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SINGLE-USE RELIABILITY COMPUTATION OF A SEMI-MARKOVIAN SYSTEM

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Abstract. Markov chain usage models were successfully used to model systems and software. The most prominent approaches are the so-called failure state models Whittaker and Thomason (1994) and the arc-based Bayesian models Sayre and Poore (2000). In this paper we propose arc-based semi-Markov usage models to test systems. We extend previous studies that rely on the Markov chain assumption to the more general semi-Markovian setting. Among the obtained results we give a closed form representation of the first and second moments of the single-use reliability. The model and the validity of the results are illustrated through a numerical example.

Keywords: system reliability; semi-Markov chain; usage model

MSC 2010: 60K15, 65R20, 90B25

1. Introduction

The themes of performance and dependability analysis of a general system have acquired high relevance and they have been extensively studied in the past. Markov chains provide a useful approach to the modeling of general systems. Indeed, Markov chain usage models provide statistical techniques for testing general systems and software, see [14] and [15].

Many variants of Markov chain usage models have been suggested in reliability literature. The most common approach is the one known as the *arc-based Bayesian model* (see [13]). These models estimate the mean and variance of the single-use reliability of a system. Usually, simulation techniques are applied with the inconvenience of requiring the generation of many test cases which may cause large time consumption. For this reason, [11] proposed the use of analytical solutions for the mean and variance of the single-use reliability.

The increasing complexity of real systems cannot be represented adequately through Markov chain models because they impose undesirable constraints. The most important inadequacies are represented by the memoryless property of Markov processes and by the rather unrealistic hypothesis of constant transition intensities between the states of the system.

Semi-Markov processes are a wide class of stochastic processes that generalize both Markov chains and renewal processes [10]. Their main advantage is that they allow to use any type of waiting time distribution function for modeling the time of switching the system from one state to another.

The semi-Markov models offer a solution to some of the drawbacks of the Markovian models and for this reason they were extensively explored in reliability studies, see e.g. [1], [3], [2], [5], [6], [9], [7].

In this work we advance a semi-Markov chain usage model in discrete time and we provide analytical solutions for the mean and variance of the single-use reliability of the system. Thus we generalize substantially the paper [11] by allowing effective application of a semi-Markov model for statistical testing of systems. The major advantage of our semi-Markov model lies in the possibility to use any type of waiting time distribution function and arc transition reliabilities depending not only on the states linked by the arc but also on the length of the sojourn time before a transition is executed. The price to pay for this model's increased flexibility is the additional complexity in the derivation of analytical representation of the mean and variance of the single-use reliability which requires now the use of the theory of geometric transform.

The paper is organized as follows: first, in Section 2 we present a short description of the semi-Markov chains and we introduce basic notation adopted in the analysis. Next, in Section 3 the single-use semi-Markov model is presented. In this section we derive the main results concerning the representation of the single-use reliability, its mean and variance. Subsequently, Section 4 demonstrates the model applied to a numerical example. Finally, Section 5 presents some concluding remarks.

2. Semi-Markov chains

In this part, the semi-Markov chain is described, following the notation given in [8].

Let us consider a finite set of states $E = \{1, 2, \ldots, S\}$ in which the system can be and a complete probability space $(\Omega, F, \mathbb{P})$ on which we define the following random variables:

$$J_n\colon\ \Omega \to E, \quad T_n\colon\ \Omega \to \mathbb{N}. \tag{2.1}$$

They denote the state occupied at the nth transition and the time of the nth transition, respectively.

The process $(J,T) = (J_n, T_n)_{n\in\mathbb{N}}$ is a discrete time homogeneous Markov Renewal Process if for all $i,j \in E$ for all $t \in \mathbb{N}$, it satisfies the following conditional independence assumption:

$$\begin{aligned} \mathbb{P}[J_{n+1} = j,\ T_{n+1} - T_n = t \mid \sigma(J_s, T_s),\ J_n = i,\ 0 \leqslant s \leqslant n] \\ = \mathbb{P}[J_{n+1} = j,\ T_{n+1} - T_n = t \mid J_n = i] := q_{ij}(t). \end{aligned} \tag{2.2}$$

The conditional probabilities $q_{ij}(t)$ for all $i,j \in E$ and $t \in \mathbb{N}$, are stored in a matrix of functions $\mathbf{q} = (q_{ij}(t))$ called the kernel of the (J,T) process, see [1]. The element $q_{ij}(t)$ represents the probability that the next visited state will be j with a sojourn time t, given that at present the process entered the state i.

The process $\{J_n\}$ is a Markov chain with state space E and transition probability matrix $\mathbf{P} = \lim_{t\to\infty} \sum_{\tau=1}^{t} \mathbf{q}(\tau)$. We shall refer to it as the embedded Markov chain.

Now it is possible to define the conditional cumulative distribution functions of the waiting time in each state i, given the subsequently occupied state j is known:

$$\begin{aligned} G_{ij}(t) &:= \mathbb{P}[T_{n+1} - T_n \leqslant t \mid J_n = i, J_{n+1} = j] \\ &= \frac{1}{p_{ij}} \sum_{s=1}^{t} q_{ij}(s) \cdot 1_{\{p_{ij}\neq 0\}} + 1_{\{p_{ij}=0\}}. \end{aligned} \tag{2.3}$$

For a fixed $N(t) = \sup\{n \in \mathbb{N} \mid T_n \leqslant t\}$ for all $t \in \mathbb{N}$, the discrete time semi-Markov chain $Z = (Z(t), t \in \mathbb{N})$ can be defined as $Z(t) = J_{N(t)}$. It represents, for each waiting time, the state occupied by the process J_n or, equivalently, the visited state at the calendar time t.

Let us assume that $T_0 = 0$, then we define for all $i,j \in E$, and $t \in \mathbb{N}$, the semi-Markov transition probabilities in the following way:

$$\varphi_{ij}(t) := \mathbb{P}[J_{N(t)} = j \mid J_0 = i]. \tag{2.4}$$

They are obtained by solving the following system of evolution equations:

$$\varphi_{ij}(t) = \delta_{ij}\left(1 - \sum_{j\in E}\sum_{\tau=1}^{t} q_{ij}(\tau)\right) + \sum_{k\in E}\sum_{s=1}^{t} q_{ik}(s)\varphi_{kj}(t-s). \tag{2.5}$$

Algorithms to solve equations (2.5) are well known, see for example [8].

3. The single use semi-Markov model

In this section, we make an extension of the Markov chain usage model proposed in [11]. We advance a semi-Markov usage model for computing system reliability. Semi-Markov processes have been extensively used in reliability studies, see e.g. [4], [1], [3], [5], [6], [7]. The interested reader can refer also to the books [10], [8], [2].

Let us assume that the system can be in one of the mutually exclusive states of $E = \{1, 2, \dots, S\}$. As usual, we assume that the state S corresponds to the model sink, which is the sole absorbing state. The system changes state according to the semi-Markov kernel (2.2), and a transition from the state i to the state j with sojourn time equal to s is executed with probability $q_{ij}(s)$.

The transition from one state to another coincides with an action in the system, for example, in software reliability estimates, the loading of a document. This action can be executed successfully or can result in a failure.

Let $R_{ij}(s)$ be the random variable called *transition reliability at s* that represents the fraction of successful transitions from the state i to the state j with sojourn time of length s. We assume that the random variables $R_{ij}(s)$ and $R_{hk}(t)$ are independent for $(i, j, s) \neq (h, k, t)$.

As we can see, the semi-Markov model may require the specification of transition reliabilities for each pair of states i, j and for each time s, because the transitions occur after a random sojourn time. In the Markov chain usage model the transition reliabilities depend only on the states i, j and not on the time because there is no randomness in the time of the next transition, see e.g. [11].

Let $F_{ij}(s) := 1 - R_{ij}(s)$ be the random variable called *transition failure rate* that represents the fraction of unsuccessful transitions from the state i to the state j with the next transition having sojourn time length s.

The single-use reliability model is completely described by the tuple $\{E, \mathbf{q}, \mathbf{R}\}$.

We are interested in the computation of expectation and variance of the single-use reliability. The single-use reliability F_i^* is the random variable that represents the fraction of times one experiences a failure prior to reaching the sink given that one starts in the state i.

Theorem 1. *For the semi-Markov usage model $\{E, \mathbf{q}, \mathbf{R}\}$, the single-use reliability satisfies the following system of equations*

$$F_i^* = \sum_{k \in E} \sum_{\gamma > 0} q_{ik}(\gamma) F_{ik}(\gamma) + \sum_{k \in E} \sum_{\gamma > 0} q_{ik}(\gamma) R_{ik}(\gamma) F_k^*, \tag{3.1}$$

where $F_S^* := 0$.

P r o o f. Let us now assume that the arc failure rate parameters $F_{ij}(s) = f_{ij}(s)$ are known and deterministic quantities. Denote by f_i^* the probability of encountering at least one failure in a random sequence of infinite length originating from the state i. Let $t^{(n)}(i,j)$ be the random variable which denotes the number of failures in a sequence of length $n+1$ originating from the state i and ending in the state j. Denote by $f_i^{(\leqslant n)} = \mathbb{P}[t^{(n)}(i,S) > 0]$. Then

$$f_i^{(\leqslant n)} = \mathbb{P}[t^{(1)}(i,J_1) = 1] + \mathbb{P}[t^{(1)}(i,J_1) = 0] \times \mathbb{P}[t^{(n-1)}(J_1,S) > 0 \mid t^{(1)}(i,J_1) = 0]. \tag{3.2}$$

Observe that

$$\begin{aligned} \mathbb{P}[t^{(1)}(i,J_1) = 1] &= \sum_{j\in E}\sum_{\gamma>0} \mathbb{P}[J_1 = j, T_1 = \gamma, F_{iJ_1}(\gamma) = 1 \mid J_0 = i] \\ &= \sum_{j\in E}\sum_{\gamma>0} \mathbb{P}[J_1 = j, T_1 = \gamma \mid J_0 = i] \cdot \mathbb{P}[F_{iJ_1}(\gamma) = 1 \mid J_1 = j, T_1 = \gamma, J_0 = i] \\ &= \sum_{j\in E}\sum_{\gamma>0} q_{ij}(\gamma) f_{ij}(\gamma). \end{aligned} \tag{3.3}$$

Moreover,

$$\begin{aligned} &\mathbb{P}[t^{(1)}(i,J_1) = 0]\mathbb{P}[t^{(n-1)}(J_1,S) > 0 \mid t^{(1)}(i,J_1) = 0] \\ &\quad = \sum_{j\in E}\sum_{\gamma>0} \mathbb{P}[F_{iJ_1}(T_1) = 0 \mid J_1 = j, T_1 = \gamma, J_0 = i] \\ &\quad\quad \times \mathbb{P}[J_1 = j, T_1 = \gamma \mid J_0 = i] \cdot \mathbb{P}[t^{(n-1)}(J_1,S) > 0 \mid J_1 = j, F_{i,j}(\gamma) = 0] \\ &\quad = \sum_{j\in E}\sum_{\gamma>0} q_{ij}(\gamma)(1 - f_{ij}(\gamma)) f_j^{(\leqslant n-1)}. \end{aligned} \tag{3.4}$$

By substitution of (3.3) and (3.4) into formula (3.2) we obtain

$$f_i^{(\leqslant n)} = \sum_{j\in E}\sum_{\gamma>0} q_{ij}(\gamma) f_{ij}(\gamma) + \sum_{j\in E}\sum_{\gamma>0} q_{ij}(\gamma)(1 - f_{ij}(\gamma)) f_j^{(\leqslant n-1)}. \tag{3.5}$$

Next, consider that for all $i \in E$ and for all $n \in \mathbb{N}$, $f_i^{(\leqslant n)} \geqslant f_i^{(\leqslant n-1)}$ and $f_i^{(\leqslant n)} \leqslant 1$. Therefore, $\{f_i^{(\leqslant n)}\}_{n\in\mathbb{N}}$ is an increasing and bounded sequence, so it admits a limit. Let us set

$$f_i^* := \lim_{n\to\infty} f_i^{(\leqslant n)} = \lim_{n\to\infty} \mathbb{P}[t^{(n)}(i,S) > 0]. \tag{3.6}$$

By taking the limit in (3.5), we get

$$f_i^* = \sum_{j\in E}\sum_{\gamma>0} q_{ij}(\gamma) f_{ij}(\gamma) + \sum_{j\in E}\sum_{\gamma>0} q_{ij}(\gamma)(1 - f_{ij}(\gamma)) f_j^*. \tag{3.7}$$

Finally, observe that the actual failure rates $F_{ij}(s)$ are unknown and consequently must be represented as random variables. Replacing the arc failure rates parameters $f_{ij}(s)$ with the corresponding random variables produces

$$F_i^* = \sum_{k\in E}\sum_{\gamma>0} q_{ik}(\gamma)F_{ik}(\gamma) + \sum_{k\in E}\sum_{\gamma>0} q_{ik}(\gamma)R_{ik}(\gamma)F_k^*. \tag{3.8}$$

□

R e m a r k 1. If the semi-Markov kernel has the special form with waiting time distributions $G_{ij}(t) = 1_{\{t=1\}}$, then it is simple to verify that $q_{ij}(t) = p_{ij}1_{\{t=1\}}$. In this particular case the semi-Markov chain specializes to a Markov chain and Theorem 1 coincides with the result proved in [12] and [11], i.e.:

$$F_i^* = \sum_{k\in E} p_{ik}F_{ik} + \sum_{k\in E} p_{ik}R_{ik}F_k^*. \tag{3.9}$$

R e m a r k 2. The semi-Markov environment allows us to consider a more in-depth description of the system behavior, because transitions occur at random times governed by whatever type of distribution function $G_{ij}(\cdot)$. Moreover, the arc failure rates $F_{ij}(s)$ may be considered to be dependent on the sojourn time length too.

It is in our best interest to compute expectation and variance of the single-use random variable F_i^*. As pointed out in [12] and reported in [11], in a Markov chain based model it is not possible to take directly the expectation of equation (3.1) because the F_i^* are not independent. To overcome this problem they suggested to use the integral forms of expectation and variance. Here, we extend this approach to the more general semi-Markovian framework.

Proposition 1. *For the semi-Markov usage model* $\{E, \mathbf{q}, \mathbf{R}\}$*, the single-use reliability admits the following representation:*

$$F_i^* = \sum_{\gamma>0}\sum_{m>0} F_i(\gamma; m), \tag{3.10}$$

where for all $\gamma \in \mathbb{N}$, $m \in \mathbb{N}\cup\{0\}$ *and* $i \neq S$

$$F_i(\gamma; m) = \begin{cases} \displaystyle\sum_{k\neq S}\sum_{s=1}^{\gamma} q_{ik}(s)R_{ik}(s)F_k(\gamma-s; m-1), & \text{if } m = 2,\dots,\gamma, \\ \displaystyle\sum_{k\neq S} q_{ik}(\gamma)F_{ik}(\gamma), & \text{if } m = 1, \\ 0, & \text{if } m > \gamma \text{ or } m = 0. \end{cases} \tag{3.11}$$

P r o o f. Theorem 1 supplies a recursive representation of the single-use reliability, see equation (3.1). If we substitute into equation (3.1) the definition (3.11) for $m=1$, we have that

$$F_i^* = \sum_{\gamma>0} F_i(\gamma;1) + \sum_{k\in E}\sum_{\gamma>0} q_{ik}(\gamma)R_{ik}(\gamma)F_k^*,$$

which can be rewritten, by applying again equation (3.11) to F_k^*, as

$$\begin{aligned}(3.12)\qquad F_i^* &= \sum_{\gamma>0} F_i(\gamma;1) + \sum_{k\in E}\sum_{\gamma>0} q_{ik}(\gamma)R_{ik}(\gamma)\\ &\quad\times\Big(\sum_{k_1\in E}\sum_{\gamma_1>0} q_{kk_1}(\gamma_1)F_{kk_1}(\gamma_1) + \sum_{k_1\in E}\sum_{\gamma_1>0} q_{kk_1}(\gamma_1)R_{kk_1}(\gamma_1)F_{k_1}^*\Big)\\ &= \sum_{\gamma>0} F_i(\gamma;1) + \sum_{\gamma>0}\sum_{k\in E}\sum_{k_1\in E}\sum_{\gamma_1>0} q_{ik}(\gamma)R_{ik}(\gamma)q_{kk_1}(\gamma_1)F_{kk_1}(\gamma_1)\\ &\quad+\sum_{k\in E}\sum_{\gamma>0} q_{ik}(\gamma)R_{ik}(\gamma)\sum_{k_1\in E}\sum_{\gamma_1>0} q_{kk_1}(\gamma_1)R_{kk_1}(\gamma_1)F_{k_1}^*.\end{aligned}$$

Now consider the second addend on the r.h.s. of (3.12),

$$\begin{aligned}(3.13)\qquad &\sum_{\gamma>0}\sum_{k\in E}\sum_{k_1\in E}\sum_{\gamma_1>0} q_{ik}(\gamma)R_{ik}(\gamma)q_{kk_1}(\gamma_1)F_{kk_1}(\gamma_1)\\ &\quad= \sum_{\gamma>0}\sum_{k\in E}\sum_{\gamma_1>0} q_{ik}(\gamma)R_{ik}(\gamma)F_k(\gamma_1;1)\\ &\quad= \sum_{t>1}\sum_{k\neq S}\sum_{\gamma=1}^{t} q_{ik}(\gamma)R_{ik}(\gamma)F_k(t-\gamma;1) = \sum_{t>0} F_i(t;2),\end{aligned}$$

where the last equality is obtained by considering the change of variable $t=\gamma+\gamma_1$ and the fact that $F_i(1;2)=0$, because $1=\gamma<m=2$.

By substitution in (3.12) we get

$$\begin{aligned}(3.14)\qquad F_i^* &= \sum_{\gamma>0} F_i(\gamma;1) + \sum_{\gamma>0} F_i(\gamma;2)\\ &\quad+\sum_{k\in E}\sum_{\gamma>0}\sum_{k_1\in E}\sum_{\gamma_1>0} q_{ik}(\gamma)R_{ik}(\gamma)q_{kk_1}(\gamma_1)R_{kk_1}(\gamma_1)F_{k_1}^*,\end{aligned}$$

and by iteration we get the claimed result. □

Let us set $F_i^{(1)}(\gamma,m) := \mathbb{E}[F_i(\gamma,m)]$ and $F_i^{(1,1)}(a,m,b,n) := \mathbb{E}[F_i(a,m)F_i(b,n)]$.

Proposition 2. *For $a, b, \gamma, n, m \in \mathbb{N}$ with $b \geqslant a$ and $n > m \geqslant 1$ it holds that*

$$F_i^{(1)}(\gamma; m) = \begin{cases} \sum_{k \neq S} q_{ik}(\gamma)\mathbb{E}[F_{ik}(\gamma)], & \text{if } m = 1, \\ \sum_{k \neq S}\sum_{s=1}^{\gamma} q_{ik}(s)\mathbb{E}[R_{ik}(s)]F_k^{(1)}(\gamma - s; m-1), & \text{if } m > 1. \end{cases} \tag{3.15}$$

$$F_i^{(1,1)}(a, m, b, n) = \begin{cases} \sum_{k \neq S}\sum_{s=1}^{a} q_{ik}(s)(\mathbb{E}[R_{ik}(s)] - \mathbb{E}[(R_{ik}(s))^2]) \\ \qquad \times F_k^{(1)}(b - s; n-1), \qquad \text{if } m = 1,\ n > m, \\ \sum_{k \neq S}\sum_{s=1}^{a} \mathbb{E}[R_{ik}^2(s)]F_k^{(1,1)}(a - s, m-1, b-s; n-1), \\ \qquad\qquad \text{if } n > m > 1. \end{cases} \tag{3.16}$$

P r o o f. Due to similarity we prove only (3.15). Let us consider a trajectory making provision for $m-1$ successful transitions and the last unsuccessful. The trajectory is composed by a sequence of states $\{k_i\}_{i=1}^m$ and transition times $\{\gamma_i\}_{i=0}^m$ with $\gamma_m = \gamma$. Consequently, the expectation of $F_i(\gamma; m)$ can be computed by using the integral form of expectation as follows:

$$\begin{aligned} F_i^{(1)}(\gamma; m) = &\sum_{k_1,\dots,k_m}\sum_{\gamma_1,\dots,\gamma_{m-1}} q_{ik_1}(\gamma_1)q_{k_1k_2}(\gamma_2)\dots q_{k_{m-1}k_m}\Bigl(\gamma - \sum_{h=1}^{m-1}\gamma_h\Bigr) \\ &\times \int_0^1 \dots \int_0^1 \mathbb{P}\Bigl[R_{ik_1}(\gamma_1) = r_1, R_{k_1k_2}(\gamma_2) = r_2, \dots, R_{k_{m-2},k_{m-1}}(\gamma_{m-1}) = r_{m-1}, \\ &R_{k_{m-1},k_m}\Bigl(\gamma - \sum_{h=1}^{m-1}\gamma_h\Bigr) = r_m\Bigr] r_1 r_2 \dots r_{m-1}(1 - r_m)\,\mathrm{d}r_1 \dots \mathrm{d}r_m. \end{aligned} \tag{3.17}$$

Now, since we assumed the random variables $R_{ij}(s)$ and $R_{hk}(t)$ to be independent we can rewrite (3.17) as follows:

$$\begin{aligned} F_i^{(1)}(\gamma; m) = &\sum_{s_1}\sum_{\gamma_1} q_{ik_1}(\gamma_1)\int_0^1 \mathbb{P}[R_{ik_1}(\gamma_1) = r_1]r_1\,\mathrm{d}r_1 \\ &\times \sum_{k_2,\dots,k_m}\sum_{\gamma_2,\dots,\gamma_{m-1}} q_{k_1k_2}(\gamma_2)\dots q_{k_{m-1}k_m}\Bigl(\gamma - \sum_{h=1}^{m-1}\gamma_h\Bigr) \end{aligned}$$

$$
\times \int_0^1 \dots \int_0^1 \mathbb{P}\Bigg[R_{k_1k_2}(\gamma_2) = r_2, \dots, R_{k_{m-2}k_{m-1}}(\gamma_{m-1}) = r_{m-1},
$$

$$
R_{k_{m-1}k_m}\left(\gamma - \sum_{h=1}^{m-1} \gamma_h\right) = r_m\Bigg] r_2 \dots r_{m-1}(1 - r_m)\,\mathrm{d}r_2 \dots \mathrm{d}r_m
$$

$$
= \sum_{k_1 \neq S} \sum_{\gamma_1=1}^{\gamma} q_{ik_1}(\gamma_1) E[R_{ik_1}(\gamma_1)] F_{s_1}^{(1)}(\gamma - \gamma_1; m-1).
$$

□

Let $\mathbf{A}$ be a square matrix of order m. By $\dot{\mathbf{A}}$ we denote the corresponding submatrix of dimension $m \times (m-1)$ obtained by deleting the last column of $\mathbf{A}$.

Denote by $\dot{\mathbf{R}}^{(1)}(\gamma) = (\mathbb{E}[R_{ij}(\gamma)])_{i=1,\dots,S;j=1,\dots,S-1}$ and consequently denote by $\dot{\mathcal{R}}_q^{(1)}(\gamma) = \dot{\mathbf{q}}(\gamma) \,\square\, \dot{\mathbf{R}}^{(1)}(\gamma)$, where $\square$ is the Hadamard (element by element) matrix product.

Definition 1. Given a matrix of functions $\mathbf{w} = (w_{ij}(t))$, $i, j \in E$, $t \in \mathbb{N} \cup \{0\}$ and a vector of functions $\mathbf{y} = (y_i(t, m))$, $m, t \in \mathbb{N} \cup \{0\}$, we define the following convolution product:

$$
(w * y)(n, m) := \sum_{t=0}^{n} \mathbf{w}(t) \cdot \mathbf{y}(n - t, m - 1).
$$

Proposition 3. *Let $\mathbf{F}^* = (F_i^*)_{i=1,2,\dots,S-1}$. Then the expectation of the single-use reliability is given by:*

$$
\mathbb{E}[\mathbf{F}^*] = (\mathbf{I} - {}^{\mathbf{g}}\dot{\mathcal{R}}(1))^{-1} \cdot {}^{g}\mathbf{H}(1), \tag{3.18}
$$

where ${}^{\mathbf{g}}\dot{\mathcal{R}}(1)$ and ${}^{g}\mathbf{H}(1)$ are the matrices of geometric transforms of $\dot{\mathcal{R}}_{\mathbf{q}}^{1}(\gamma)$ and $\mathbf{H}(\gamma) := \dot{q}(\gamma) \,\square\, \mathbb{E}[\mathbf{F}(\gamma)] \cdot \mathbf{1}_S$, respectively, evaluated at $z = 1$ and $\mathbf{1}_S$ is the unitary vector.

P r o o f. From Proposition (2), for $m > 1$, we know that:

$$
\begin{aligned}
\mathbf{F}^{(1)}(\gamma; m) &= \sum_{s=1}^{\gamma} \dot{\mathcal{R}}_q^{(1)}(\gamma) \cdot \mathbf{F}^{(1)}(\gamma - s; m - 1) \\
&= (\dot{\mathcal{R}}_q^{(1)} * \mathbf{F}^{(1)})(\gamma, m).
\end{aligned} \tag{3.19}
$$

For $m = 1$ we have

$$
\mathbf{F}^{(1)}(\gamma; 1) = \dot{q}(\gamma) \,\square\, \mathbb{E}[\mathbf{F}(\gamma)] \cdot \mathbf{1}_S = \mathbf{H}(\gamma). \tag{3.20}
$$

Denote by

$$(3.21)\qquad {}^g\mathbf{F}^{(1)}(z,m) = \sum_{\gamma=0}^{\infty} \mathbf{F}^{(1)}(\gamma;m)z^{\gamma}$$

the geometric transform of the vector of functions $\mathbf{F}^{(1)}(\gamma;m)$.

Since the geometric transform of a convolution corresponds to transform multiplication, we have from (3.19) that

$${}^g\mathbf{F}^{(1)}(z;m) = {}^g\dot{\mathcal{R}}_q(z) \cdot {}^g\mathbf{F}^{(1)}(z;m-1)$$

and

$${}^g\mathbf{F}^{(1)}(z;1) = {}^g\mathbf{H}(z).$$

Then

$${}^g\mathbf{F}^{(1)}(z;2) = {}^g\dot{\mathcal{R}}_q(z) \cdot {}^g\mathbf{F}^{(1)}(z;1) = {}^g\dot{\mathcal{R}}_q(z) \cdot {}^g\mathbf{H}(z),$$

and inductively we get

$${}^g\mathbf{F}^{(1)}(z;m) = ({}^g\dot{\mathcal{R}}_q(z))^{m-1} \cdot {}^g\mathbf{H}(z).$$

Now since the geometric transform evaluated at $z = 1$ is just the infinite sum of the discrete function, we have that

$${}^g\mathbf{F}^{(1)}(1;m) = \sum_{\gamma=0}^{\infty} \mathbf{F}^{(1)}(\gamma;m),$$

and consequently

$$\sum_{m>0}^{\infty} {}^g\mathbf{F}^{(1)}(1;m) = \sum_{m>0}^{\infty} \sum_{\gamma=0}^{\infty} \mathbf{F}^{(1)}(\gamma;m)] = \mathbb{E}[\mathbf{F}^*],$$

where the last equality follows from Proposition 1 and the fact that $\mathbf{F}^{(1)}(0,m) = 0$. This leads to the following representation:

$$\begin{aligned}\mathbb{E}[\mathbf{F}^*] &= \sum_{m=1}^{\infty} {}^g\mathbf{F}^{(1)}(1;m) = \sum_{m=1}^{\infty} ({}^g\dot{\mathcal{R}}_q(1))^{m-1} \cdot {}^g\mathbf{H}(1) \\ &= (\mathbf{I} - {}^g\dot{\mathcal{R}}_q(1))^{-1} \cdot {}^g\mathbf{H}(1).\end{aligned}$$

□

It should be pointed out that if $m = 0$ we know from (3.11) that $F_i^{(1)}(\gamma;0) = 0$ and consequently $\mathbb{E}[\mathbf{F}^*] = 0$.

Proposition 4. *The variance of the single-use reliability is given by:*

$$V[\mathbf{F}^*] = (\mathbf{I} - {}^g\dot{\mathcal{R}}_q^{(2)}(1))^{-1} \cdot {}^g\mathbf{H}^{(2)}(1) + 2(\mathbf{I} - {}^g\dot{\mathcal{R}}_q^{(2)}(1))^{-1} \cdot {}^g(\dot{\mathcal{R}}_q^{(1)} - \dot{\mathcal{R}}_q^{(2)})(1) \cdot (\mathbf{I} - {}^g\dot{\mathcal{R}}_q^{(1)}(1))^{-1} \cdot {}^g\mathbf{H}(1) - ((\mathbf{I} - {}^g\dot{\mathcal{R}}_q(1))^{-1} \cdot {}^g\mathbf{H}(1))^2. \tag{3.22}$$

P r o o f. To compute the variance we need to evaluate only the second order moment of the single-use reliability because the first order moment has been already determined in Proposition 3.

From Proposition 1 we know that $F_i^* = \sum_{\gamma>0}\sum_{m>0} F_i(\gamma; m)$. Then the second order moment is given by

$$\mathbb{E}[(F_i^*)^2] = \mathbb{E}\Big[\Big(\sum_{m>0}\sum_{\gamma\geqslant m} F_i(\gamma; m)\Big)^2\Big].$$

For simplicity of notation, set $A_m = \sum_{\gamma\geqslant m} F_i(\gamma; m)$. Then we have

$$\mathbb{E}[(F_i^*)^2] = \mathbb{E}\Big[\Big(\sum_{m=1}^{\infty} A_m\Big)^2\Big] = \mathbb{E}\Big[\sum_{m=1}^{\infty}(A_m)^2 + 2\sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty} A_m A_n\Big] = \sum_{m=1}^{\infty}\mathbb{E}[(A_m)^2] + 2\sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty}\mathbb{E}[A_m A_n]. \tag{3.23}$$

Let's start to compute the first addend of (3.23). Denote by $\mathbf{F}^{(2)}(\gamma, m) = (\mathbb{E}[(F_i(\gamma, m))^2])_{i=1,\dots,S-1}$. Similar computations as those executed in Proposition 2 give

$$\mathbf{F}^{(2)}(\gamma, m) := \begin{cases} \mathbf{H}^{(2)}(\gamma) := \dot{\mathbf{q}}(\gamma) \,\square\, \mathbb{E}[\mathbf{F}^{(2)}(\gamma)] \cdot \mathbf{1}_{S-1}, & \text{if } m = 1, \\ (\dot{\mathcal{R}}_q^{(2)} * \mathbf{F}^{(2)})(\gamma, m-1), & \text{if } m > 1. \end{cases} \tag{3.24}$$

If we denote by $\dot{\mathcal{F}}^{(2)}(\gamma) = \dot{\mathbf{q}}(\gamma) \,\square\, \mathbb{E}[\mathbf{F}^{(2)}(\gamma)]$, then $\mathbf{F}^{(2)}(\gamma, 1) = \dot{\mathcal{F}}^{(2)}(\gamma) \cdot \mathbf{1}_{S-1}$ and ${}^g\mathbf{F}^{(2)}(z, 1) = {}^g\mathbf{H}^{(2)}(z)$, consequently

$${}^g\mathbf{F}^{(2)}(z, 2) = {}^g\dot{\mathcal{R}}_q^{(2)}(z) \cdot {}^g\mathbf{F}^{(2)}(z, 1) = {}^g\dot{\mathcal{R}}_q^{(2)}(z) \cdot {}^g\mathbf{H}^{(2)}(z),$$

and by induction

$${}^g\mathbf{F}^{(2)}(z, m) = ({}^g\dot{\mathcal{R}}_q^{(2)}(z))^{m-1} \cdot {}^g\mathbf{H}^{(2)}(z).$$

Then, if we evaluate the sum of this geometric transform at $z = 1$, we obtain

$$\begin{aligned} \text{(3.25)} \qquad \sum_{m=1}^{\infty} {}^{g}\mathbf{F}^{(2)}(1, m) &= \sum_{m=1}^{\infty} ({}^{g}\dot{\mathcal{R}}_q^{(2)}(1))^{m-1} \cdot {}^{g}\mathbf{H}^{(2)}(1) \\ &= (\mathbf{I} - {}^{g}\dot{\mathcal{R}}_q^{(2)}(1))^{-1} \cdot {}^{g}\mathbf{H}^{(2)}(1). \end{aligned}$$

Next step consists in computing the term

$$\text{(3.26)} \qquad \sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} \mathbb{E}[A_m A_n] = \sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} \sum_{a \geqslant m} \sum_{b \geqslant n} \mathbb{E}[F_i(a, m) F_i(b, n)].$$

To this end it is worth noting that the quantity $F_i^{(1,1)}(a, m, b, n) = \mathbb{E}[F_i(a, m) \times F_i(b, n)]$ was evaluated in formula (3.16). This formula can be expressed in matrix form as follows:

$$\mathbf{F}^{(1,1)}(a, m, b, n) := \mathbb{E}[\mathbf{F}^{(1)}(a, m) \mathbf{F}^{(1)}(b, n)],$$

where

$$\text{(3.27)} \quad \mathbf{F}^{(1,1)}(a, m, b, n) := \begin{cases} ((\dot{R}_q^{(1)} - \dot{R}_q^{(2)}) * \mathbf{F}^{(1)})(b, n-1), & \text{if } m = 1, n > m, \\ (\dot{R}_q^{(2)} * \mathbf{F}^{(1,1)})(a, m-1, b, n-1), & \text{if } m > 1,\ n > m. \end{cases}$$

Now let us consider the double geometric transform

$$\begin{aligned} {}^{g}\mathbf{F}^{(1,1)}(z_1, m, z_2, n) &= \sum_{a \geqslant 0} \sum_{b \geqslant 0} \mathbf{F}^{(1,1)}(a, m, b, n) z_1^a z_2^b \\ &= \sum_{a \geqslant 0} \sum_{b \geqslant 0} \sum_{\gamma=1}^{a} \dot{R}_q^{(2)}(\gamma) \mathbf{F}^{(1,1)}(a-\gamma, m-1, b-\gamma, n-1) z_1^{a-\gamma} z_2^{b-\gamma} \\ &= \sum_{a \geqslant 0} \sum_{b \geqslant 0} \sum_{\gamma=1}^{a} \dot{R}_q^{(2)}(\gamma) z_1^{\gamma} z_2^{\gamma} \mathbf{F}^{(1,1)}(a-\gamma, m-1, b-\gamma, n-1) z_1^{a-\gamma} z_2^{b-\gamma} \\ &= \sum_{\gamma=1}^{\infty} \dot{R}_q^{(2)}(\gamma) z_1^{\gamma} z_2^{\gamma} \sum_{a=\gamma}^{\infty} \sum_{b=\gamma}^{\infty} \mathbf{F}^{(1,1)}(a-\gamma, m-1, b-\gamma, n-1) z_1^{a-\gamma} z_2^{b-\gamma} \\ &= {}^{g}R_q^{(2)}(z_1, z_2) \cdot {}^{g}\mathbf{F}^{(1,1)}(z_1, m-1, z_2, n-1), \end{aligned}$$

where we defined ${}^{g}R_q^{(2)}(z_1, z_2) = \sum_{\gamma=1}^{\infty} \dot{R}_q^{(2)}(\gamma) z_1^{\gamma} z_2^{\gamma}$.

Now let us fix $n > m \geqslant 1$. Then

$$\begin{aligned} {}^{g}\mathbf{F}^{(1,1)}(z_1, m, z_2, n) &= {}^{g}R_q^{(2)}(z_1, z_2) \cdot {}^{g}\mathbf{F}^{(1,1)}(z_1, m-1, z_2, n-1) \\ &= {}^{g}R_q^{(2)}(z_1, z_2) \cdot {}^{g}R_q^{(2)}(z_1, z_2) \cdot {}^{g}\mathbf{F}^{(1,1)}(z_1, m-2, z_2, n-2), \end{aligned}$$

and by induction we obtain

$$
\begin{aligned}
(3.28)\qquad & {}^{g}\mathbf{F}^{(1,1)}(z_1,m,z_2,n) \\
&= ({}^{g}R_q^{(2)}(z_1,z_2))^{m-1}\cdot {}^{g}\mathbf{F}^{(1,1)}(z_1,1,z_2,n-m+1) \\
&= ({}^{g}R_q^{(2)}(z_1,z_2))^{m-1}\cdot ({}^{g}(R_q^{(1)}-R_q^{(2)})(z_1,z_2))\cdot {}^{g}\mathbf{F}^{(1,1)}(z_1,0,z_2,n-m).
\end{aligned}
$$

Finally, by noting that

$$
{}^{g}\mathbf{F}^{(1,1)}(z_1,0,z_2,n-m) = ({}^{g}R_q^{(1)}(z_2))^{n-m-1}\cdot {}^{g}\mathbf{H}(z_2),
$$

we get

$$
\begin{aligned}
(3.29)\qquad & {}^{g}\mathbf{F}^{(1,1)}(z_1,m,z_2,n) \\
&= ({}^{g}R_q^{(2)}(z_1,z_2))^{m-1}\cdot {}^{g}(R_q^{(1)}-R_q^{(2)})(z_1,z_2)\cdot ({}^{g}R_q^{(1)}(z_2))^{n-m-1}\cdot {}^{g}\mathbf{H}(z_2).
\end{aligned}
$$

Evaluating (3.29) at $z_1 = z_2 = 1$, we have

$$
{}^{g}\mathbf{F}^{(1,1)}(1,m,1,n) = \sum_{a\geqslant m}\sum_{b\geqslant n}\mathbf{F}^{(1,1)}(a,m,b,n) = \sum_{a\geqslant m}\sum_{b\geqslant n}\mathbb{E}[F_i(a,m)F_i(b,n)],
$$

and therefore by (3.27) and (3.28) we come to

$$
\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty}\sum_{a\geqslant m}\sum_{b\geqslant n}\mathbb{E}[F_i(a,m)F_i(b,n)] &= \sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty}{}^{g}\mathbf{F}^{(1,1)}(1,m,1,n) \\
&= \sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty}({}^{g}R_q^{(2)}(1,1))^{m-1}\cdot {}^{g}(R_q^{(1)}-R_q^{(2)})(1,1)\cdot({}^{g}R_q^{(1)}(1))^{n-m-1}\cdot {}^{g}\mathbf{H}(1) \\
&= \sum_{m=1}^{\infty}({}^{g}R_q^{(2)}(1,1))^{m-1}\cdot {}^{g}(R_q^{(1)}-R_q^{(2)})(1,1)\sum_{n=m+1}^{\infty}({}^{g}R_q^{(1)}(1))^{n-m-1}\cdot {}^{g}\mathbf{H}(1) \\
&= \sum_{m=1}^{\infty}({}^{g}R_q^{(2)}(1,1))^{m-1}\cdot {}^{g}(R_q^{(1)}-R_q^{(2)})(1,1)\cdot(I-{}^{g}R_q^{(1)}(1))^{-1}\cdot {}^{g}\mathbf{H}(1),
\end{aligned}
$$

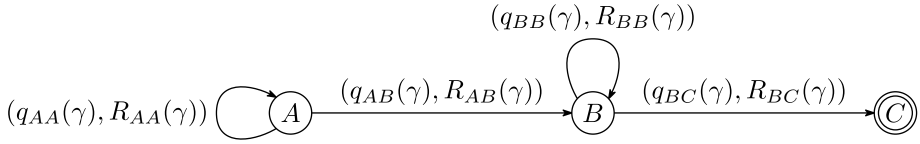
and considering that ${}^{g}R_q^{(2)}(1,1) = {}^{g}R_q^{(2)}(1)$ and ${}^{g}R_q^{(1)}(1,1) = {}^{g}R_q^{(1)}(1)$ we obtain

$$
\begin{aligned}
(3.30)\qquad & \sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty}{}^{g}\mathbf{F}^{(1,1)}(1,m,1,n) \\
&= (I-{}^{g}R_q^{(2)}(1))^{-1}\cdot {}^{g}(R_q^{(1)}-R_q^{(2)})(1)\cdot(I-{}^{g}R_q^{(1)}(1))^{-1}\cdot {}^{g}\mathbf{H}(1).
\end{aligned}
$$

A substitution of (3.25) and (3.30) in (3.23) completes the proof. □

4. Numerical example

In this section, a numerical example which illustrates the previous results is presented. We consider a system composed of three states, i.e., $E = \{A, B, C\}$, where state C corresponds to the model sink which is the sole absorbing state of the system. The evolution among the states is described by the following graph:



To describe the probabilistic behavior of the system we have to specify a semi-Markov kernel. To this end, first we assume that the transition probability matrix of the embedded Markov chain is described by the matrix $\mathbf{P}$ below:

$$\mathbf{P} = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0.60 & 0.40 & 0.00 \\ 0.00 & 0.20 & 0.80 \\ 0.00 & 0.00 & 1.00 \end{pmatrix} \end{array} \tag{4.1}$$

and second, we choose the matrix $\mathbf{G}$ of the conditional waiting time distribution functions as follows:

$$\begin{aligned} &G_{11}(\cdot) = \text{cdf}(\text{Weibul})(2,2),\ G_{12}(\cdot) = \text{cdf}(\text{Weibul})(2,3), \\ &G_{22}(\cdot) = \text{cdf}(\text{Weibul})(1,3),\ G_{23}(\cdot) = \text{cdf}(\text{Weibul})(2,3), \\ &G_{13}(\cdot) = G_{21}(\cdot) = G_{31}(\cdot) = G_{32}(\cdot) = G_{33}(\cdot) = \text{cdf}(\text{Unit distribution}) \end{aligned} \tag{4.2}$$

The symbol $\text{cdf}(\text{Weibul})(x, y)$ means the cumulative distribution function of a discrete Weibull distribution with parameters x and y and cdf(Unit distribution) denotes the cumulative distribution function of the unit distribution.

The r.v. X has a unit distribution if $\Pr[X \leqslant k] = 1$ for all $k \in \mathbb{N}$.

The r.v. X has a discrete Weibul(x, y) distribution if for all $k \in \mathbb{N}$

$$\mathbb{P}[X \leqslant k] = 1 - \mathrm{e}^{-(k/x)^y}. \tag{4.3}$$

The unit distribution describes the waiting time in state i before making a transition into j when $p_{ij} = 0$ and when state i coincides with the sink. Notice that when $p_{ij} = 0$, the corresponding conditional waiting time distribution can be defined as arbitrary. Consequently, this choice does not affect the results.

When $p_{ij} > 0$, we consider discrete Weibull distributions. The reason for this choice is that Weibull distributions are flexible and frequently used in the modeling of waiting times.

For simplicity we assume that the transition reliabilities are constant in time, that is, $R_{ij}(s) = R_{ij}$ for all $s \in \mathbb{N}$ and they are fixed as follows:

$$\mathbf{R} = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0.30 & 0.40 & 0.00 \\ 0.00 & 0.20 & 0.30 \\ 0.00 & 0.00 & 1.00 \end{pmatrix} \end{array}. \tag{4.4}$$

We computed the expectation of the single-use reliability function $\mathbb{E}[\mathbf{F}^*]$ which results in 0.9512 and 0.7500 for a system starting from state A and B, respectively.

We executed a scenario-sensitivity analysis by changing some of the input parameters. First of all we computed the values of the expectation of the single-use reliability in response to changes in the values of the transition probabilities of the embedded Markov chain (4.1). More precisely, we considered a new transition matrix

$$\mathbf{P} = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} x & 1-x & 0.00 \\ 0.00 & y & 1-y \\ 0.00 & 0.00 & 1.00 \end{pmatrix} \end{array}, \tag{4.5}$$

where x and y are allowed to change values between 0.10 to 1.00. The other parameters stay unchanged as in (4.2) and (4.4).

The results of the expected single-use reliability with initial state A and initial state B are summarized in Figure 1.

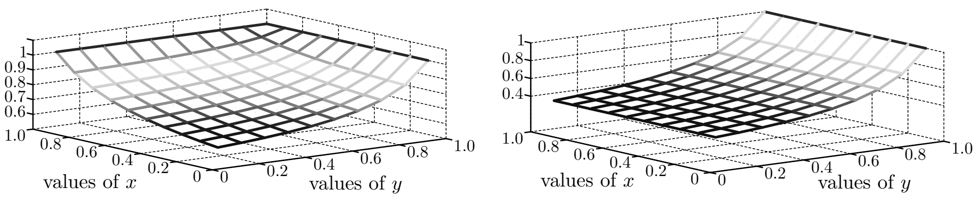


Figure 1. Single-use reliability depending on P for the initial state A (left panel) and state B (right panel).

The left panel of Figure 1 reveals that $\mathbb{E}[\mathbf{F}^*_A]$ increase monotonically in x and y. This means that the expected fraction of times one experiences a failure prior to reaching the sink (state C) increases with respect to variables x and y. Indeed, higher values of x (y) increase the probability of traversing the loop of the state

A (B) repeatedly and, hence, increase the possibility of having a failure during the execution of one of these loops. It should be noted that the increase of $\mathbb{E}[\mathbf{F}_A^*]$ is not linear in both x and y. The right panel shows that $\mathbb{E}[\mathbf{F}_B^*]$ does not depend on x because it is not possible to transit from the state B to the state A. The monotonicity with respect to y is confirmed in this case too.

We executed the sensitivity analysis by computing the expected values of the single-use reliability in response to changes in the values of the transition reliability matrix R. More precisely, we considered the transition reliability matrix

$$\mathbf{R} = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} z & 1-z & 0.00 \\ 0.00 & w & 1-w \\ 0.00 & 0.00 & 1.00 \end{pmatrix} \end{array} \tag{4.6}$$

where z and w are allowed to change values between 0.10 to 1.00. The other parameters stay unchanged as in (4.1) and (4.2).

The results of the expected single-use reliability with initial state A and initial state B are summarized in Figure 2.

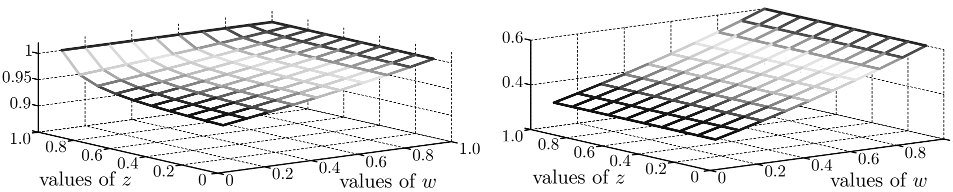


Figure 2. Single-use reliability depending on R for the initial state A (left panel) and state B (right panel).

The left panel of Figure 2 illustrates the dependence of the single-use reliability on the transition reliability matrix. The $\mathbb{E}[\mathbf{F}_A^*]$ increases monotonically in z and w. The right panel shows that $\mathbb{E}[\mathbf{F}_B^*]$ does not depend on z, because it is not possible to transit from the state B to the state A. The monotonicity with respect to w is confirmed in this case, too.

Figure 1 and Figure 2 show that the results are sensitive to the choice of the initial state. This is true because the system has different probabilities of reaching the model sink without failures depending on the starting state of the system. Indeed, in our example, the probabilities depend on the semi-Markov kernel and on the transition reliability matrix which are both markedly sensitive to the states of the system. Finally, it should be noticed that the application could be also done in the case that the transition probabilities are time-inhomogeneous. In this case the results would be sensitive also to the initial time because the kernel and the transition reliability matrix are time-varying.

5. Conclusion

The primary goal of this analysis is the study of general semi-Markovian usage model and the proposition of analytical computations of the expectation and variance of the system's single-use reliability. The analysis requires the use of the theory of the geometric trasform and extends previous contribution that relies on the Markov chain approach. The increased model complexity is rewarded by an increased model flexibility which allows the possibility of considering transition reliabilities that depend on the states of the system and on the length of stay in the initial state. From a more practical point of view, this means that the transition reliability depends not only on the specific arc (system instruction) to be executed but also on the time of execution that is in general random.

The determination of analytical solutions concerning moments of the single-use reliability avoid the use of simulation-based techniques which may be very long and time consuming even for small models.

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