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# NEW BOUNDS FOR THE MINIMUM EIGENVALUE OF THE FAN PRODUCT OF TWO $M$-MATRICES

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*Abstract.* In this paper, we mainly use the properties of the minimum eigenvalue of the Fan product of $M$-matrices and Cauchy-Schwarz inequality, and propose some new bounds for the minimum eigenvalue of the Fan product of two $M$-matrices. These results involve the maximum absolute value of off-diagonal entries of each row. Hence, the lower bounds for the minimum eigenvalue are easily calculated in the practical examples. In theory, a comparison is given in this paper. Finally, to illustrate our results, a simple example is also considered.

*Keywords*: Fan product; minimum eigenvalue; $M$-matrix

*MSC 2010*: 15A18, 15A42

## 1. Introduction

For convenience, the set $\{1, 2, \ldots, n\}$ is denoted by $\mathbb{N}$, where $n$ is any positive integer. A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called a nonnegative (positive) matrix if $a_{ij} \geqslant 0$ ($a_{ij} > 0$). A matrix $A \in \mathbb{R}^{n\times n}$ is called a nonsingular $M$-matrix [1] if there exists $P \geqslant 0$ and $\alpha > 0$ such that

$$A = \alpha I - P \quad \text{and} \quad \alpha > \varrho(P),$$

where $\varrho(P)$ is the spectral radius (Perron root) of the nonnegative matrix $P$ and $I$ is the $n \times n$ identity matrix. Denote by $\mathscr{M}_n$ the set of all $n \times n$ nonsingular $M$-matrices. Denote

$$\tau(A) = \min\{\operatorname{Re}\lambda\colon\ \lambda \in \sigma(A)\},$$

where $\sigma(A)$ denotes the spectrum of $A$. If $A \in \mathscr{M}_n$, then

$$\tau(A) = \frac{1}{\varrho(A^{-1})}$$

is a positive real eigenvalue, and the corresponding eigenvector is nonnegative [3].

A matrix $A$ is irreducible if there does not exist a permutation matrix $P$ such that

$$PAP^T = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix},$$

where $A_{1,1}$ and $A_{2,2}$ are square matrices.

Let $A, B \in \mathbb{C}^{n\times n}$. The Fan product of $A$ and $B$ is denoted by $A \star B \equiv C = (c_{ij}) \in \mathbb{C}^{n\times n}$ and is defined by

$$c_{ij} = \begin{cases} -a_{ij}b_{ij}, & i \neq j, \\ a_{ii}b_{ii}, & i = j. \end{cases}$$

If $A, B \in \mathscr{M}_n$, then $A \star B$ is a $M$-matrix. Let $A, B \in \mathscr{M}_n$. In [2], Fang gave a lower bound for $\tau(A \star B)$ as follows:

$$\tau(A \star B) \geqslant \min_{1\leqslant i\leqslant n} \{a_{ii}\tau(B) + b_{ii}\tau(A) - \tau(A)\tau(B)\}. \tag{1.1}$$

In [4], Liu and Chen gave a sharper lower bound for $\tau(A \star B)$ as follows:

$$\begin{aligned} \tau(A \star B) \geqslant \tfrac{1}{2}\min_{i\neq j}\{a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\ + 4(a_{ii} - \tau(A))(b_{ii} - \tau(B))(a_{jj} - \tau(A))(b_{jj} - \tau(B))]^{1/2}\}. \end{aligned} \tag{1.2}$$

In this paper, our aim is to propose some new lower bounds for the minimum eigenvalue of the Fan product of two $M$-matrices.

## 2. Some lower bounds for the minimum eigenvalue of the Fan product of $M$-matrices

**Lemma 2.1** ([1]). *If $A \in \mathscr{M}_n$ is irreducible, $Az \geqslant kz$ for a nonnegative nonzero vector $z$, then $k \leqslant \tau(A)$.*

**Lemma 2.2** (Cauchy-Schwarz inequality). *For any vectors $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$ and $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$, it holds that*

$$\left|\sum_{i=1}^n u_i v_i\right| \leqslant \left(\sum_{i=1}^n u_i^2\right)\left(\sum_{i=1}^n v_i^2\right).$$

**Theorem 2.1.** *If $A = (a_{ij}) \in \mathscr{M}_n$ and $B = (b_{ij}) \in \mathscr{M}_n$, then*

$$\tau(A \star B) \geqslant \min_{1\leqslant i\leqslant n} \{(a_{ii} - \alpha_i)b_{ii} + \alpha_i\tau(B)\}, \tag{2.1}$$

*where $\alpha_i = \max\limits_{k\neq i}\{|a_{ik}|\}$, for all $i \in \mathbb{N}$.*

P r o o f. It is easy to see that (2.1) holds with equality for $n=1$. Next, we assume that $n \geqslant 2$. Two cases will be discussed in the following.

*Case 1.* If $A \star B$ is irreducible, then $A$ and $B$ are irreducible. Hence, there exists a positive vector $v=(v_1, v_2, \ldots, v_n)^T$ such that

$$Bv = \tau(B)v.$$

So, we have

$$b_{ii}v_i - \sum_{j\neq i} |b_{ij}|v_j = \tau(B)v_i, \quad \forall i \in \mathbb{N},$$

i.e.,

$$\sum_{j\neq i} |b_{ij}|v_j = [b_{ii} - \tau(B)]v_i, \quad \forall i \in \mathbb{N}. \tag{2.2}$$

Let $\alpha_i = \max_{k\neq i}\{|a_{ik}|\}$, for all $i \in \mathbb{N}$. Denote $C = A \star B$. For all $i \in \mathbb{N}$, by (2.2), we have

$$\begin{aligned}(Cv)_i &= a_{ii}b_{ii}v_i - \sum_{j\neq i} |a_{ij}||b_{ij}|v_j = a_{ii}b_{ii}v_i - \sum_{j\neq i} |b_{ij}||a_{ij}|v_j \\ &\geqslant a_{ii}b_{ii}v_i - \alpha_i \sum_{j\neq i} |b_{ij}|v_j = a_{ii}b_{ii}v_i - \alpha_i[b_{ii} - \tau(B)]v_i \\ &= [(a_{ii} - \alpha_i)b_{ii} + \alpha_i\tau(B)]v_i.\end{aligned}$$

By Lemma 2.1, we obtain

$$\tau(A \star B) \geqslant \min_{1\leqslant i\leqslant n} \{(a_{ii} - \alpha_i)b_{ii} + \alpha_i\tau(B)\}.$$

*Case 2.* If $A \star B$ is reducible, let $T = (t_{ij})$ be the permutation matrix such that $t_{12} = t_{23} = \ldots = t_{n-1,n} = t_{n,1} = 1$ and the remaining $t_{ij} = 0$. Then there exists a positive real number $\varepsilon$ such that $A - \varepsilon T$ and $B - \varepsilon T$ are two irreducible $M$-matrices, i.e., $(A - \varepsilon T) \star (B - \varepsilon T)$ is irreducible. Apply Case 1 and then use the continuity argument to complete the proof. □

Since the Fan product is commutative, the inequality (2.1) remains correct if $A$ and $B$ are switched. Moreover, the following result can be immediately obtained.

**Theorem 2.2.** *If $A = (a_{ij}) \in \mathscr{M}_n$ and $B = (b_{ij}) \in \mathscr{M}_n$, then*

$$\tau(A \star B) \geqslant \min_{1\leqslant i\leqslant n} \{(b_{ii} - \beta_i)a_{ii} + \beta_i\tau(A)\}, \tag{2.3}$$

*where $\beta_i = \max_{k\neq i}\{|b_{ik}|\}$, for all $i \in \mathbb{N}$.*

From Theorem 2.1 and Theorem 2.2 we can obtain the following result.

**Theorem 2.3.** *If $A = (a_{ij}) \in \mathscr{M}_n$ and $B = (b_{ij}) \in \mathscr{M}_n$, then*

$$\tau(A \star B) \geqslant \max\{\min_{1\leqslant i\leqslant n}\{(a_{ii} - \alpha_i)b_{ii} + \alpha_i\tau(B)\}, \min_{1\leqslant i\leqslant n}\{(b_{ii} - \beta_i)a_{ii} + \beta_i\tau(A)\}\},$$

*where $\alpha_i = \max_{k\neq i}\{|a_{ik}|\}$ and $\beta_i = \max_{k\neq i}\{|b_{ik}|\}$, for all $i \in \mathbb{N}$.*

**Theorem 2.4.** *If $A = (a_{ij}) \in \mathscr{M}_n$ and $B = (b_{ij}) \in \mathscr{M}_n$, then*

$$\tau(A \star B) \geqslant \min_{1\leqslant i\leqslant n}\{a_{ii}b_{ii} - \alpha_i^{1/2}\beta_i^{1/2}[a_{ii} - \tau(A)]^{1/2}[b_{ii} - \tau(B)]^{1/2}\}, \tag{2.4}$$

*where $\alpha_i = \max_{k\neq i}\{|a_{ik}|\}$ and $\beta_i = \max_{k\neq i}\{|b_{ik}|\}$, for all $i \in \mathbb{N}$.*

P r o o f. It is easy to see that (2.4) holds with equality for $n = 1$. Next, we assume that $n \geqslant 2$. Two cases will be discussed in the following.

*Case 1.* If $A \star B$ is irreducible, then $A$ and $B$ are irreducible. There exist two positive vectors $u = (u_1^2, u_2^2, \ldots, u_n^2)^T$ and $v = (v_1^2, v_2^2, \ldots, v_n^2)^T$ such that

$$Au = \tau(A)u,$$

and

$$Bv = \tau(B)v.$$

In order to prove the following, let $u_i > 0$ and $v_i > 0$ for all $i \in \mathbb{N}$. Hence, we have

$$a_{ii}u_i^2 - \sum_{j\neq i}|a_{ij}|u_j^2 = \tau(A)u_i^2, \quad \forall i \in \mathbb{N},$$

and

$$b_{ii}v_i^2 - \sum_{j\neq i}|b_{ij}|v_j^2 = \tau(B)v_i^2, \quad \forall i \in \mathbb{N},$$

i.e.,

$$\sum_{j\neq i}|a_{ij}|u_j^2 = [a_{ii} - \tau(A)]u_i^2, \quad \forall i \in \mathbb{N}, \tag{2.5}$$

and

$$\sum_{j\neq i}|b_{ij}|v_j^2 = [b_{ii} - \tau(B)]v_i^2, \quad \forall i \in \mathbb{N}. \tag{2.6}$$

Let $\alpha_i = \max_{k\neq i}\{|a_{ik}|\}$ and $\beta_i = \max_{k\neq i}\{|b_{ik}|\}$, for all $i \in \mathbb{N}$. Define a positive vector $z = (z_1, z_2, \ldots, z_n)^T$, where

$$z_i = u_iv_i, \quad \forall i \in \mathbb{N}.$$

Denote $C = A \star B$. For all $i \in \mathbb{N}$, by Lemma 2.2 and equalities (2.5), (2.6), we have

$$\begin{aligned}
(Cz)_i &= a_{ii}b_{ii}z_i - \sum_{j\neq i} |a_{ij}||b_{ij}|z_j = a_{ii}b_{ii}z_i - \sum_{j\neq i} |b_{ij}||a_{ij}|u_j v_j \\
&\geqslant a_{ii}b_{ii}z_i - \Big(\sum_{j\neq i} |a_{ij}|^2 u_j^2\Big)^{1/2} \Big(\sum_{j\neq i} |b_{ij}|^2 v_j^2\Big)^{1/2} \\
&\geqslant a_{ii}b_{ii}z_i - \alpha_i^{1/2}\beta_i^{1/2}\Big(\sum_{j\neq i} |a_{ij}| u_j^2\Big)^{1/2} \Big(\sum_{j\neq i} |b_{ij}| v_j^2\Big)^{1/2} \\
&= a_{ii}b_{ii}z_i - \alpha_i^{1/2}\beta_i^{1/2}[a_{ii} - \tau(A)]^{1/2}[b_{ii} - \tau(B)]^{1/2}u_i v_i \\
&= (a_{ii}b_{ii} - \alpha_i^{1/2}\beta_i^{1/2}[a_{ii} - \tau(A)]^{1/2}[b_{ii} - \tau(B)]^{1/2})z_i.
\end{aligned}$$

By Lemma 2.1, we get

$$\tau(A \star B) \geqslant \min_{1\leqslant i\leqslant n} \{a_{ii}b_{ii} - \alpha_i^{1/2}\beta_i^{1/2}[a_{ii} - \tau(A)]^{1/2}[b_{ii} - \tau(B)]^{1/2}\}.$$

*Case 2.* If $A \star B$ is reducible, the proof is similar to the one of Theorem 2.1. □

## 3. Example

In this section, we will show an example to illustrate our results.

**Example 3.1** ([4]). Consider two $3 \times 3$ $M$-matrices as follows.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -0.5 \\ -0.5 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -0.25 & -0.25 \\ -0.5 & 1 & -0.25 \\ -0.25 & -0.5 & 1 \end{bmatrix}$$

By direct calculation, $\tau(A) = 0.5402$, $\tau(B) = 0.3432$ and $\tau(A \star B) = 0.9377$. According to inequalities (1.1) and (1.2), we have

$$\tau(A \star B) \geqslant \min_{1\leqslant i\leqslant 3} \{a_{ii}\tau(B) + b_{ii}\tau(A) - \tau(A)\tau(B)]\} = 0.6980$$

and

$$\begin{aligned}
\tau(A \star B) \geqslant \tfrac{1}{2} \min_{i\neq j} \{&a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\
&+ 4(a_{ii} - \tau(A))(b_{ii} - \tau(B))(a_{jj} - \tau(A))(b_{jj} - \tau(B))]^{1/2}\} = 0.7654.
\end{aligned}$$

According to inequalities (2.1), (2.3) and (2.4), we have

$$\begin{aligned}\tau(A\star B) &\geqslant \min_{1\leqslant i\leqslant 3}\{(a_{ii}-\alpha_i)b_{ii}+\alpha_i\tau(B)\} = 0.6716,\\ \tau(A\star B) &\geqslant \min_{1\leqslant i\leqslant 3}\{(b_{ii}-\beta_i)a_{ii}+\beta_i\tau(A)\} = 0.7701,\end{aligned}$$

and

$$\tau(A\star B) \geqslant \min_{1\leqslant i\leqslant 3}\{a_{ii}b_{ii}-\alpha_i^{1/2}\beta_i^{1/2}[a_{ii}-\tau(A)]^{1/2}[b_{ii}-\tau(B)]^{1/2}\} = 0.7252,$$

respectively.

Although we can not prove that our results are sharper than the ones of [2], [4] in theory, we can see that our results are sharper than the ones of [2], [4] for some matrices from Example 3.1.

**Addendum.** After this paper was accepted, I learned that Theorem 2.3 is the same as Theorem 2 in the paper H. Li: New estimation of the eigenvalue bounds of the Hadamard product and the Fan product of matrices, Henan Science, 30 (2012), 680–683; but my results are independent and obtained by a different method.

*References*


[1] *A. Berman, R. J. Plemmons*: Nonnegative Matrices in the Mathematical Sciences. Classics in Applied Mathematics 9, SIAM, Philadelphia, 1994.

[2] *M. Z. Fang*: Bounds on eigenvalues of the Hadamard product and the Fan product of matrices. Linear Alegebra. Appl. *425* (2007), 7–15.

[3] *R. A. Horn, C. R. Johnson*: Topics in Matrix Analysis. Cambridge University Press, Cambridge, 1991.

[4] *Q. B. Liu, G. L. Chen*: On two inequalities for the Hadamard product and the Fan product of matrices. Linear Algebra Appl. *431* (2009), 974–984.



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