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A HYBRID MEAN VALUE INVOLVING TWO-TERM EXPONENTIAL SUMS AND POLYNOMIAL CHARACTER SUMS

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Abstract. Let $q \geqslant 3$ be a positive integer. For any integers m and n, the two-term exponential sum $C(m,n,k;q)$ is defined by $C(m,n,k;q) = \sum\limits_{a=1}^{q} e((ma^k+na)/q)$, where $e(y) = \mathrm{e}^{2\pi \mathrm{i} y}$. In this paper, we use the properties of Gauss sums and the estimate for Dirichlet character of polynomials to study the mean value problem involving two-term exponential sums and Dirichlet character of polynomials, and give an interesting asymptotic formula for it.

Keywords: Dirichlet character of polynomials; two-term exponential sums; hybrid mean value; asymptotic formula

MSC 2010: 11L40, 11F20

1. Introduction

Let $q \geqslant 3$ be a positive integer. For any integers m and n, the two-term exponential sum $C(m,n,k;q)$ is defined by

$$C(m,n,k;q) = \sum_{a=1}^{q} e\Big(\frac{ma^k+na}{q}\Big),$$

where $e(y) = \mathrm{e}^{2\pi \mathrm{i} y}$.

Various properties of $C(m,n,k;q)$ were investigated by many authors (see [1], [3],

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[4], [5], [6], [7]). For example 'Gauss's classical work' (referred in [1]) proved the remarkable formula

$$C(1,0,2;q) = \frac{1}{2}\sqrt{q}(1+\mathrm{i})(1+e(-q/4)) = \begin{cases} \sqrt{q}, & \text{if } q \equiv 1 \bmod 4, \\ 0, & \text{if } q \equiv 2 \bmod 4, \\ \mathrm{i}\sqrt{q}, & \text{if } q \equiv 3 \bmod 4, \\ (1+\mathrm{i})\sqrt{q}, & \text{if } q \equiv 0 \bmod 4, \end{cases}$$

where $\mathrm{i}^2 = -1$.

In fact the exact value of $|C(m,n,2;q)|$ is $\sqrt{q}$, if $(2m,q) = 1$ (see e.g. Apostol's related work [1]). Cochrane and Zheng [5] show for the general sum that

$$|C(m,n,k;q)| \leqslant k^{\omega(q)} q^{1/2},$$

where $\omega(q)$ denotes the number of distinct prime divisors of q.

The main purpose of this paper is to study the asymptotic properties of the hybrid mean value

$$\sum_{n=1}^{p-1} |C(m,n,k;p)|^2 \cdot \left| \sum_{a=1}^{p-1} \chi(ma+\bar{a}) \right|^2, \tag{1}$$

where χ is any non-principal even character mod p, and $a \cdot \bar{a} \equiv 1 \bmod p$.

In fact, if χ is an odd character mod p, then we have the identity

$$\sum_{a=1}^{p-1} \chi(ma+\bar{a}) = \sum_{a=1}^{p-1} \chi(-ma+\overline{-a}) = -\sum_{a=1}^{p-1} \chi(ma+\bar{a}) \text{ or } \sum_{a=1}^{p-1} \chi(ma+\bar{a}) = 0.$$

So we only consider the case that χ is an even character mod p in (1).

For any integer a with $(a,p) = 1$ we know from Euler's theorem that $a^{p-2} \equiv \bar{a} \bmod p$. So the sum

$$\sum_{a=1}^{p-1} \chi(ma+\bar{a})$$

is a special case of a general polynomial character sums

$$\sum_{a=N+1}^{N+M} \chi(f(a)), \tag{2}$$

where M and N are any positive integers, and $f(x)$ is a polynomial.

It is a very important and difficult problem in analytic number theory to give a sharper upper bound estimate for (2). But for some special cases, such as $f(x) = x$, Pólya and Vinogradov's classical work (see Theorem 8.21 of [1]) proved that for any non-principal character χ mod q, we have

$$\sum_{a=N+1}^{N+M} \chi(a) \ll q^{1/2} \ln q.$$

If $q = p$ is an odd prime, then Weil (see [10]) obtained the following result:

Let χ be a qth-order character mod p. If $f(x)$ is not a perfect qth power mod p, then we have the estimate

$$\sum_{x=1}^{p} \chi(f(x)) \leqslant kp^{1/2}, \qquad \sum_{x=N+1}^{N+M} \chi(f(x)) \leqslant kp^{1/2} \ln p, \tag{3}$$

where k denotes the degree of the polynomial $f(x)$. Some related results can also be found in [2], [8], [11] and [12].

But for a hybrid mean value such as (1), it seems that no one has yet studied the asymptotic properties of the hybrid mean value, at least we have not seen any related result. This problem is significant, it can reflect the close relations between the two sums. Although each of the two-term exponential sums and polynomial character sums has no precise estimates, their mean value is well behaved. The main purpose of this paper is to show this point. That is, we shall prove the following:

Theorem. *Let p be an odd prime, k be any integer with $k \neq 0, 1$. Then for any non-principal even character χ mod p and any integer n with $(n, p) = 1$, we have the asymptotic formula*

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} e\Big(\frac{ma^k + na}{p}\Big) \right|^2 \cdot \left| \sum_{b=1}^{p-1} \chi(mb + \bar{b}) \right|^2 = \begin{cases} 2p^3 + O(|k|p^2), & \text{if } k \text{ is an even number,} \\ 2p^3 + O(|k|p^{\frac{5}{2}}), & \text{if } k \text{ is an odd number.} \end{cases}$$

Taking $k = -1$ in our theorem we may immediately deduce the following:

Corollary. *Let $p > 3$ be a prime. Then for any non-principal even character χ mod p, we have the asymptotic formula*

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} e\Big(\frac{ma + \bar{a}}{p}\Big) \right|^2 \cdot \left| \sum_{b=1}^{p-1} \chi(mb + \bar{b}) \right|^2 = 2p^3 + O(p^2).$$

For a general integer $q \geqslant 3$, whether there exists an asymptotic formula for

$$\sum_{m=1}^{q} |C(m,n,k;q)|^2 \cdot \Bigg| \sum_{\substack{b=1\\(b,q)=1}}^{q} \chi(mb+\bar{b})\Bigg|^2$$

and

$$\sum_{m=1}^{p-1} |C(m,n,k;p)|^{2r} \cdot \Bigg|\sum_{b=1}^{p-1} \chi(mb+\bar{b})\Bigg|^2,$$

where k, $r \geqslant 3$ are integers and $(n,q)=(p,n)=1$, are two open problems.

2. Several lemmas

In this section, we shall give several lemmas, which are necessary in the proof of our theorem.

Lemma 1. *Let p be an odd prime, χ be any non-principal even character* mod p. *Then for any integer m with $(m,p)=1$, we have the identity*

$$\sum_{a=1}^{p-1} \chi(ma+\bar{a}) = \frac{\chi_1(m)\tau^2(\bar{\chi}_1)}{\tau(\bar{\chi})}\Big(1+\Big(\frac{m}{p}\Big)\Big(\frac{\tau(\bar{\chi}_1\chi_2)}{\tau(\bar{\chi}_1)}\Big)^2\Big),$$

where $\tau(\chi) = \sum\limits_{a=1}^{p-1} \chi(a)e(\frac{a}{p})$ denotes the classical Gauss sums, $\chi=\chi_1^2$, $(\frac{*}{p})=\chi_2$ denotes the Legendre symbol.

Proof. Since χ is a non-principal even character mod p, we know that $\chi(-1)=1$. Therefore, there exists one and only one primitive character χ_1 mod p such that $\chi=\chi_1^2$. Then from the properties of Gauss sums we have

$$\begin{aligned}
\sum_{a=1}^{p-1} \chi(ma+\bar{a}) &= \frac{1}{\tau(\bar{\chi})}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \bar{\chi}(b)e\Big(\frac{b(ma+\bar{a})}{p}\Big)\\
&= \frac{1}{\tau(\bar{\chi})}\sum_{a=1}^{p-1} \bar{\chi}(a)\sum_{b=1}^{p-1} \bar{\chi}(b)e\Big(\frac{b(ma^2+1)}{p}\Big)\\
&= \frac{1}{\tau(\bar{\chi})}\sum_{b=1}^{p-1} \bar{\chi}(b)e\Big(\frac{b}{p}\Big)\sum_{a=1}^{p-1} \bar{\chi}(a)e\Big(\frac{bma^2}{p}\Big)\\
&= \frac{1}{\tau(\bar{\chi})}\sum_{b=1}^{p-1} \bar{\chi}(b)e\Big(\frac{b}{p}\Big)\sum_{a=1}^{p-1} \bar{\chi}_1(a^2)e\Big(\frac{bma^2}{p}\Big)
\end{aligned}$$

$$= \frac{1}{\tau(\bar\chi)}\sum_{b=1}^{p-1}\bar\chi(b)e\Big(\frac{b}{p}\Big)\sum_{a=1}^{p-1}(1+\chi_2(a))\bar\chi_1(a)e\Big(\frac{bma}{p}\Big)$$
$$= \frac{1}{\tau(\bar\chi)}\sum_{b=1}^{p-1}\bar\chi(b)e\Big(\frac{b}{p}\Big)\Big(\sum_{a=1}^{p-1}\bar\chi_1(a)e\Big(\frac{bma}{p}\Big)+\sum_{a=1}^{p-1}\chi_2(a)\bar\chi_1(a)e\Big(\frac{bma}{p}\Big)\Big)$$
$$= \frac{1}{\tau(\bar\chi)}\sum_{b=1}^{p-1}\bar\chi(b)e\Big(\frac{b}{p}\Big)(\chi_1(bm)\tau(\bar\chi_1)+\chi_1(bm)\chi_2(bm)\tau(\bar\chi_1\chi_2))$$
$$= \frac{\chi_1(m)\tau(\bar\chi_1)}{\tau(\bar\chi)}\Big(\sum_{b=1}^{p-1}\bar\chi_1(b)e\Big(\frac{b}{p}\Big)+\chi_2(m)\frac{\tau(\bar\chi_1\chi_2)}{\tau(\bar\chi_1)}\sum_{b=1}^{p-1}\bar\chi_1(b)\chi_2(b)e\Big(\frac{b}{p}\Big)\Big)$$
$$= \frac{\chi_1(m)\tau^2(\bar\chi_1)}{\tau(\bar\chi)}\Big(1+\Big(\frac{m}{p}\Big)\Big(\frac{\tau(\bar\chi_1\chi_2)}{\tau(\bar\chi_1)}\Big)^2\Big),$$

where we have used the identities $\chi=\chi_1^2$ and $\sum\limits_{a=1}^{p-1}\bar\chi_1(a)e(ma/p)=\chi_1(m)\tau(\bar\chi_1)$.

This proves Lemma 1. □

Note. It is clear that for any non-principal even character $\chi \bmod p$, we have $|\tau(\chi)|=\sqrt{p}$. So from Lemma 1 we can deduce the upper bound

$$\Big|\sum_{a=1}^{p-1}\chi(ma+\bar a)\Big|\leqslant 2\sqrt{p}. \tag{4}$$

This estimate is interesting, because it immediately recovers the Weil bound.

Lemma 2. *Let p be an odd prime, χ be any non-principal even character* mod p. *Then for any integer m with $(m,p)=1$, we have the identity*

$$\Big|\sum_{a=1}^{p-1}\chi(ma+\bar a)\Big|^2=2p+\Big(\frac{m}{p}\Big)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\Big(\frac{b(b-1)(a^2b-1)}{p}\Big),$$

where $(\frac{*}{p})$ denotes the Legendre symbol.

Proof. Let $am+\bar a=u$. Then for any $(m,p)=1$, we have

$$\sum_{a=1}^{p-1}\chi(ma+\bar a)=\sum_{u=1}^{p-1}\chi(u)\sum_{\substack{a=1\\ am+\bar a\equiv u \bmod p}}^{p-1}1=\sum_{u=1}^{p-1}\chi(u)\sum_{\substack{a=1\\ a^2m^2-amu+m\equiv 0 \bmod p}}^{p-1}1 \tag{5}$$
$$=\sum_{u=1}^{p-1}\chi(u)\sum_{\substack{a=0\\ (2am-u)^2\equiv u^2-4m \bmod p}}^{p-1}1=\sum_{u=1}^{p-1}\chi(u)\sum_{\substack{a=0\\ a^2\equiv u^2-4m \bmod p}}^{p-1}1.$$

Note that for any fixed integer $u^2 - 4m$, the number of solutions of the congruent equation $x^2 \equiv u^2 - 4m \bmod p$ is $1 + ((u^2 - 4m)/p)$, so from (5) we have

$$
\begin{aligned}
(6) \qquad \sum_{a=1}^{p-1} \chi(ma + \bar{a}) &= \sum_{u=1}^{p-1} \chi(u)\Big(1 + \Big(\frac{u^2 - 4m}{p}\Big)\Big) \\
&= \sum_{u=1}^{p-1} \chi(u)\Big(\frac{u^2 - 4m}{p}\Big) = \chi(2) \sum_{u=1}^{p-1} \chi(u)\Big(\frac{u^2 - m}{p}\Big).
\end{aligned}
$$

Now from (6) and the properties of reduced residue systems mod p we have

$$
\begin{aligned}
(7) \qquad \Big|\sum_{a=1}^{p-1} \chi(ma + \bar{a})\Big|^2 &= \Big|\sum_{u=1}^{p-1} \chi(u)\Big(\frac{u^2 - m}{p}\Big)\Big|^2 \\
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi(a\bar{b})\Big(\frac{a^2 - m}{p}\Big)\Big(\frac{b^2 - m}{p}\Big) \\
&= \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \Big(\frac{a^2 b^2 - m}{p}\Big)\Big(\frac{b^2 - m}{p}\Big) \\
&= \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \Big(1 + \Big(\frac{b}{p}\Big)\Big)\Big(\frac{a^2 b - m}{p}\Big)\Big(\frac{b - m}{p}\Big) \\
&= \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \Big(\frac{(a^2 b - 1)(b - 1)}{p}\Big) \\
&\quad + \Big(\frac{m}{p}\Big) \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \Big(\frac{(a^2 b - 1)b(b - 1)}{p}\Big).
\end{aligned}
$$

Knowing that $\chi(-1) = 1$, from the properties of the complete residue system mod p we also have

$$
\begin{aligned}
(8) \quad \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \Big(\frac{(a^2 b - 1)(b - 1)}{p}\Big) &= \sum_{a=1}^{p-1} \chi(a) \sum_{b=0}^{p-1} \Big(\frac{(2a^2 b - a^2 - 1)^2 - (a^2 - 1)^2}{p}\Big) \\
&= \sum_{a=1}^{p-1} \chi(a) \sum_{b=0}^{p-1} \Big(\frac{b^2 - (a^2 - 1)^2}{p}\Big)
\end{aligned}
$$

and

$$
(9) \qquad \sum_{a=1}^{p} \Big(\frac{a^2 + n}{p}\Big) = \begin{cases} -1, & \text{if } (n, p) = 1; \\ p - 1, & \text{if } (n, p) = p. \end{cases}
$$

(This formula can be found in Hua's book [9], §7.8, Theorem 8.2).

Combining (8) and (9) we can deduce the identity

$$\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\Big(\frac{(a^2b-1)(b-1)}{p}\Big) = 2(p-1) - \sum_{a=2}^{p-2}\chi(a) = 2p. \tag{10}$$

Now Lemma 2 follows from (7) and (10). □

Lemma 3. *Let p be an odd prime. Then for any non-principal even character $\chi \bmod p$, we have the estimate*

$$\left|\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\Big(\frac{b(a^2b-1)(b-1)}{p}\Big)\right| \leqslant 2p.$$

P r o o f. For any character $\chi \bmod p$, from the properties of the Gauss sums $\tau(\chi)$ we know that $|\tau(\chi)| \leqslant \sqrt{p}$. Then from (4) and Lemma 2 we may immediately obtaine the estimate

$$\left|\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\Big(\frac{b(a^2b-1)(b-1)}{p}\Big)\right| = \left|\left|\sum_{a=1}^{p-1}\chi(ma+\bar{a})\right|^2 - 2p\right| \leqslant 2p.$$

This proves Lemma 3. □

3. Proof of the theorem

In this section, we shall use the lemmas from Section 2 to complete the proof of our theorem. For any non-principal character $\chi \bmod p$ and any integer $(n,p)=1$, note that

$$\begin{aligned} |C(m,n,k;p)|^2 &= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} e\Big(\frac{m(a^k-b^k)+n(a-b)}{p}\Big) \\ &= p-1+\sum_{a=2}^{p-1}\sum_{b=1}^{p-1} e\Big(\frac{mb^k(a^k-1)+nb(a-1)}{p}\Big). \end{aligned} \tag{11}$$

Let $\chi_2 = (\frac{*}{p})$ denote the Legendre symbol. If $k \neq 0$ is an even number, then from (11), (3), Lemma 2, Lemma 3 and the properties of Gauss sums we have

$$
\begin{aligned}
(12) \qquad & \sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1} e\Big(\frac{ma^k+na}{p}\Big)\right|^2 \cdot \left|\sum_{b=1}^{p-1}\chi(mb+\bar{b})\right|^2 \\
&= 2p(p-1)^2 + 2p\sum_{m=1}^{p-1}\sum_{a=2}^{p-1}\sum_{b=1}^{p-1} e\Big(\frac{mb^k(a^k-1)+nb(a-1)}{p}\Big) \\
&\quad + (p-1)\sum_{m=1}^{p-1}\Big(\frac{m}{p}\Big)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\Big(\frac{b(b-1)(a^2b-1)}{p}\Big) \\
&\quad + \Big(\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\Big(\frac{b(b-1)(a^2b-1)}{p}\Big)\Big) \\
&\quad\quad \times \Big(\sum_{c=2}^{p-1}\sum_{d=1}^{p-1}\sum_{m=1}^{p-1}\Big(\frac{m}{p}\Big)e\Big(\frac{md^k(c^k-1)+nd(c-1)}{p}\Big)\Big) \\
&= 2p(p-1)^2 + 2p(p-2) - 2p^2\sum_{\substack{a=2 \\ a^k\equiv 1 \bmod p}}^{p-1} 1 \\
&\quad + \Big(\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\Big(\frac{b(b-1)(a^2b-1)}{p}\Big)\Big) \\
&\quad\quad \times \tau(\chi_2)\Big(\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\Big(\frac{d^k(c^k-1)}{p}\Big)e\Big(\frac{nd(c-1)}{p}\Big)\Big) \\
&= 2p^3 + O(|k|p^2) + O\Big(p\cdot\sqrt{p}\Big|\sum_{c=1}^{p-1}\Big(\frac{c^{|k|}-1}{p}\Big)\sum_{d=1}^{p-1}e\Big(\frac{nd(c-1)}{p}\Big)\Big|\Big) \\
&= 2p^3 + O(|k|p^2).
\end{aligned}
$$

If k is an odd number and $k \neq 1$, since

$$
\sum_{c=1}^{p-1}\Big(\frac{c^k-1}{p}\Big)\Big(\frac{c-1}{p}\Big) = \sum_{c=1}^{p-1}\Big(\frac{(c^k-1)(c-1)}{p}\Big) \ll k\sqrt{p}, \quad \text{if} \quad k>1,
$$

and

$$
\sum_{c=1}^{p-1}\Big(\frac{c^k-1}{p}\Big)\Big(\frac{c-1}{p}\Big) = \sum_{c=1}^{p-1}\Big(\frac{c(c^{|k|}-1)(c-1)}{p}\Big) \ll |k|\sqrt{p}, \quad \text{if} \quad k<0,
$$

from the process of proving (12) we have

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1} e\Big(\frac{ma^k+na}{p}\Big)\right|^2\cdot\left|\sum_{b=1}^{p-1}\chi(mb+\bar b)\right|^2 \tag{13}$$
$$= 2p^3 + O(|k|p^2) + O\bigg(p\cdot\sqrt{p}\bigg|\sum_{c=2}^{p-1}\Big(\frac{c^k-1}{p}\Big)\sum_{d=1}^{p-1}\Big(\frac{d}{p}\Big)e\Big(\frac{nd(c-1)}{p}\Big)\bigg|\bigg)$$
$$= 2p^3 + O(|k|p^2) + O\bigg(p^2\cdot\bigg|\sum_{c=1}^{p-1}\Big(\frac{c^k-1}{p}\Big)\Big(\frac{c-1}{p}\Big)\bigg|\bigg)$$
$$= 2p^3 + O\big(|k|p^{5/2}\big).$$

Now our theorem follows from (12) and (13).

Note that if we take $k=-1$ in (13), then

$$\left|\sum_{c=1}^{p-1}\Big(\frac{\bar c-1}{p}\Big)\Big(\frac{c-1}{p}\Big)\right| = \left|\sum_{c=1}^{p-1}\Big(\frac{\bar c}{p}\Big)\Big(\frac{1-c}{p}\Big)\Big(\frac{c-1}{p}\Big)\right| = 1,$$

from (13) we can also deduce the asymptotic formula

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1} e\Big(\frac{m\bar a+a}{p}\Big)\right|^2\cdot\left|\sum_{b=1}^{p-1}\chi(mb+\bar b)\right|^2$$
$$= 2p^3 + O(p^2) + O\bigg(p^2\cdot\bigg|\sum_{c=1}^{p-1}\Big(\frac{\bar c-1}{p}\Big)\Big(\frac{c-1}{p}\Big)\bigg|\bigg)$$
$$= 2p^3 + O(p^2).$$

This completes the proof of our corollary. □

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