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OPTIMAL CONTROL SOLUTION FOR PENNES' EQUATION USING STRONGLY CONTINUOUS SEMIGROUP

Alaeddin Malek and Ghasem Abbasi

A distributed optimal control problem on and inside a homogeneous skin tissue is solved subject to Pennes' equation with Dirichlet boundary condition at one end and Rubin condition at the other end. The point heating power induced by conducting heating probe inserted at the tumour site as an unknown control function at specific depth inside biological body is preassigned. Corresponding pseudo-port Hamiltonian system is proposed. Moreover, it is proved that bioheat transfer equation forms a contraction and dissipative system. Mild solution for bioheat transfer equation and its adjoint problem are proposed. Controllability and exponentially stability for the related system is proved. The optimal control problem is solved using strongly continuous semigroup solution and time discretization. Mathematical simulations for a thermal therapy in the presence of point heating power are presented to investigate efficiency of the proposed technique.

Keywords: optimal control, Pennes' bioheat equation, semigroup theory, thermal therapy, hyperthermia

Classification: 93E12, 62A10

1. INTRODUCTION

Temperature raising into the burning ranges $40-45\,^{\circ}\mathrm{C}$ is useful in surgical procedures for selective removal of target tissues. The primary objective of hyperthermia is to raise the temperature of the diseased tissue to a therapeutic value, and then thermally destroy it. The microwave, the ultrasound, and the laser are popular apparatus used to deposit a spatial heating for treating the tumour in the deep biological body. Deng et al. performed several closed form of analytical solutions in bioheat transfer problems, with transient heating on the skin surface or inside biological bodies by inserting a heating probe at the tumour region using Green's function method [5]. Malek and his co-authors [14, 16, 17] developed various kinds of finite difference techniques and pseudo-spectral or collocation discretization to solve the 1D, 2D and 3D heat transfer equations for problems that obey Fourier and non-Fourier laws. Karaa et al. [12] have developed an implicit numerical study of a 3D bioheat transfer problem with different

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spatial heating. Deng and his co-author [4] have presented several closed form analytical solutions to 3D bioheat transfer problems with or without phase change. In the recent years Heidari and his co-authors [10, 11] have performed several closed forms of analytical solutions for different heat transfer problems. Malek et al. [15] proposed an efficient mathematical algorithm for solving an optimal control problem of the Burgers' equation by using conjugate gradient method. In hyperthermia treatment, the temperature of the tumour inside the tissue is raised to its beneficial therapeutic value at the final time of treatment cycle. This process must be done without overheating the healthy tissue by controlling microwave power level and surface cooling temperature. A thermal dose optimization problem in hyperthermia was studied analytically by Loulou and his co-author [13] using conjugate gradient method via the numerically solutions of the direct, adjoint and variation problems in finite control volume method. With the aid of conjugate gradient method, a distributed optimal control problem for a system described by bioheat equation in a homogeneous plane tissue due to induced microwave was investigated by Dhar et al. [6, 7, 8] via the numerical solution of the corresponding problem in finite sin transform method. An inverse problem of temperature optimization in hyperthermia was investigated numerically by Aghayan et al. [1] via controlling the overall heat transfer coefficient of the cooling system using conjugate gradient method. Cheng et al. [3] performed an investigation for fast temperature optimization for heating system in hyperthermia with large number of physical sources with the aid of the successive over-relaxation finite difference method. In this paper, approximately controllability of the bioheat transfer equation is proved by strongly continuous semigroup (C_0-semigroup) theory. It is shown that C_0-semigroup generated by Pennes' equation is a contraction semigroup and exponentially stable. It means that, the solution to the Pennes' equation converges to zero exponentially as $t \to \infty$. It is proved that, the Pennes' equation is dissipative and pseudo-port Hamiltonian. Thus, the derivative of the energy system along solution is less or equal to the product of the input (heating power) times the output (temperature response). If we apply no heating power and surface cooling to the system, the energy will remain constant. Analytical form solution of the Pennes' equation (Bioheat equation) and its adjoint problem in the presence of internal heat source and surface cooling temperature is proposed. Optimal control problem in the piecewise function space is solved mathematically by using conjugate gradient method and C_0-semigroup theory.

2. PROBLEM DESCRIPTION

In bioheat transfer for the one-dimensional problem, heat propagates in the direction perpendicular to the tissue surface. For a point conducting heating probe at $z = z_1$ inside of the domain (where the tumour is located), the heat transfer process can be expressed by the Pennes' bioheat transfer equation as [5],

$$\rho c \frac{\partial T(z,t)}{\partial t} = k \frac{\partial^2 T(z,t)}{\partial z^2} + w(T_a - T(z,t)) + Q(t)\delta(z - z_1) + Q_m, \tag{1}$$

$$T(z,t) = T_0, \qquad \text{on} \quad t = 0, \quad 0 \le z \le l, \tag{2}$$

$$k \frac{\partial T(z,t)}{\partial z} = h\left(T(z,t) - u\right), \qquad \text{on} \quad z = 0, \quad 0 < t \le t_f, \tag{3}$$

$$T(z,t) = T_a, \qquad \text{on} \quad z = l, \quad 0 < t \le t_f, \tag{4}$$

where T_0 and T_a are initial and arterial temperatures at the boundary, the parameters ρ, c, k are, the density, heat capacity, and thermal conductivity of the tissue, w is product of flow and heat capacity of blood, Q_m is the rate of metabolic heat generation, u is the constant temperature of the surface cooling medium, h is the heat transfer coefficient between the skin and the ambient air. l is the thickness of the plate, t_f is the total time of the process. The point-heating source is $Q(t)\delta(z-z_1)$ where, $Q(t)$ is the heating power, $\delta(z-z_1)$ is the Dirac function, z_1 is the position of the point-heating source. With a specified point-heating source $Q(t)\delta(z-z_1)$, the temperature response $T(x,t)$ can be computed in the whole depth of the tissue $z \in [0,l]$. Figure 1 reflects a typical cancer hyperthermia where an apparatus deposits heat through inserting a heating probe in the deep tumor site while a surface cooling water is simultaneously adopted to prevent the surface healthy tissues from possible burn injury.

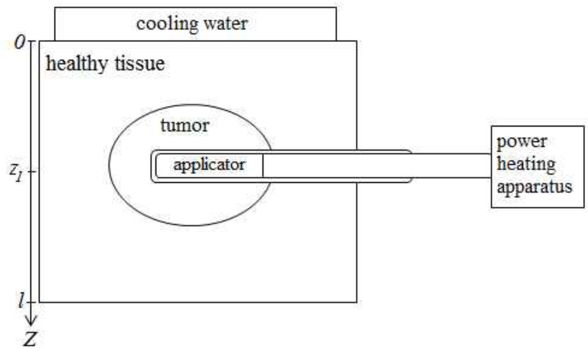


Fig. 1. An illustration of a hyperthermia configuration.

Here, we propose a robust procedure to solve inverse problem of Pennes' equation with the help of semigroups theory. For T_d as the desired thermal at the depth point $z = z_1$, where the tumour is located, by controlling the heating power $Q(t)$ induced by conducting heating probe, the Pennes' equation can be recast as an optimal control problem. It is obvious that this will happen at the end of thermal treatment cycle $t = t_f$[5]. Then the hyperthermia problem is reduced to how to choose heating power $Q(t)$ such that it minimize the following objective functional

$$\mathcal{J}(Q) = \frac{1}{2}\int_0^l (T_d - T(z,t_f))^2\, \delta(z-z_1)\, \mathrm{d}z + \frac{1}{2}\int_0^{t_f} Q^2(t)\, \mathrm{d}t, \tag{5}$$

subject to the constraints defined by problem (1)$-$(4). The first term designates the square deviation of the temperature T_d from $T(z,t_f)$ at $z = z_1$.

Let us to introduce functional $\mathcal{L}$

$$\mathcal{L} = \frac{1}{2}\int_0^l (T_d - T(z,t_f))^2\delta(z-z_1)\, \mathrm{d}z + \frac{1}{2}\int_0^{t_f} Q^2(t)\, \mathrm{d}t + \int_0^{t_f}\int_0^l \psi(z,t)$$
$$\times \left(\rho c\frac{\partial T(z,t)}{\partial t} - k\frac{\partial^2 T(z,t)}{\partial z^2} - w(T_a - T(z,t)) - Q(t)\delta(z-z_1) - Q_m\right) \mathrm{d}z\mathrm{d}t,$$

where $\psi(z,t)$ is the auxiliary function. By considering Q_m and u as constants, the first variation of the functional $\mathcal{L}$ can be written as

$$\Delta\mathcal{L} = -\int_0^l (T_d - T(z,t_f))\,\delta(z-z_1)\Delta T(z,t_f)\,\mathrm{d}z + w\int_0^{t_f}\int_0^l \psi(z,t)\Delta T(z,t)\,\mathrm{d}z\,\mathrm{d}t \quad (6)$$

$$-\int_0^{t_f}\left(k\frac{\partial\psi(0,t)}{\partial z} - h\psi(0,t)\right)\Delta T(0,t)\,\mathrm{d}t + k\int_0^{t_f}\frac{\partial}{\partial z}\psi(l,t)\Delta T(l,t)\,\mathrm{d}t \quad (7)$$

$$-k\int_0^{t_f}\int_0^l \frac{\partial^2}{\partial z^2}\psi(z,t)\Delta T(z,t)\,\mathrm{d}z\,\mathrm{d}t - \int_0^{t_f}\int_0^l \psi(z,t)\delta(z-z_1)\Delta Q(t)\,\mathrm{d}z\,\mathrm{d}t \quad (8)$$

$$-k\int_0^{t_f}\psi(l,t)\Delta T_z(l,t)\,\mathrm{d}t - \rho c\int_0^l \psi(z,0)\Delta T(z,0)\,\mathrm{d}z + \int_0^{t_f} Q(t)\Delta Q(t)\,\mathrm{d}t \quad (9)$$

$$+\rho c\int_0^l \psi(z,t_f)\Delta T(z,t_f)\,\mathrm{d}z - \rho c\int_0^l\int_0^{t_f}\frac{\partial\psi(z,t)}{\partial t}\Delta T(z,t)\,\mathrm{d}t\,\mathrm{d}z. \quad (10)$$

Now, according to Eqs. (2) and (4) taking $\Delta T(z,0)$, $\Delta T(l,t)$ equal to zero, and assuming $\Delta\mathcal{L}$ to vanish for any $\Delta T_z(l,t)$, $\Delta T(z,t)$, $\Delta T(0,t)$, $\Delta T(z,t_f)$ and $\Delta Q(t)$, auxiliary function $\psi(z,t)$ can be represented as an adjoint function. Then $\psi(z,t)$ is the solution of the following adjoint problem,

$$\rho c\frac{\partial\psi(z,t)}{\partial t} = -k\frac{\partial^2\psi(z,t)}{\partial z^2} + w\psi(z,t), \quad (11)$$

$$\psi(z,t) = (T_d - T(z,t))\,\delta(z-z_1), \qquad \text{on} \quad t=t_f, \quad 0\le z\le l, \quad (12)$$

$$k\frac{\partial\psi(z,t)}{\partial z} = h\psi(z,t), \qquad \text{on} \quad z=0, \quad 0<t\le t_f, \quad (13)$$

$$\psi(z,t) = 0, \qquad \text{on} \quad z=l, \quad 0<t\le t_f. \quad (14)$$

According to Pennes' equation and its adjoint problem all of the terms in right hand side of (6) will vanish unless

$$\int_0^{t_f}\left(Q(t) - \int_0^l \psi(z,t)\delta(z-z_1)\,\mathrm{d}z\right)\Delta Q(t)\,\mathrm{d}t.$$

By Conjugate gradient method the optimal function for the heating power $Q(t)$ stands as [13]

$$Q(t) = \int_0^l \psi(z,t)\delta(z-z_1)\,\mathrm{d}z = \psi(z_1,t). \quad (15)$$

Now, if the optimal heating power $\psi(z_1,t)$ is adopted, the position for the highest temperature will just stay at the site of the point source will happen. By this way we have desired temperature T_d at the final time t_f in the position z_1. This is beneficial for hyperthermia therapy since design makers can then selectively program the deep regional tumour heating strategy. In the next section, we give some properties of generator and C_0-semigroup corresponding to the bioheat transfer equation, e. g., exponentially stable and contraction and dissipative.

3. STRONGLY CONTINUOUS SEMIGROUP THEORY

Define $T(z,t) = U(z,t) + V(z)$ for

$$V(z) = \frac{h}{k+lh}\left((T_a - u)z + (\frac{kT_a}{h} + lu)\right), \tag{16}$$

where $V(z)$ satisfy in the boundary conditions (3) and (4). Then $U(z,t)$ satisfies in the following problem

$$\frac{\partial U(z,t)}{\partial t} = \frac{k}{\rho c}\frac{\partial^2 U(z,t)}{\partial z^2} - \frac{w}{\rho c}U(z,t) + \frac{Q(t)}{\rho c}\delta(z - z_1) + \frac{1}{\rho c}\left(w(T_a - V(z)) + Q_m\right), \tag{17}$$

$$U(z,t) = T_0 - V(z), \qquad \text{on} \quad t = 0, \quad 0 \le z \le l, \tag{18}$$

$$k\frac{\partial U(z,t)}{\partial z} = hU(z,t), \qquad \text{on} \quad z = 0, \quad 0 < t \le t_f, \tag{19}$$

$$U(z,t) = 0, \qquad \text{on} \quad z = l, \quad 0 < t \le t_f. \tag{20}$$

For $t \in (0, t_f)$, choose $\boldsymbol{L}_2(0,l)$ as the state space. Let the trajectory segment $U(\cdot,t) = \{U(z,t) | \, 0 \le z \le l\}$ be the state and regard heating power $Q(t)$ as the input.

Corollary 3.1. If

$$AU = \frac{1}{\rho c}\left(k\frac{\mathrm{d}^2U}{\mathrm{d}z^2} - wU\right) \quad \text{and} \quad B = \frac{\delta(z - z_1)}{\rho c}I,$$

$$\boldsymbol{D}(A) = \{U(\cdot,t) \in \boldsymbol{L}_2(0,l) | \, U \text{ and } \frac{\mathrm{d}U}{\mathrm{d}z} \text{ are absolutely continuous,} \tag{21}$$

$$\frac{\mathrm{d}^2U}{\mathrm{d}z^2} \in \boldsymbol{L}_2(0,l) \text{ and } U \text{ satisfies in boundary conditions (19) and (20)}\},$$

where I is identity operator, then problem (17) – (20) form an inhomogeneous abstract differential equation

$$\frac{\partial U(z,t)}{\partial t} = AU(z,t) + BQ(t) + f(z), \ \ t \ge 0, \ U(z,0) = T_0 - V(z), \tag{22}$$

where $f(z) = \frac{1}{\rho c}(w(T_a - V(z)) + Q_m)$.

Theorem 3.2. The eigenvalues and corresponding eigenfunctions for the operator A defined in (21) are given by

$$\lambda_n = -\frac{w + kp_n^2}{\rho c},$$

$$U_n = \sin p_n(l - z), \quad 0 \le z \le l, \ n = 1, 2, \ldots$$

in which p_n for $n = 1, 2, \ldots$, are positive real roots of,

$$p\cot(pl) = \frac{-h}{k}. \tag{23}$$

Proof. The eigenvalue problem

$$AU = \lambda U,$$

is equivalent to

$$\frac{1}{\rho c}\Big(k\frac{\mathrm{d}^2U}{\mathrm{d}z^2} - wU\Big) = \lambda U. \tag{24}$$

Thus, from Corollary 3.1,

$$\{\sin p_n(l-z)\,|\,0 \le z \le l\}, \tag{25}$$

forms an orthogonal basis for $U(z,t)$ with the inner product $\langle\cdot,\cdot\rangle$ on $\boldsymbol{L}_2(0,l)$ space. Substituting basis (25) in Eq. (24) yields

$$\lambda_n = -\frac{1}{\rho c}(kp_n^2 + w), \quad U_n = \sin p_n(l-z),$$

which proves the theorem. □

Theorem 3.3. The linear operator A in (21) is a closed, densely defined and self-adjoint operator.

Proof. It is clear that A is a closed and densely defined. For all $\phi,\psi \in \boldsymbol{D}(A)$ by using integrating by parts, one can show $\langle A\phi,\psi\rangle = \langle\phi, A\psi\rangle$ and for adjoint operator A^* we have $\boldsymbol{D}(A^*) = \boldsymbol{D}(A)$. □

Theorem 3.4. Let $\{\phi_n|\, n = 1,2,\ldots\}$ be an orthonormal basis for eigenfunctions of operator A. Then for

$$S(t)U = \sum_{n=1}^{\infty} e^{\lambda_n t}\langle U,\phi_n\rangle\phi_n(z), \tag{26}$$

$(S(t))_{t\ge0}$ forms the C_0-semigroup with the unique infinitesimal generator A.

Proof. The growth bound of operator A: $\omega_0 = \sup\{Re(\lambda)|\,\lambda \in \sigma(A)\}$, is a negative value thus, $(S(t))_{t\ge0}$ forms a C_0-semigroup. From the Hille–Yosida Theorem [2], A generates the unique C_0-semigroup $(S(t))_{t\ge0}$. □

Corollary 3.5. The C_0-semigroup $(S(t))_{t\ge0}$ is a contraction C_0-semigroup. In other words, $\|S(t)\| \le 1$ for every $t \ge 0$.

Theorem 3.6. Suppose that A is the infinitesimal generator of the C_0-semigroup $(S(t))_{t\ge0}$ on the Hilbert space $\boldsymbol{L}_2(0,l)$. Then the $(S(t))_{t\ge0}$ is exponentially stable.

Proof. From [2] it is sufficient to show that, there exists a positive operator $P \in \boldsymbol{L}_2(0,l)$ such that

$$\langle AU, PU\rangle + \langle PU, AU\rangle \le -\langle U,U\rangle, \qquad \text{for all } U \in \boldsymbol{D}(A).$$

Let $P = \frac{\rho c}{2w} I$ as a positive operator on $\boldsymbol{L}_2(0,l)$ where ρ, c, w are positive. Thus,

$$\begin{aligned}
\langle AU, PU\rangle + \langle PU, AU\rangle &= \frac{\rho c}{w}\langle AU, U\rangle \\
&= \frac{\rho c}{w}\left(-\frac{k}{\rho c}\left.\frac{\partial U(z,t)}{\partial z}\right|_{z=0} U(0,t) - \frac{k}{\rho c}\langle \frac{\partial U}{\partial z}, \frac{\partial U}{\partial z}\rangle - \frac{w}{\rho c}\langle U, U\rangle\right) \\
&= \frac{\rho c}{w}\left(-\frac{h}{\rho c}U^2(0,t) - \frac{k}{\rho c}\langle \frac{\partial U}{\partial z}, \frac{\partial U}{\partial z}\rangle - \frac{w}{\rho c}\langle U, U\rangle\right) \\
&\leq -\langle U, U\rangle
\end{aligned} \tag{27}$$

so $(S(t))_{t\geq 0}$ is exponentially stable. □

Since $(S(t))_{t\geq 0}$ is exponentially stable, then the solution to the abstract differential equation (22) tends to zero exponentially fast as $t \to \infty$.

Corollary 3.7. The infinitesimal generator $A : \boldsymbol{D}(A) \subset \boldsymbol{L}_2(0,l) \to \boldsymbol{L}_2(0,l)$ is dissipative, since from (27)

$$Re\langle AU, U\rangle \leq 0 \quad \text{for all } U \in \boldsymbol{D}(A).$$

Thus, the system (22) can be described as a dissipation system.

Note that an inhomogeneous abstract differential equation (22) is not port-Hamiltonian in the classical sense [9]. However, it can be written as non-standard port-Hamiltonian structure that we call it pseudo-port Hamiltonian system.

Theorem 3.8. The variation of internal energy for the inhomogeneous abstract differential equation (22) is less or equal to the power provided to the system through the boundary and its internal points.

Proof. Define the energy as:

$$\mathcal{H}(U) = \frac{1}{2}\int_0^l U^2(\xi,t)\,\mathrm{d}\xi,$$

then rate of the internal energy with respected to time is as follows:

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} &= \int_0^l U\frac{\partial U}{\partial t}\,\mathrm{d}\xi = \int_0^l U\left(\frac{k}{\rho c}\frac{\partial^2 U}{\partial \xi^2} - \frac{w}{\rho c}U + Q(t)\delta(\xi - z_1) + f(\xi)\right)\mathrm{d}\xi \\
&= \frac{k}{\rho c}\int_0^l [\frac{\partial}{\partial \xi}(U\frac{\partial U}{\partial \xi}) - (\frac{\partial U}{\partial \xi})^2]\,\mathrm{d}\xi - \frac{1}{\rho c}\int_0^l \big(wU^2 - Q(t)\delta(\xi - z_1)U - f(\xi)U\big)\,\mathrm{d}\xi \\
&\leq \frac{k}{\rho c}U\frac{\partial U}{\partial \xi}|_0^l + \frac{1}{\rho c}Q(t)U(z_1,t) + \frac{1}{\rho c}\int_0^l f(\xi)U\,\mathrm{d}\xi.
\end{aligned}$$

This is an energy sub-balance inequality and interprets that variation of internal energy is less or equal to the power provided to the system through (i) The boundary (see, the first right hand side (RHS) term), (ii) Internal source heating (see, the second RHS term) and (iii) Metabolic heat generation, initial, arterial and blood temperature (see,

the third RHS term).
This system is not port-Hamiltonian (energy balance equality does not hold). We call it as pseudo-port Hamiltonian since it has the energy sub-balance inequality property. This proves the theorem. □

In the following, we prove controllability of the bioheat transfer problem.

Theorem 3.9. (Curtain and Zwart [2]) Assume that $\lambda_n = -\frac{w+kp_n^2}{\rho c}$ and $\phi_n(z) = \frac{\sqrt{2}\sin p_n(l-z)}{\sqrt{l-\frac{\sin 2p_n l}{2p_n}}}$ for $n = 1, 2, \ldots$, are eigenvalues and eigenfunctions corresponding to operator A in (21). Let $(S(t))_{t\geq 0}$ form the C_0-semigroup corresponding to A. The state linear system (22) is approximately controllable on $[0, t_f]$ if and only if the following condition holds for a given $U \in \boldsymbol{L}_2(0, l)$,

$$\text{if } B^* S^*(\tau)U = 0 \text{ on } [0, t_f] \text{ then } U = 0, \tag{28}$$

where $B^*, S^*(t)$ are adjoint operators of $B, S(t)$ that is defined in (21) and (26).

Proof. From (21) and (26), one can rewrite $B^* S^*(\tau)U = 0$ as

$$\frac{\delta(z-z_1)}{\rho c}\sum_{n=1}^{\infty}\exp(\lambda_n\tau)\langle U,\phi_n\rangle\phi_n(z) = 0, \text{ on } [0, t_f], \tag{29}$$

or

$$\frac{1}{\rho c}\sum_{n=1}^{\infty}\exp(\lambda_n\tau)\langle U,\phi_n\rangle\delta(z-z_1)\phi_n(z) = 0, \text{ on } [0, t_f]. \tag{30}$$

Thus,

$$\sum_{n=1}^{\infty}\exp(\lambda_n\tau)\langle U,\phi_n\rangle\phi_n(z_1) = 0, \text{ on } [0, t_f]. \tag{31}$$

Since λ_n's are distinct thus, $\{\exp(\lambda_n\tau)\}_{n=1}^{\infty}$ are linearly independent on $[0, t_f]$ therefore

$$\langle U,\phi_n\rangle\phi_n(z_1) = 0, \quad n = 1, 2, \ldots,$$

since $\phi_n(z_1) \neq 0$ thus, $\langle U,\phi_n\rangle = 0$ for all n. This proves $U = 0$, because $\phi_n(z)$ for $n = 1, 2, \ldots$, are orthonormal basis in $\boldsymbol{L}_2(0, l)$. □

In the next section, we compute an optimal strategy for inserting heating power $Q(t)$ using mild solutions for the problems given by (1) – (4) and (11) – (14).

4. OPTIMAL CONTROL SOLUTION

The following theorem gives conditions under which the mild solution for inhomogeneous abstract differential equation (22) are existed.

Theorem 4.1. (Curtain and Zwart [2]) Let assumptions in Theorem 3.9 hold, If $Q(t) \in \boldsymbol{L}_2([0,t_f],\mathbb{R})$, then equation (22) possesses an unique mild solution

$$U(z,t) = S(t)\{T_0 - V(z)\} + \int_0^t S(t-s)\{BQ(s) + f(\xi))\}\,\mathrm{d}s. \tag{32}$$

Now, according to Theorem 4.1 and Eq. (16) the mild solution for problem (1)-(4) is

$$\begin{aligned} T(z,t) =& V(z) + \sum_{n=1}^{\infty} e^{\lambda_n t} \int_0^l (T_0 - V(\xi))\phi_n(\xi)\,\mathrm{d}\xi\,\phi_n(z) \\ &+ \frac{1}{\rho c}\sum_{n=1}^{\infty}\int_0^t e^{\lambda_n(t-s)} Q(s)\,\mathrm{d}s\,\phi_n(z_1)\phi_n(z) \\ &+ \frac{1}{\rho c}\sum_{n=1}^{\infty}\int_0^t e^{\lambda_n(t-s)}\,\mathrm{d}s \int_0^l \{w(T_a - V(\xi)) + Q_m\}\phi_n(\xi)\,\mathrm{d}\xi\phi_n(z). \end{aligned} \tag{33}$$

Note that here, the adjoint problem is different from the standard initial boundary value problem in that the final time condition at $t = t_f$ is specified instead of the customary initial condition $t = 0$. By defining a new time variable $\tau = t_f - t$ and using Theorem 4.1 one may write solution to the adjoint problem (11) – (14) in the form

$$\psi(z,\tau) = (T_d - T(z_1,t_f))\sum_{n=1}^{\infty} e^{\lambda_n \tau}\phi_n(z_1)\phi_n(z). \tag{34}$$

Now, we determine optimal control $Q(t)$ using Eq. (15) with the aid of Eqs. (16), (33) and (34).

5. MATHEMATICAL SIMULATIONS

The interval $(0, t_f)$ is discretized into m subdomains at specific $m+1$ times $0 = t_0$, $t_1, t_2, \ldots, t_m = t_f$ such that $t_i = \frac{i\,t_f}{m}$. We assume that $Q(t)$ in each subinterval $[t_i, t_{i+1})$ is a piecewise constant function of time. Mathematical simulations are done in the environment of Mathematica 9, on a standard PC (Intel(R) Core(TM) i5 CPU 2.67 GHz, 4 GB RAM).

Practical problem in clinics for the point heating control is solved mathematically, where heat was deposited by a controlled conducting heating probe in the deep tumour site. Here, the thermophysical properties for homogeneous tissue have been chosen as $\rho = 1000$ kgm^{-3}, $l = 0.01$ m, $z_1 = 0.006$ m, $T_a = 37\,°$C, $c = 4200$ Jkg^{-1}/°C, $k = 0.5$ Wm^{-1}/°C, $h = 100$ Wm^{-2}/°C, $T_d = 43\,°$C, $w = 2100$ Wm^{-3}/°C, $T_0 = 25\,°$C, $Q_m = 33800$ Wm^{-3}, $t_f = 600$ s, 800 s, 1000 s, $u = 20\,°$C [5].

Figure 2 displays distribution of the optimal heating power $Q(t)$ versus time(s) under three different final times of treatment cycle t_f=600 s, 800 s and 1000 s. In Figure 2, it is seen that the value of optimal heating power will decrease steadily in the beginning and it will increases over time and ultimately approaches to the value zero at the final time. Figures 3(a, c, e) display the steady state temperature distributions for the tissue across its depth under different final times t_f=600 s, 800 s and 1000 s acted by the corresponding

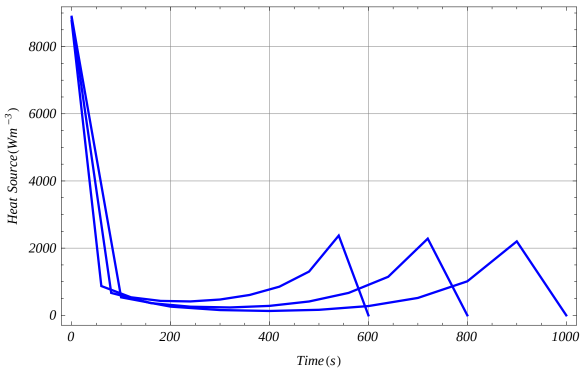


Fig. 2. The optimal heating power $Q(t)$ versus time at the final times t_f=600 s, 800 s and 1000 s, where the optimal control problem: minimizing (5) subject to (1)–(4) at the point $z = 0.006$ m is solved.

heating power as calculated and is given in Figure 2. It is shown that by applying the novel optimal control method based on C_0-semigroup theory the temperature values at the location of the tumour ($z = z_1$) attain 43 °C (desired temperature) with machine accuracy. These figures verifies the controllability of the Pennes' equation. Mathematical simulations show that the temperature in the left side of the location of the tumour due to the surface cooling by the flowing medium, in Figures 3(a, c, e), are always less than the tumour temperature. In the right side of the tumour, the temperature of the tissue steadily decreases to 37 °C (arterial temperature). Thus, the overheating of the healthy tissue is avoided. This simulation is coincide with the basic concepts of hyperthermia treatment. Figures 3(a, c, e) show that independent of the fixed final time t_f the solution converges to the desired treatment at the specific tumour site ($z_1 = 0.006$ m). Figures 3(b, d, f) depict how temperature response decreases after the corresponding final times, whereas there is no heating power at the tumour site and there is a surface cooling $u = 20$ °C. These graphs show that in the absence of the heating power the temperature response decreases faster due to the surface cooling. These figures verifies the exponential stability, dissipativity and pseudo-port Hamiltonian of the Pennes' equation.

This simulation show that the maximum value of the heating power will increase at the smaller final times, in spite of that the tissue temperature around the position $z = z_1$ has less values (see Figure 4(b)) compared with Figures 3(a, c, e). Note that, in this case ($t_f = 100$ s) the desired temperature is achieved in the position $z = z_1$ by the optimal heating power Figure 4(a).

The optimal heating power for the two different tumour positions $z = 0.002$ m and $z = 0.008$ m compared in Figure 5(a). This shows that the desired temperature at the points close to the tissue surface needs more energy, as it is expected.

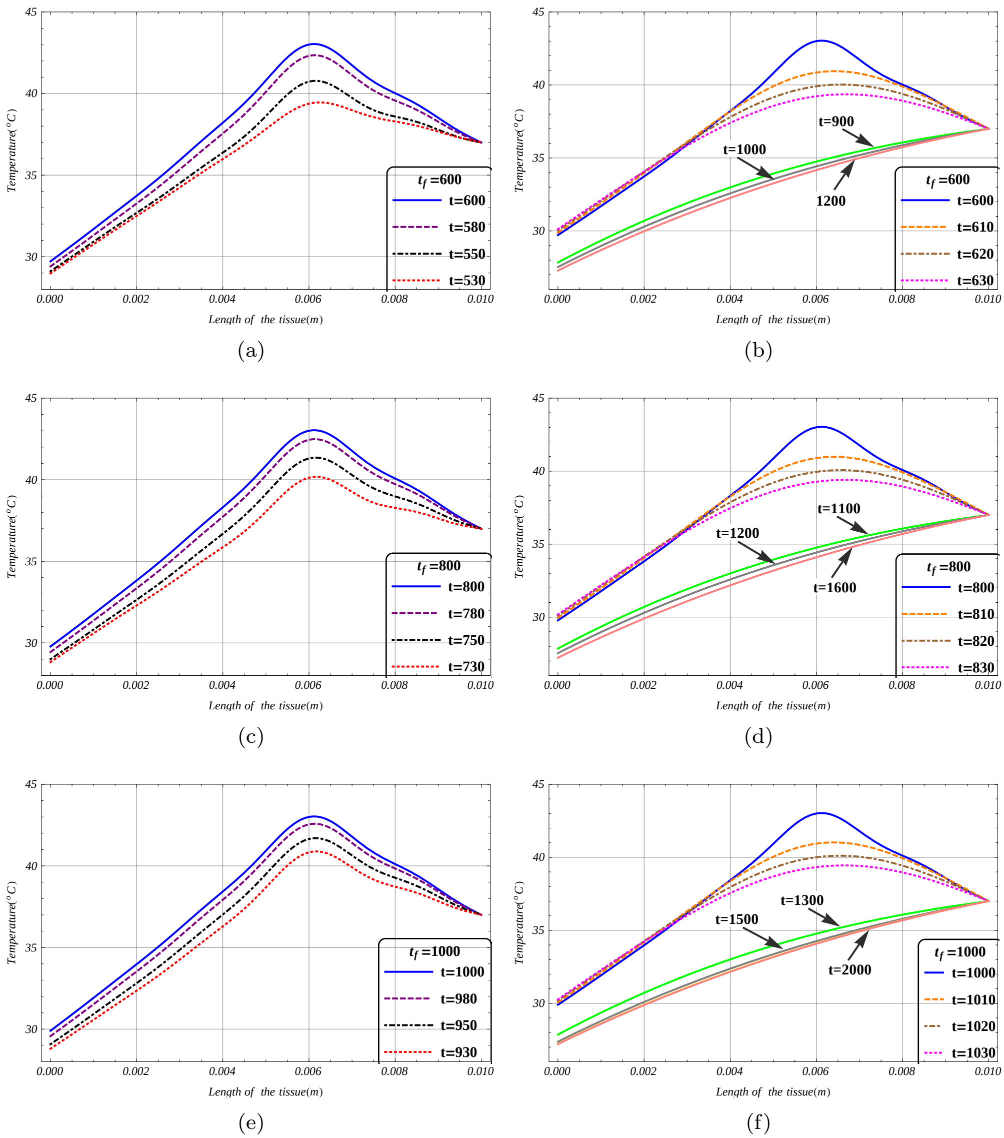
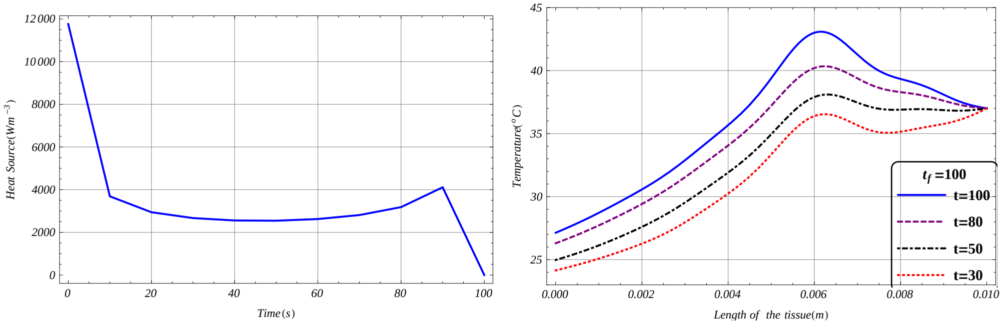
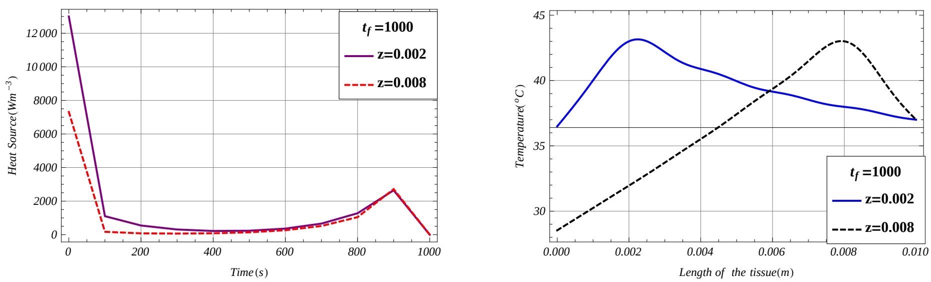


Fig. 3. Temperature (°C) of the tissue along the depth at the final times, t_f=600 s, 800 s and 1000 s respectively. (a, c, e): Gradually convergent profile for temperature distribution before and at the corresponding final times. (b, d, f): Cooling process for the heat distribution through the tissue depth, after and at the corresponding final times, when the heating power is off.



(a) (b)

Fig. 4. At the final time t_f=100 s (a): The optimal heating power versus time. (b): Temperature response (°C) along the depth of the tissue at the position $z = 0.006$ m.



(a) (b)

Fig. 5. For two depth positions z =0.002 m, 0.008 m (a): The optimal heating power versus time. (b): Temperature response (°C) of the tissue along the tissue depth.

6. CONCLUSION

The aim of this paper is to solve the optimal control problem of the Pennes' bioheat transfer with Dirichlet boundary condition at one end, and Rubin condition inclusive surface cooling term at the other end. The control function is the heating power $Q(t)$, which is placed by a Dirac function at a position z_1 inside of the domain. In Section 3 main results deal with properties of the operator A, which is used to establish the abstract form of the related original partial differential equation. In Theorem 3.2 the eigenvalues are computed, and the well-posedness is shown in Theorem 3.3. It is shown that A is a Riesz spectral operator by Theorem 3.4. Theorem 3.6 proves the exponential stability of the open-loop system. Section 4 deals with mild solution of the inhomogeneous abstract differential equation. Section 5 contains some mathematical simulations which indicate high convergence speed and accuracy of recent method.

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