varphi_3(e_{i,i+1}) \equiv e_{i,i+1} \pmod{Z} \quad \text{for } i = 1, 2, 3, \dots .$$

One should note that

$$\varphi_3(e_{13}) \equiv r e_{13} \pmod{Z} \quad \text{and} \quad \varphi_3(e_{24}) \equiv q e_{24} \pmod{Z},$$

where $r = (a_{12}^{(1)} a_{23}^{(2)})^{-1}$, $q = (a_{23}^{(2)} a_{34}^{(3)})^{-1}$. By $[e_{13} + e_{12}, e_{34} - e_{24}] = 0$ we have that $[\varphi_3(e_{13}) + \varphi_3(e_{12}), \varphi_3(e_{34}) - \varphi_3(e_{24})] = 0$, which leads to $r = q$. Denote $\tau_{r^{-1}} \circ \varphi_3$

by φ_4. Then we have $\varphi_4(e_{ij}) \equiv e_{ij} \pmod Z$ for all $(i,j) \in \Phi - \{(1,n)\}$. Assume that

$$\varphi_4(e_{ij}) = e_{ij} + r_{ij}e_{1n} \quad \forall\, (i,j) \in \Phi.$$

A similar discussion (as in the proof of Lemma 2.8) shows that $1 + r_{1n}$ is invertible. Let f be the linear function defined by $f\colon\ x = (x_{ij}) \in \mathscr{N} \mapsto \sum_{(i,j)\in\Phi} x_{ij}r_{ij}$. Then $1 + f(e_{1n}) = 1 + r_{1n}$ is invertible, and $\varphi_4(x) = x + f(x)e_{1n}$ for $x \in \mathscr{N}$. This implies that φ_4 is exactly the central quasi-automorphism θ_f of $\mathscr{N}$. Finally, we have that

$$\varphi = \lambda_{a,b,c,d} \circ \sigma_z \circ \varrho_{s,t} \circ \tau_r \circ \theta_f, \quad \text{where } z = x^{-1}y^{-1}.$$

□

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