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RICCI-FLAT LEFT-INVARIANT LORENTZIAN METRICS ON 2-STEP NILPOTENT LIE GROUPS

Mohammed Guediri and Mona Bin-Asfour

Abstract. The purpose of this paper is to investigate Ricci-flatness of left-invariant Lorentzian metrics on 2-step nilpotent Lie groups. We first show that if $\langle\,,\rangle$ is a Ricci-flat left-invariant Lorentzian metric on a 2-step nilpotent Lie group N, then the restriction of $\langle\,,\rangle$ to the center of the Lie algebra of N is degenerate. We then characterize the 2-step nilpotent Lie groups which can be endowed with a Ricci-flat left-invariant Lorentzian metric, and we deduce from this that a Heisenberg Lie group H_{2n+1} can be endowed with Ricci-flat left-invariant Lorentzian metric if and only if $n=1$. We also show that the free 2-step nilpotent Lie group on m generators $N_{m,2}$ admits a Ricci-flat left-invariant Lorentzian metric if and only if $m=2$ or $m=3$, and we determine all Ricci-flat left-invariant Lorentzian metrics on the free 2-step nilpotent Lie group on 3 generators $N_{3,2}$.

1. Introduction

A pseudo-Riemannian manifold is called flat (resp. Ricci-flat) if its Riemannian curvature tensor (resp. Ricci tensor) vanishes identically. Ricci-flat metrics play a crucial role in string theory, since they appear as the fixed-points of the so-called Ricci-flow. In 3 dimensions, the Ricci tensor completely determines the curvature tensor, and therefore Ricci-flatness in 3 dimensions implies flatness. This is not true in higher dimensions.

It is well known that every left-invariant pseudo-Riemannian metric on a commutative Lie group is flat. In [19], it has been shown that a Lie group G admits a flat left-invariant Riemannian metric if and only if its Lie algebra $\mathcal{G}$ splits as an orthogonal direct sum $\mathcal{G}=\mathcal{B}\oplus\mathcal{A}$ where $\mathcal{B}$ is a commutative subalgebra, $\mathcal{A}$ is a commutative ideal, and where the linear transformation ad_b is skew-adjoint for every $b\in\mathcal{B}$. Lie groups which can carry a flat left-invariant Lorentzian metric (i.e., with signature $-,+,\dots,+$) have been classified in [11] (see also [6] and [15]).

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On the other hand, it has been shown in [1] that a Ricci-flat left invariant Riemannian metric on a Lie group is necessarily flat. This is not true in the general pseudo-Riemannian case, and the following question then arises naturally: Which Lie groups can carry a Ricci-flat left-invariant Lorentzian metric?

In this paper we are interested in the particular case when the Lie group is 2-step nilpotent and the metric is Lorentzian. We recall here that, although they are close to being abelian, 2-step nilpotent Lie groups possess a very rich geometry (see [3], [4], [5], [7], [8], [9], [10], [12], [13], [14], [15], [16], [17], [18]).

In [14], it has been shown that a 2-step nilpotent Lie group N admits a flat left-invariant Lorentzian metric if and only if N is a trivial central extension of the three-dimensional Heisenberg group H_3. Ricci-flat left-invariant pseudo-Riemannian metrics on 2-step nilpotent Lie groups have been studied in [5], using a different method from ours.

The paper is organized as follows. In Section 1, we will review all necessary definitions of curvatures of a left-invariant pseudo-Riemannian metric on a general Lie group. In Section 2, we will compute the curvatures of a left-invariant Lorentzian metric on a 2-step nilpotent Lie group. Section 3 is devoted to determining all 2-step nilpotent Lie groups that can admit a Ricci-flat left-invariant Lorentzian metric. We will first show that if $\langle\,,\rangle$ is a Ricci-flat left-invariant Lorentzian metric on a 2-step nilpotent Lie group N, then the restriction of $\langle\,,\rangle$ to the center of the Lie algebra of N is degenerate. We then will characterize the 2-step nilpotent Lie groups which can be endowed with a Ricci-flat left-invariant Lorentzian metric, and we will deduce from this that a Heisenberg Lie group H_{2n+1} can be endowed with Ricci-flat left-invariant Lorentzian metric if and only if $n = 1$. In Section 4, we will study Ricci-flatness for left-invariant Lorentzian metrics on free 2-step nilpotent Lie groups. Using a result in [22] which asserts that a free 2-step nilpotent Lie group on m generators admits a bi-invariant metric if and only if $m = 3$, we will first show that this bi-invariant metric is unique and has signature $(-,-,-,+,+,+)$. We will then determine all Ricci-flat left-invariant Lorentzian metrics on the free 2-step nilpotent Lie group on 3 generators $N_{3,2}$ in terms of the eigenvalues of the self-adjoint endomorphism that relies each left-invariant Lorentzian metric on $N_{3,2}$ to the (unique) bi-invariant metric. We will close this section by proving that the free 2-step nilpotent Lie group on m generators $N_{m,2}$ admits a Ricci-flat left-invariant Lorentzian metric if and only if $m = 2$ or $m = 3$.

2. Curvatures of left-invariant metrics on Lie groups

Let G be a Lie group with Lie algebra $\mathcal{G}$. Each scalar product (i.e., nondegenerate symmetric bilinear form) $\langle\,,\rangle$ on $\mathcal{G}$ can be extended in a unique way to a semi-Riemannian metric, also denoted $\langle\,,\rangle$, on G so that the left translations L_g are isometries of the semi-Riemannian homogeneous manifold $(G, \langle\,,\rangle)$. Such a metric is called the left-invariant metric determined by the scalar product $\langle\,,\rangle$, and it satisfies

$$\langle X, Y\rangle_g = \left\langle D_g L_{g^{-1}}(X), D_g L_{g^{-1}}(Y)\right\rangle_e ,$$

for all X, $Y \in T_g G$.

In this case, the function $g \mapsto \langle X, Y \rangle_g$ is constant for any left-invariant vector fields $X, Y \in \mathcal{G}$. In particular, we have $X \langle Y, Z \rangle = 0$ for all $X, Y, Z \in \mathcal{G}$, from which we deduce that for each $X \in \mathcal{G}$ the map $Y \mapsto \nabla_X Y$ is skew-symmetric. These observations lead to the simplification ∇ as follows

$$\nabla_X Y = \frac{1}{2} \left\{ [X, Y] - (\mathrm{ad}_X)^* Y - (\mathrm{ad}_Y)^* X \right\} , \tag{1}$$

where $(\mathrm{ad}_X)^*$ denotes the adjoint of ad_X with respect to $\langle \, , \rangle$. The *Riemannian curvature tensor* R is defined in terms of ∇ by the following formula:

$$R(X, Y) Z = \nabla_{[X,Y]} Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z , \tag{2}$$

for all X, Y, $Z \in \mathcal{G}$.

Let $g \in G$. If $\mathcal{P}$ is a plane in $T_g G$, then we say that $\mathcal{P}$ is nondegenerate if the restriction $\langle \, , \rangle_{|\mathcal{P}}$ of $\langle , \rangle$ to $\mathcal{P}$ is nondegenerate. This means that $\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2 \neq 0$, for any basis $\{X, Y\}$ of $\mathcal{P}$. A nondegenerate plane $\mathcal{P}$ is said to be spacelike or timelike if $\langle \, , \rangle_{|\mathcal{P}}$ is positive definite or indefinite (i.e., it has signature $(-, +)$), respectively.

The *sectional curvature* of a nondegenerate plane $\mathcal{P} = \mathrm{span}\{X, Y\}$ spanned by X, $Y \in T_g G$ is defined by the formula:

$$K(X, Y) = \frac{\langle R(X, Y) X, Y \rangle}{\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2} . \tag{3}$$

We say that $(G, \langle \, , \rangle)$ is *flat* if the Riemannian curvature tensor R vanishes, or equivalently if the sectional curvature K is identically zero.

For $g \in G$ and X, $Y \in T_g G$, the *Ricci tensor* $\mathrm{Ric}(X, Y)$ evaluated at (X, Y) is defined to be the trace of the linear map of $T_g G$ given by $\xi \longrightarrow R(X, \xi) Y$. If $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of $T_e G \cong \mathcal{G}$ and X, $Y \in \mathcal{G}$, then

$$\mathrm{Ric}(X, Y) = \sum_{i=1}^{n} \epsilon_i \langle R(X, e_i) Y, e_i \rangle , \quad \text{where} \quad \epsilon_i = \langle e_i, e_i \rangle . \tag{4}$$

We should note that, since it is symmetric in the sense that $\mathrm{Ric}(X, Y) = \mathrm{Ric}(Y, X)$, the Ricci tensor is completely determined by the quantities $\mathrm{Ric}(e_i, e_i)$ for all vectors i, $1 \leq i \leq n$. We say that $(G, \langle \, , \rangle)$ is *Ricci-flat* if the Ricci curvature Ric is identically zero. In the Riemannian case, it is shown in [1] that a left-invariant Riemannian metric on a Lie group is Ricci-flat if and only if it is flat.

3. Curvatures for 2-step nilpotent Lie groups

Recall that a non-abelian Lie algebra $\mathcal{N}$ is called 2-step nilpotent if $[\mathcal{N}, \mathcal{N}]$ is contained in the center $\mathcal{Z}$ of $\mathcal{N}$, or equivalently, if $\mathrm{ad}_X^2 = 0$ for all $X \in \mathcal{N}$. A connected Lie group N is called 2-step nilpotent if its Lie algebra $\mathcal{N}$ is 2-step nilpotent. The standard example of a 2-step nilpotent Lie group is the Heisenberg group H_{2n+1} of dimension $2n+1$, which is defined as the vector space $H_{2n+1} = \mathbb{R} \times \mathbb{C}^n$ endowed with the group law

$$(z, v) \cdot (z', v') = \left(z + z' + \frac{1}{2} B(v, v'), v + v'\right),$$

where B is the nondegenerate alternating $\mathbb{R}$-bilinear form

$$B(v, v') = \sum_{i=1}^{n} x_i y_i' - y_i x_i',$$

with $v = \left(x_i + \sqrt{-1} y_i\right)_{1 \leq i \leq n}, v' = \left(x_i' + \sqrt{-1} y_i'\right)_{1 \leq i \leq n}$ and $z, z' \in \mathbb{R}$. Its Lie algebra $\mathcal{H}_{2n+1}$ has a basis $\{Z, X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ such that all brackets are zeros except $[X_i, Y_i] = Z$ for $1 \leq i \leq n$.

A 2-step nilpotent Lie algebra $\mathcal{N}$ with center $\mathcal{Z}$ is said to be nonsingular if $[X, \mathcal{N}] = \mathcal{Z}$ for all $X \in \mathcal{N} \setminus \mathcal{Z}$. A 2-step nilpotent Lie group N is nonsingular provided its Lie algebra $\mathcal{N}$ is nonsingular. For example, the Heisenberg group H_{2n+1} is nonsingular.

Throughout this paper, N will be a 2-step nilpotent Lie group with Lie algebra $\mathcal{N}$ and $\langle , \rangle$ a left-invariant Lorentzian metric on N, that is, we shall assume that $\langle , \rangle$ has signature $(-, +, \ldots, +)$. In this case, we recall that a tangent vector X to N is said to be spacelike, timelike, or lightlike (null) if $\langle X, X \rangle$ is > 0, < 0, or $= 0$, respectively. Similarly, a subspace W of $\mathcal{N}$ is said to be nondegenerate or degenerate according to whether the restriction of $\langle , \rangle$ to W is nondegenerate or degenerate, respectively. Therefore, if $\mathcal{Z}$ denotes the center of $\mathcal{N}$, then we shall distinguish two cases.

3.1. **The center is nondegenerate.** In this case, let $\mathcal{V}$ denote the orthogonal complement of $\mathcal{Z}$ in $\mathcal{N}$ relative to $\langle , \rangle$, and write $\mathcal{N}$ as an orthogonal direct sum

$$\mathcal{N} = \mathcal{V} \overset{\perp}{\oplus} \mathcal{Z}.$$

For each $Z \in \mathcal{Z}$, we define a skew-symmetric linear map $j(Z) : \mathcal{V} \to \mathcal{V}$ by

$$j(Z) X = \operatorname{ad}_X^* Z, \quad \text{for all} \quad X \in \mathcal{V},$$

where $\operatorname{ad}_X^* Z$ denotes the adjoint of ad_X relative to $\langle , \rangle$. Equivalently, the map $j(Z)$ is given by

$$\langle j(Z) X, Y \rangle = \langle [X, Y], Z \rangle, \quad \text{for all} \quad X, Y \in \mathcal{V}.$$

This endomorphism was defined first by Kaplan to study Riemannian 2-step nilmanifolds of Heisenberg type ([16], [17], [18]). In the Riemannian case, being skew-symmetric with respect to a positive definite inner product, $j(z)$ has all its nonzero characteristic roots purely imaginary, where by a characteristic root of an operator J we mean any eigenvalue of the complexified operator J^c associated to J. However, in the Lorentzian case, a skew-symmetric operator of an indefinite inner product might have nonzero real eigenvalues, and its complexified operator might contain Jordan blocks.

The following propositions can be obtained by using formulas (1), (2), (3), and (4).

Proposition 1. *With the notation above, if $x,\ y \in \mathcal{V}$ and $z,\ z' \in \mathcal{Z}$, we have*

1. $\nabla_x y = \frac{1}{2}\left[x, y\right]$.
2. $\nabla_x z = \nabla_z x = -\frac{1}{2}\ j\left(z\right)$.
3. $\nabla_z z = 0$.

Proposition 2. *With the notation above, if $x_1,\ x_2,\ x_3 \in \mathcal{V}$ and $z_1,\ z_2 \in \mathcal{Z}$, we have*

1. $R\left(x_1, x_2\right) x_3 = -\frac{1}{2} j\left(\left[x_1, x_2\right]\right) x_3 + \frac{1}{4} j\left(\left[x_2, x_3\right]\right) x_1 - \frac{1}{4} j\left(\left[x_1, x_3\right]\right) x_2$.
2. $R\left(x_1, x_2\right) z_1 = \frac{1}{4}\left\{\left[x_1, j\left(z_1\right) x_2\right] - \left[x_2, j\left(z_1\right) x_1\right]\right\}$.
3. $R\left(x_1, z_1\right) x_2 = \frac{1}{4}\left[x_1, j\left(z_1\right) x_2\right]$.
4. $R\left(x_1, z_1\right) z_2 = \frac{1}{4}\left\{j\left(z_1\right) \circ j\left(z_2\right) x_1\right\}$.
5. $R\left(z_1, z_2\right) x_1 = -\frac{1}{4}\left\{j\left(z_1\right) \circ j\left(z_2\right) x_1 - j\left(z_2\right) \circ j\left(z_1\right) x_1\right\}$.
6. $R\left(z_1, z_2\right) z_3 = 0$.

Proposition 3. *With the notation above, we have*

1. *If $z_1,\ z_2 \in \mathcal{Z}$ are orthonormal, then $K\left(z_1, z_2\right) = 0$.*
2. *If $x, y \in \mathcal{V}$ are orthonormal, then $K\left(x, y\right) = -\frac{3}{4}\epsilon\left\langle\left[x, y\right], \left[x, y\right]\right\rangle$, where $\epsilon = -1$ or $\epsilon = 1$ depending up on whether the plane $\mathcal{P} = \operatorname{span}\left\{x, y\right\}$ is timelike or spacelike, respectively.*
3. *If $x \in \mathcal{V}$ and $z \in \mathcal{Z}$ are orthonormal, then $K\left(x, z\right) = \frac{1}{4}\epsilon\left\langle j\left(z\right) x, j\left(z\right) x\right\rangle$, where $\epsilon = -1$ or $\epsilon = 1$ depending up on whether the plane $\mathcal{P} = \operatorname{span}\left\{x, y\right\}$ is timelike or spacelike, respectively.*

Proposition 4. *With the notation above, if $\left\{e_1, \ldots, e_m\right\}$ is an orthonormal basis of $\mathcal{V}$ and $\left\{z_{1+m}, \ldots, z_n\right\}$ is an orthonormal basis of $\mathcal{Z}$, then for any $x,\ y \in \mathcal{V}$ and $z,\ z_1,\ z_2 \in \mathcal{Z}$, we have*

1. $\operatorname{Ric}\left(z, x\right) = 0$.
2. $\operatorname{Ric}\left(z_1, z_2\right) = \frac{1}{4}\sum_{i=1}^{m} \epsilon_i \left\langle j\left(z_2\right) e_i, j\left(z_1\right) e_i\right\rangle$, *where $\epsilon_i = \left\langle e_i, e_i\right\rangle$.*
3. $\operatorname{Ric}\left(x, y\right) = -\frac{1}{2}\sum\limits_{i=1+m}^{n} \epsilon_i^{'} \left\langle j\left(z_i\right) y, j\left(z_i\right) x\right\rangle$, *where $\epsilon_i^{'} = \left\langle z_i, z_i\right\rangle$.*

As a consequence of the above proposition, we have

Corollary 5. *With the notation above, the scalar curvature of $(N, \langle\ ,\ \rangle)$ is*

$$\rho = -\frac{1}{4}\sum_{i=1}^{m}\sum_{j=m+1}^{n} \epsilon_i \epsilon_j^{'} \left\langle j\left(z_j\right) e_i, j\left(z_j\right) e_i\right\rangle\ ,$$

where $\epsilon_i = \left\langle e_i, e_i\right\rangle$ and $\epsilon_j' = \left\langle z_j, z_j\right\rangle$.

3.2. **The center is degenerate.** If the center is degenerate, then it turns out that the maps $j(Z)$ defined above have no meaning in this case. To overcome the difficulties in this case, we will proceed as in [13]. We first observe that, as a vector subspace, $\mathcal{Z}$ is tangent to the light cone at the identity e of N. Thus, one can write $\mathcal{Z} = \mathcal{Z}_1 \oplus \mathbb{R}b$ with $\mathcal{Z}_1$ a Euclidean subspace, and since the orthogonal complement $\mathcal{Z}_1^{\perp}$ of $\mathcal{Z}_1$ in $\mathcal{N}$ is Lorentzian, one can find another null vector $c \in \mathcal{Z}_1^{\perp}$ such that $\langle b, c\rangle = 1$. Hence, $\mathcal{N}$ can be written as an orthogonal decomposition of the form

$$\mathcal{N} = \mathcal{Z}_1 \oplus \mathcal{U}_1 \oplus \operatorname{span}\{b, c\} ,$$

with $\mathcal{U}_1$ a Euclidean subspace of $\mathcal{N}$.

For each $z_1 + yc \in \mathcal{Z}_1 \oplus \mathbb{R}c$, we can define a linear skew-symmetric map $j(z_1 + yc) \in \operatorname{End}(\mathcal{U}_1 \oplus \operatorname{span}\{b, c\})$ by

$$j(z_1 + yc)\, X = \operatorname{ad}_X^{*}(z_1 + yc) \quad \text{for all} \quad X \in \mathcal{U}_1 \oplus \operatorname{span}\{b, c\} ,$$

where $\operatorname{ad}_X^{*} Z$ denotes as above the adjoint of ad_X relative to $\langle , \rangle$. However, we will need to modify this definition because, as we can easily check, we have $j(z_1 + yc)\, b = 0$, $j(z_1 + yc)\, c \in \mathcal{U}_1$, and $j(z_1 + yc)\, X_1 \in \mathcal{U}_1 \oplus \mathbb{R}b$ for all $X_1 \in \mathcal{U}_1$. Instead, we will adopt the following definition

$$j \colon \mathcal{Z}_1 \oplus \mathbb{R}c \longrightarrow \operatorname{Hom}(\mathcal{U}_1 \oplus \mathbb{R}c, \mathcal{U}_1 \oplus \mathbb{R}b) ,$$

where $\operatorname{Hom}(\mathcal{U}_1 \oplus \mathbb{R}c, \mathcal{U}_1 \oplus \mathbb{R}b)$ denotes the set of all homomorphisms (i.e., linear maps) from $\mathcal{U}_1 \oplus \mathbb{R}c$ to $\mathcal{U}_1 \oplus \mathbb{R}b$.

The following propositions can be obtained by using formulas (1), (2), (3), and (4).

Proposition 6. *With the notation above, if z_1, $z_2 \in \mathcal{Z}_1$ and u_1, $u_2 \in \mathcal{U}_1$, we have*

1. $\nabla_b = \nabla_{z_1} z_2 = 0$.
2. $\nabla_c c = -j(c)\, c$.
3. $\nabla_c z_1 = \nabla_{z_1} c = -\frac{1}{2} j(z_1)\, c$.
4. $\nabla_{u_1} z_1 = \underset{z_1}{\nabla} u_1 = -\frac{1}{2} j(z_1)\, u_1$.
5. $\nabla_c u_1 = \frac{1}{2}\left[[c, u_1] - j(c)\, u_1\right]$.
6. $\nabla_{u_1} c = -\frac{1}{2}\left[[c, u_1] + j(c)\, u_1\right]$.
7. $\nabla_{u_1} u_2 = \frac{1}{2}[u_1, u_2]$.

Proposition 7. *With the notation above, if z, z_1, z_2, $z_3 \in \mathcal{Z}_1$ and u, u_1, u_2, $u_3 \in \mathcal{U}_1$, we have*

1. $R(u_1, z_1)\, u_2 = \frac{1}{4}\left([u_1, j(z_1)\, u_2] - \operatorname{ad}_{u_1}^{*} j(z_1)\, u_2\right)$.
2. $R(u_1, c)\, u_2 = -\frac{1}{2} ad_{u_2}^{*}[u_1, c] + \frac{1}{4}\operatorname{ad}_{u_1}^{*}[c, u_2] - \frac{1}{4}\operatorname{ad}_{c}^{*}[u_1, u_2] + \frac{1}{4}[u_1, j(c)\, u_2]$.
3. $R(z, c)\, u = -\frac{1}{4}[c, j(z)\, u] - \frac{1}{4} j(z)\, j(c)\, u + \frac{1}{4} j(c)\, j(z)\, u$.
4. $R(c, u)\, z = \frac{1}{4}\left[[j(z)\, c, u] - j(c)\, j(z)\, u + [c, j(z)\, u]\right]$.
5. $R(u, z)\, c = \frac{1}{4} j(z)\, j(c)\, u + \frac{1}{4}[u, j(z)\, c]$.

6. $R(c,z)\,c = \frac{1}{4}\left[[c, j(z)\,c] - j(c)\,j(z)\,c + 2j(z)\,j(c)\,c\right]$.
7. $R(z_1,z_2)\,z_3 = 0$.
8. $R(z_1,z_2)\,c = -\frac{1}{4}\left\{j(z_1)\circ j(z_2)\,c - j(z_2)\circ j(z_1)\,c\right\}$.
9. $R(z_1,c)\,z_2 = -\frac{1}{4} j(z_1)\circ j(z_2)\,c$.
10. $R(z_1,z_2)\,u_1 = -\frac{1}{4}\left\{\mathrm{ad}^*_{j(z_2)u_1} z_1 - \mathrm{ad}^*_{j(z_1)u_1} z_2\right\}$.
11. $R(z_1,u)\,z_2 = -\frac{1}{4} ad^*_{j(z_2)u} z_1$.
12. $R(u_1,u_2)\,u_3 = -\frac{1}{2} ad^*_{u_3}[u_1,u_2] + \frac{1}{4}\mathrm{ad}^*_{u_1}[u_2,u_3] - \frac{1}{4}\mathrm{ad}^*_{u_2}[u_1,u_3]$.
13. $R(u_1,u_2)\,z_1 = \frac{1}{4}\left([u_1, j(z_1)\,u_2] - [u_2, j(z_1)\,u_1]\right)$.
14. $R(u_1,u_2)\,c = -\frac{1}{2}\mathrm{ad}^*_c[u_1,u_2] - \frac{1}{4}\mathrm{ad}^*_{u_1}[c,u_2] + \frac{1}{4}\mathrm{ad}^*_{u_2}[c,u_1] + \frac{1}{4}[u_1, j(c)\,u_2]$ $-\frac{1}{4}[u_2, j(c)\,u_1]$.
15. $R(b,x)\,y = R(v,w)\,b = 0$, for all $x,\ y,\ v,\ w \in \mathcal{N}$.

As a consequence of the above proposition, we have

Corollary 8. *With the notation above, if $u,\ u_1,\ u_2 \in \mathcal{U}_1$ and $z,\ z_1,\ z_2 \in \mathcal{Z}_1$, we have*

1. $\langle R(z_1,z_2)\,z_1, z_2\rangle = 0$.
2. $\langle R(u,z)\,u, z\rangle = \frac{1}{4}\|j(z)\,u\|^2$.
3. $\langle R(c,z)\,c, z\rangle = \frac{1}{4}\|j(z)\,c\|^2$.
4. $\langle R(u,c)\,u, c\rangle = -\frac{3}{4}\|[u,c]\|^2 + \frac{1}{4}\|j(c)\,u\|^2$.
5. $\langle R(u_1,u_2)\,u_1, u_2\rangle = -\frac{3}{4}\|[u_1,u_2]\|^2$.
6. $\langle R(b,x)\,b, y\rangle = 0$, *for all* $x,\ y \in \mathcal{N}$.

Proposition 9. *With the notation above, if $v = \frac{b+c}{\sqrt{2}}$ and $w = \frac{b-c}{\sqrt{2}}$, we have*

1. $R(v,w) = 0$.
2. $\langle R(z_1,v)\,z_2, v\rangle = \langle R(z_1,w)\,z_2, w\rangle = \frac{1}{8}\langle j(z_2)\,c, j(z_1)\,c\rangle$, *for all* z_1, $z_2 \in \mathcal{Z}_1$.
3. $\langle R(z,v)\,u, v\rangle = \langle R(z,w)\,u, w\rangle = -\frac{1}{4}\langle j(z)\,u, j(c)\,c\rangle + \frac{1}{8}\langle j(c)\,u, j(z)\,c\rangle$, *for all* $z \in \mathcal{Z}_1$, $u \in \mathcal{U}_1$.
4. $\langle R(u_1,v)\,u_2, v\rangle = \langle R(u_1,w)\,u_2, w\rangle = -\frac{3}{8}\langle [u_1,c], [u_2,c]\rangle$ $+\frac{1}{8}\langle j(c)\,u_2, j(c)\,u_1\rangle$, *for all* $u_1,\ u_2 \in \mathcal{U}_1$.
5. $\langle R(v,z)\,w, z\rangle = -\frac{1}{8}\|j(z)\,c\|^2$, *for all* $z \in \mathcal{Z}_1$.
6. $\langle R(v,u)\,w, u\rangle = \frac{3}{8}\|[u,c]\|^2 - \frac{1}{8}\|j(c)\,u\|^2$, *for all* $u \in \mathcal{U}_1$.

As a consequence of the above proposition, we have

Corollary 10. *With the notation above, if $v = \frac{b+c}{\sqrt{2}}$, $w = \frac{b-c}{\sqrt{2}}$, $u,\ u_1,\ u_2 \in \mathcal{U}_1$, and $z,\ z_1,\ z_2 \in \mathcal{Z}_1$, we have*

1. *If $z_1,\ z_2$ are orthonormal, then* $K(z_1,z_2) = 0$.

2. *If u_1, u_2 are orthonormal, then* $K(u_1,u_2) = -\frac{3}{4}\left\|[u_1,u_2]\right\|^2 \leq 0$.
3. *If u and z are orthonormal, then* $K(u,z) = \frac{1}{4}\left\|j(z)u\right\|^2 \geq 0$.
4. *If v, w are orthonormal, then* $K(v,w) = 0$.
5. *If v and z are orthonormal, then* $K(z,v) = \frac{1}{8}\left\|j(z)c\right\|^2$.
6. *If w and z are orthonormal, then* $K(z,w) = -\frac{1}{8}\left\|j(z)c\right\|^2$.
7. *If u and v are orthonormal, then* $K(u,v) = -\frac{3}{8}\left\|[u,c]\right\|^2 + \frac{1}{8}\left\|j(c)u\right\|^2$.
8. *If u and w are orthonormal, then* $K(u,w) = \frac{3}{8}\left\|[u,c]\right\|^2 - \frac{1}{8}\left\|j(c)u\right\|^2$.

Proposition 11. *With the notation above, if $\{U_1,U_2,\ldots,U_m\}$ is an orthonormal basis of $\mathcal{U}_1$ and $\{Z_{1+m},Z_{2+m},\ldots,Z_{n-2}\}$ is an orthonormal basis of $\mathcal{Z}_1$, then for all $v=\frac{b+c}{\sqrt{2}}$, $w=\frac{b-c}{\sqrt{2}}$, u, u_1, $u_2\in\mathcal{U}_1$, and z, z_1, $z_2\in\mathcal{Z}_1$, we have*

1. $\operatorname{Ric}(b,x) = 0$, *for all* $x\in\mathcal{N}$.
2. $\operatorname{Ric}(u,z) = 0$.
3. $\operatorname{Ric}(z_1,z_2) = \frac{1}{4}\sum\limits_{i=1}^{m}\langle j(z_1)U_i, j(z_2)U_i\rangle$.
4. $\operatorname{Ric}(u_1,u_2) = -\frac{1}{2}\sum\limits_{i=1+m}^{n-2}\langle j(Z_i)u_1, j(Z_i)u_2\rangle$.
5. $\operatorname{Ric}(c) = -\frac{1}{2}\sum\limits_{j=1+m}^{n-2}\left\|j(Z_j)c\right\|^2 + \frac{1}{4}\sum\limits_{i=1}^{m}\left\|j(c)U_i\right\|^2$.
6. $\operatorname{Ric}(u,c) = -\frac{1}{2}\sum\limits_{i=1+m}^{n-2}\langle j(Z_i)c, j(Z_i)u\rangle$.
7. $\operatorname{Ric}(z,c) = \frac{1}{4}\sum\limits_{i=1}^{m}\langle j(c)U_i, j(z)U_i\rangle$.
8. $\operatorname{Ric}(v,w) = -\frac{1}{8}\sum\limits_{i=1}^{m}\left\|j(c)U_i\right\|^2 + \frac{1}{4}\sum\limits_{j=1+m}^{n-2}\left\|j(Z_i)c\right\|^2$.
9. $\operatorname{Ric}(v) = Ric(w) = -\frac{1}{4}\sum\limits_{j=1+m}^{n-2}\left\|j(Z_i)c\right\|^2 + \frac{1}{8}\sum\limits_{i=1}^{m}\left\|j(c)U_i\right\|^2$.
10. $\operatorname{Ric}(u,v) = -\frac{1}{2\sqrt{2}}\sum\limits_{i=1+m}^{n-2}\langle j(Z_i)c, j(Z_i)u\rangle$.
11. $\operatorname{Ric}(u,w) = \frac{1}{2\sqrt{2}}\sum\limits_{i=1+m}^{n-2}\langle j(Z_i)c, j(Z_i)u\rangle$.
12. $\operatorname{Ric}(z,v) = \frac{1}{4\sqrt{2}}\sum\limits_{i=1}^{m}\langle j(c)U_i, j(z)U_i\rangle$.
13. $\operatorname{Ric}(z,w) = -\frac{1}{4\sqrt{2}}\sum\limits_{i=1}^{m}\langle j(c)U_i, j(z)U_i\rangle$.

As a consequence of the above proposition, we have

Corollary 12. *With the notation above, the scalar curvature of* $(N, \langle\, , \,\rangle)$ *is*

$$\rho = -\frac{1}{4}\sum_{i=1}^{m}\sum_{i=1+m}^{n-2}\left\| j\left(Z_i\right) U_i \right\|^2 .$$

In the following example, we will show using direct computation that the $(2n+1)$-dimensional Heisenberg group H_{2n+1} admits a Ricci-flat left-invariant Lorentzian metric if and only if $n = 1$, and if $\langle\, , \rangle$ is such a metric on H_3 then its restriction to the center of H_3 is necessarily degenerate.

Example 13. Let $\langle\, , \rangle$ be a left-invariant Lorentzian metric $\langle\, , \rangle$ on Heisenberg group H_{2n+1}.

<u>*Case 1.*</u> Assume first that the center of H_{2n+1} is nondegenerate. In this case, it is not difficult to see that there exists an orthonormal basis $\{X_1, Y_1, \ldots, X_n, Y_n, Z\}$ of the Lie algebra $\mathcal{H}_{2n+1}$ of H_{2n+1} such that $[X_i, Y_i] = Z$, $1 \le i \le n$, and all other brackets are zeros. With the notation of Subsection 3.1 and relative to this basis, we have

$$j\left(Z\right) = \langle Z, Z\rangle \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} .$$

By setting $\epsilon_i = \langle X_i, X_i\rangle$ and $\epsilon_i^{'} = \langle Y_i, Y_i\rangle$, we easily verify that the non-trivial Ricci curvatures are

$$\mathrm{Ric}\left(Z, Z\right) = \frac{1}{2}\sum_{i=1}^{n}\epsilon_i \epsilon_i^{'} ,$$

$$\mathrm{Ric}\left(X_i, X_i\right) = -\frac{1}{2}\left\langle Z, Z\right\rangle \epsilon_i^{'} ,$$

$$\mathrm{Ric}\left(Y_i, Y_i\right) = -\frac{1}{2}\left\langle Z, Z\right\rangle \epsilon_i .$$

It follows that $(H_{2n+1}, \langle, \rangle)$ is non Ricci-flat.

<u>*Case 2.*</u> Assume now that the center of H_{2n+1} is degenerate. In this case, it is not difficult to see that there exists a pseudo-orthonormal basis $\{X_1, Y_1, \ldots, X_n, Y_n, Z\}$ of $\mathcal{H}_{2n+1}$ with all scalar products are zeros except $\langle Z, X_1\rangle = \langle X_i, X_i\rangle = \langle Y_j, Y_j\rangle = 1$ for $2 \le i \le n$ and $1 \le j \le n$, such that $[X_i, Y_i] = Z$, $1 \le i \le n$, and all other brackets are zeros. We can easily verify that the non-trivial Ricci curvatures are

$$\mathrm{Ric}\left(v\right) = \mathrm{Ric}\left(w\right) = -\,\mathrm{Ric}\left(v, w\right) = \frac{1}{4}\left(n-1\right) ,$$

where $v = \frac{1}{\sqrt{2}}\left(X_1 + Z\right)$, $w = \frac{1}{\sqrt{2}}\left(X_1 - Z\right)$. It follows that $(H_{2n+1}, \langle\, , \rangle)$ is Ricci-flat if and only if $n = 1$. In this case, the metric is even flat.

4. Main results

It has been shown in [14] that a 2-step nilpotent Lie group N admits a flat left-invariant Lorentzian metric if and only if N is a trivial central extension of the three-dimensional Heisenberg group H_3. At this point, a natural question arises:

What are the 2-step nilpotent Lie groups which can admit a Ricci-flat left-invariant Lorentzian metric?

The aim of this section is to answer this question by identifying the 2-step nilpotent Lie groups which can admit a Ricci-flat left-invariant Lorentzian metric (compare [5], Theorem 5.1). For this, we shall first characterize 2-step nilpotent Lie groups endowed with a Ricci-flat left-invariant Lorentzian metric.

Lemma 14. Let N be a 2-step nilpotent Lie group with Lie algebra $\mathcal{N}$, and let $\langle\, ,\rangle$ be a Ricci-flat left-invariant Lorentzian metric on N. Then, the restriction of $\langle\, ,\rangle$ to the center $\mathcal{Z}$ of $\mathcal{N}$ is degenerate.

Proof. Assume now that $(N, \langle, \rangle)$ is Ricci-flat, and assume to the contrary that the restriction of $\langle\, ,\rangle$ to $\mathcal{Z}$ is nondegenerate. In this case, we write as usual

$$\mathcal{N} = \mathcal{V} \overset{\perp}{\oplus} \mathcal{Z}\,,$$

where $\mathcal{V}$ denotes the orthogonal complement of $\mathcal{Z}$ relative to $\langle\, ,\rangle$. Let $\{e_1, \ldots, e_q\}$ be an orthonormal basis of $\mathcal{V}$ and $\{z_1, \ldots, z_p\}$ an orthonormal basis of $\mathcal{Z}$. Then, we have

$$\begin{aligned}
[e_i, e_j] &= \sum_{k=1}^{p} \epsilon_k \left\langle [e_i, e_j]\,, z_k \right\rangle z_k \\
&= \sum_{k=1}^{p} \epsilon_k \left\langle j\left(z_k\right) e_i, e_j \right\rangle z_k\,,
\end{aligned}$$

where $\epsilon_k = \langle z_k, z_k \rangle$. Setting $a_{ikj} = \epsilon_k \langle j(z_k) e_i, e_j \rangle$ and $\epsilon_i^{'} = \langle e_i, e_i \rangle$, we get after computing that

$$\text{Ric}\,(e_i, e_i) = -\frac{1}{2} \sum_{j=1}^{q} \sum_{k=1}^{p} \epsilon_j^{'} \epsilon_k a_{ikj}^2\,, \qquad 1 \leq i \leq q, \tag{5}$$

$$\text{Ric}\,(z_k, z_k) = \frac{1}{4} \sum_{i,j=1}^{q} \epsilon_i^{'} \epsilon_j^{'} a_{ikj}^2, \qquad 1 \leq k \leq p\,. \tag{6}$$

If $\mathcal{V}$ is spacelike, then we get by (6) that $a_{ikj} = 0$ for all i, j such that $1 \leq i$, $j \leq q$ and $1 \leq k \leq p$. This implies that $[x, y] = 0$ for all $x, y \in \mathcal{N}$ (i.e., $\mathcal{N}$ is abelian), a contradiction. Thus, $\mathcal{V}$ is necessarily timelike. So, without loss of generality we can assume that $\epsilon_1^{'} = -1$ and $\epsilon_i^{'} = 1$ for all $i \geq 2$. Since $a_{iki} = 0$ for all i, k such that $1 \leq i \leq q$ and $1 \leq k \leq p$, then (5) yields

$$\text{Ric}\,(e_1, e_1) = -\frac{1}{2} \sum_{j=2}^{q} \sum_{k=1}^{p} a_{1kj}^2\,,$$

from which we deduce, by Ricci-flatness, that

$$a_{1kj} = 0\,, \quad \text{for all } j,\, k \text{ such that } 2 \leq j \leq q \text{ and } 1 \leq k \leq p\,. \tag{7}$$

Recalling that $a_{ikj} = -a_{jki}$ for all i, j, k such that $1 \le i$, $j \le q$ and $1 \le k \le p$, and substituting (7) into (5) and (6) we get

$$\mathrm{Ric}\,(e_i, e_i) = -\frac{1}{2}\sum_{j=2}^{q}\sum_{k=1}^{p} a_{ikj}^2\,, \qquad 2 \le i \le q\,,$$

$$\mathrm{Ric}\,(z_k, z_k) = \frac{1}{4}\sum_{i,j=2}^{q} a_{ikj}^2\,, \qquad 1 \le k \le p\,,$$

respectively.

We deduce, by Ricci-flatness, that $a_{ikj} = 0$ for all i, j, k such that $2 \le i$, $j \le q$ and $1 \le k \le p$. It follows that $[x, y] = 0$ for all x, $y \in \mathcal{N}$, that is $\mathcal{N}$ is abelian, a contradiction. We deduce that the restriction of $\langle\,,\rangle$ to the center $\mathcal{Z}$ is degenerate, as desired. $\square$

The following theorem which is the main result of the paper will characterize the 2-step nilpotent Lie groups which can be endowed with a Ricci-flat left-invariant Lorentzian metric.

Theorem 15. *A connected 2-step nilpotent Lie group N admits a Ricci-flat left-invariant Lorentzian metric if and only if N is a product of the form $\mathbb{R}^n \times G$, so that the Lie algebra of G has a basis $\{b, z_1, \ldots, z_p, c, e_1, \ldots, e_q\}$ satisfying*

$$[c, e_i] = a_i b + \sum_{k=1}^{p} c_{ik} z_k\,, \qquad 1 \le i \le q,$$

$$[e_i, e_j] = a_{ij} b\,, \qquad 1 \le i,\ j \le q,$$

with $\sum\limits_{i,j=1}^{q} a_{ij}^2 = 2\sum\limits_{k=1}^{p}\sum\limits_{i=1}^{q} c_{ik}^2$. Moreover, for each such a metric, there exists a pseudo-orthonormal basis like above with $\langle c, c\rangle = \langle b, b\rangle = 0$ and $\langle b, c\rangle = \pm 1$. In particular, the restriction of the metric to the commutator subgroup N' is degenerate.

Proof. Let N be a 2-step nilpotent Lie group with a Ricci-flat left-invariant Lorentzian metric, and let $\mathcal{N}$ be the Lie algebra of N and $\mathcal{Z}$ the center of $\mathcal{N}$. By Lemma 14, the restriction of $\langle\,,\rangle$ to $\mathcal{Z}$ is degenerate. It follows that $\mathcal{Z}$ is an orthogonal direct sum of the form $\mathcal{Z} = \mathcal{Z}_1 \oplus \mathbb{R}b$, with b a null vector and $\mathcal{Z}_1$ is a spacelike subspace. Since the orthogonal complement $\mathcal{Z}_1^{\perp}$ of $\mathcal{Z}_1$ in $\mathcal{N}$ relative to $\langle\,,\rangle$ is timelike, we may choose a second null vector $c \in \mathcal{Z}_1^{\perp}$ so that $\langle c, b\rangle = 1$. Thus, we get the orthogonal direct sum

$$\mathcal{N} = \mathcal{Z}_1 \oplus \mathcal{U}_1 \oplus \mathrm{span}\,\{b, c\}\,,$$

where $\mathcal{U}_1$ is a spacelike subspace of $\mathcal{N}$ with $\dim \mathcal{U}_1 \ge 1$.

Let $\{z_1, z_2, \ldots, z_p, e_1, e_2, \ldots, e_q\}$ be an orthonormal basis of $\mathcal{Z}_1 \oplus \mathcal{U}_1$, with $z_1, z_2, \ldots, z_p$ span $\mathcal{Z}_1$ and $e_1, e_2, \ldots, e_q$ span $\mathcal{U}_1$. Write

$$[c, e_i] = a_i b + \sum_{k=1}^{p} c_{ik} z_k \,, \tag{8}$$

$$[e_i, e_j] = a_{ij} b + \sum_{k=1}^{p} c_{ikj} z_k \,, \qquad 1 \leq i,\ j \leq q \,, \tag{9}$$

with

$$\begin{aligned} a_i &= \langle j\,(c)\, c, e_i \rangle \,, \\ c_{ik} &= \langle j\,(z_k)\, c, e_i \rangle \,, \\ a_{ij} &= \langle j\,(c)\, e_i, e_j \rangle \,, \\ c_{ikj} &= \langle j\,(z_k)\, e_i, e_j \rangle \qquad 1 \leq i,\ j \leq q,\ 1 \leq k \leq p \,, \end{aligned}$$

where $j\,(z_i)$ and $j\,(c)$ are defined as in Subsection 3.2 of the above section.

An easy computation shows that

$$\operatorname{Ric}(c, c) = -\frac{1}{2} \sum_{k=1}^{p} \sum_{i=1}^{q} c_{ik}^2 + \frac{1}{4} \sum_{i,j=1}^{q} a_{ij}^2 \,, \tag{10}$$

$$\operatorname{Ric}(b, b) = 0 \,,$$

$$\operatorname{Ric}(z_k, z_k) = \frac{1}{4} \sum_{i,j=1}^{q} c_{ikj}^2 \,, \qquad 1 \leq k \leq p \,, \tag{11}$$

$$\operatorname{Ric}(e_i, e_i) = -\frac{1}{2} \sum_{j=1}^{q} \sum_{k=1}^{p} c_{ikj}^2 \,, \qquad 1 \leq i \leq q \,.$$

Since we are assuming that $(N, \langle\,,\rangle)$ is Ricci-flat, then we deduce from (10) and (11) that $\sum_{i,j=1}^{q} a_{ij}^2 = 2 \sum_{k=1}^{p} \sum_{i=1}^{q} c_{ik}^2$ and $c_{ikj} = 0$ for all i, j such that $1 \leq i,\ j \leq q$ and $1 \leq k \leq p$, respectively. Thus, $\mathcal{N}$ is as desired, and the proof of the theorem is complete. □

As a straightforward consequence of the above theorem, we have the following result that we have previously reported in Example 13.

Corollary 16. *H_{2n+1} admits a Ricci-flat left-invariant Lorentzian metric if and only if $n = 1$.*

Proof. As we have seen in Example 13, Case 2, any left-invariant Lorentzian metric $\langle\,,\rangle$ on H_3 for which the center is degenerate is flat (see also [14] or [20]). Conversely, by Theorem 15, if H_{2n+1} admits a Ricci-flat left-invariant Lorentzian metric, then its Lie algebra $\mathcal{H}_{2n+1}$ has a pseudo-orthonormal basis $\{b, z_1, \ldots, z_p, c, e_1, \ldots, e_q\}$, with $\langle c, c\rangle = \langle b, b\rangle = 0$ and $\langle b, c\rangle = \pm 1$, and which satisfies

$$[c, e_i] = a_i b + \sum_{k=1}^{p} c_{ik} z_k\,, \qquad 1 \le i \le q\,,$$
$$[e_i, e_j] = a_{ij} b\,, \qquad 1 \le i,\ j \le q\,,$$

with $\sum\limits_{i,j=1}^{q} a_{ij}^2 = 2 \sum\limits_{k=1}^{p} \sum\limits_{i=1}^{q} c_{ik}^2$. Here, we have $p + q + 2 = 2n + 1$.

Since the center of $\mathcal{H}_{2n+1}$ is degenerate and one-dimensional, it follows that $c_{ik} = 0$ for all i, k, which in turn implies that $a_{ij} = 0$ for all i, j Thus, the structure of $\mathcal{H}_{2n+1}$ reduces as follows

$$[c, e_i] = a_i b\,, \qquad 1 \le i \le q\,,$$
$$[e_i, e_j] = 0\,, \qquad 1 \le i,\ j \le q\,.$$

Since $\mathcal{H}_{2n+1}$ is not abelian, then $a_{i_0} \neq 0$ for some i_0, say $i_0 = 1$ to simplify. By changing e_i with $e_i' = e_i - \frac{a_i}{a_1} e_1$ for all $i \in \{2, \dots, q\}$ which is still an element of $\mathcal{U}_1$, we see that $[c, e_1] = a_1 b$ and all other brackets are zeros. In other words, $\mathcal{H}_{2n+1}$ is isomorphic to the product $\mathbb{R}^{2(n-1)} \times \mathcal{H}_3$, where $\mathcal{H}_3$ is the Heisenberg algebra. However, this can happen only if $n = 1$, as desired. $\square$

5. Ricci-flatness for free 2-step nilpotent Lie groups

In this section, we shall determine the Ricci-flat left-invariant Lorentzian metrics on free 2-step nilpotent Lie groups that admit a bi-invariant metric. We shall first see that there is only one such a group, namely the free 2-step nilpotent Lie group on 3 generators $N_{3,2}$. We shall then classify the Ricci-flat metrics on $N_{3,2}$ that come from a bi-invariant metric. In order to state our results precisely we need to recall first a few definitions and facts concerning free 2-step nilpotent Lie groups.

Definition 17. A 2-step nilpotent Lie algebra $\mathcal{N}_{m,2}$ is said to be free on $m \ge 2$ generators if there exists a generating set $\{e_1, e_2, \dots, e_m\}$ for $\mathcal{N}_{m,2}$ with the following property: Let $\mathcal{N}_{m,2}^*$ be any 2-step nilpotent Lie algebra, and let $\{e_1', e_2', \dots, e_m'\}$ be any subset of m elements in $\mathcal{N}_{m,2}^*$. Then, there exists a unique Lie algebra homomorphism $T\colon \mathcal{N}_{m,2} \to \mathcal{N}_{m,2}^*$ such that $T(e_i) = e_i'$, $1 \le i \le m$. A 2-step nilpotent Lie group $N_{m,2}$ is said to be free on $m \ge 2$ generators if its Lie algebra $\mathcal{N}_{m,2}$ is so.

The following results are well known (see [9]).

Proposition 18. *For every integer $m \ge 2$, there exists a free 2-step nilpotent Lie algebra $\mathcal{N}_{m,2}$ on m generators which is unique up to a Lie algebra isomorphism. Moreover,* $\dim \mathcal{N}_{m,2} = \frac{m(m+1)}{2}$.

Proposition 19. *Let $m \ge 2$ be an integer, and let $\mathcal{N}_{m,2}$ be the free 2-step nilpotent Lie algebra on m generators. Let $\mathcal{Z}_{m,2}$ denote the center of $\mathcal{N}_{m,2}$. Then*

1. $\dim \mathcal{Z}_{m,2} = \frac{m(m-1)}{2}$.

2. *If* $\{e_1,\ldots,e_m\}$ *is any generating set for* $\mathcal{N}_{m,2}$ *with* m *elements, then* $\{[e_i,e_j] : 1\leq i<j\leq m\}$ *is a basis for* $\mathcal{Z}_{m,2}$. *In particular,* $\mathcal{Z}_{m,2} = [\mathcal{N}_{m,2},\mathcal{N}_{m,2}]$.

The following result determines the free nilpotent Lie algebras which can admit a bi-invariant metric (see [22]).

Theorem 20. *Let* $\mathcal{N}_{m,k}$ *be a free* k*-step nilpotent Lie algebra in* m *generators. Then,* $\mathcal{N}_{m,k}$ *admits a bi-invariant metric if and only if* $(m,k)=(3,2)$ *or* $(m,k)=(2,3)$.

Before we present the main result of this section, we shall first establish some useful lemmas.

Lemma 21. Up to isometry, $N_{3,2}$ admits a unique bi-invariant metric $\langle\,,\rangle$. It has signature $(3,3)$, and its restriction to the center $\mathcal{Z}_{3,2}$ of $\mathcal{N}_{3,2}$ vanishes identically (i.e., $\mathcal{Z}_{3,2}$ is totally degenerate).

Proof. Let $\langle\,,\rangle$ be a bi-invariant metric on $\mathcal{N}_{3,2}$, and let e_1, e_2, e_3 be generators of $\mathcal{N}_{3,2}$. By setting $e_4=[e_1,e_2]$, $e_5=[e_1,e_3]$, and $e_6=[e_2,e_3]$, the vectors $e_1,e_2,\ldots,e_6$ form a basis of $\mathcal{N}_{3,2}$. The bi-invariance of $\langle\,,\rangle$ shows that

$$\begin{aligned}\langle e_1,e_i\rangle &= 0\,, \quad \text{for } i=4,5\,,\\ \langle e_2,e_i\rangle &= 0\,, \quad \text{for } i=4,6\,,\\ \langle e_3,e_i\rangle &= 0\,, \quad \text{for } i=5,6\,,\end{aligned}$$

and

$$\langle e_i,e_j\rangle = 0\,, \quad \text{for } i,j\in\{4,5,6\}\,.$$

In particular, the center of $\mathcal{N}_{3,2}$ is totally degenerate. Moreover, we have $\langle e_1,[e_2,e_3]\rangle = \langle e_2,[e_3,e_1]\rangle = \langle e_3,[e_1,e_2]\rangle$, that is $\langle e_1,e_6\rangle = -\langle e_2,e_5\rangle = \langle e_3,e_4\rangle$. If we denote this value by λ, we see that $\lambda\neq 0$ since $\langle\,,\rangle$ is nondegenerate.

Now, by changing e_1, e_2, e_3 if necessary, we can without loss of generality assume that e_1, e_2, e_3 are null and orthogonal to each other. It follows that the signature of $\langle\,,\rangle$ is $(3,3)$.

It is easy to see that e_1, e_2, e_3 should be null vectors and span a totally degenerate subspace. By changing the basis $\{e_1,\ldots,e_6\}$ with $\left\{\frac{e_1}{\sqrt{|\lambda|}},\ldots,\frac{e_6}{\sqrt{|\lambda|}}\right\}$ we see that for this new basis all scalar products are zeros except $\langle e_1',e_6'\rangle = -\langle e_2',e_5'\rangle = \langle e_3',e_4'\rangle = 1$. In other words, we have shown that for any bi-invariant metric $\langle\,,\rangle$ on $N_{3,2}$, there exists a basis $\{e_1,\ldots,e_6\}$ of $\mathcal{N}_{3,2}$ such that e_1, e_2, e_3 are generators, $e_4=[e_1,e_2]$, $e_5=[e_1,e_3]$, $e_6=[e_2,e_3]$, and $e_1,\ldots,e_6$ are null vectors with $\langle e_i,e_j\rangle=0$ for all i, j except $\langle e_1,e_6\rangle = -\langle e_2,e_5\rangle = \langle e_3,e_4\rangle = \pm 1$. This shows that $\langle\,,\rangle$ is unique up to an isometry. □

Lemma 22. Let $N_{3,2}$ be the free 2-step nilpotent Lie group on 3 generators, endowed with a bi-invariant metric $\langle\,,\rangle$. For any pseudo-orthonormal basis $\{e_1,e_2,\ldots,e_6\}$ of $\mathcal{N}_{3,2}$ such that all inner products are zeros except $\langle e_1,e_6\rangle = -\langle e_2,e_5\rangle = \langle e_3,e_4\rangle = 1$, with e_4, e_5, e_6 spanning the center $\mathcal{Z}_{3,2}$ of $\mathcal{N}_{3,2}$, we have

$$[e_1,e_2]=\mu e_4\,, \quad [e_1,e_3]=\mu e_5\,, \quad [e_2,e_3]=\mu e_6\,, \quad \text{for some } \mu\neq 0.$$

Proof. Let e_1, e_2, e_3, e_4, e_5, e_6 be a pseudo-orthonormal basis for $\mathcal{N}_{3,2}$ such that all products are zero except $\langle e_1, e_6\rangle = -\langle e_2, e_5\rangle = \langle e_3, e_4\rangle = 1$, with e_4, e_5, e_6 spanning the center $\mathcal{Z}_{3,2}$. In this case, we see that e_1, e_2, e_3 are generators for $\mathcal{N}_{3,2}$. Setting

$$\begin{aligned}
[e_1, e_2] &= \alpha_1 e_4 + \alpha_2 e_5 + \alpha_3 e_6\,, \\
[e_1, e_3] &= \beta_1 e_4 + \beta_2 e_5 + \beta_3 e_6\,, \\
[e_2, e_3] &= \gamma_1 e_4 + \gamma_2 e_5 + \gamma_3 e_6\,,
\end{aligned}$$

we deduce, after an easy computation which uses the bi-invariance of $\langle\,,\rangle$, that

$$[e_1, e_2] = \mu e_4\,, \quad [e_1, e_3] = \mu e_5\,, \quad [e_2, e_3] = \mu e_6\,, \quad \text{for some } \mu \neq 0.$$

□

Lemma 23. Let $N_{3,2}$ be the free 2-step nilpotent Lie group on 3 generators endowed with a bi-invariant metric $\langle\,,\rangle$. Then, for any orthonormal basis $\{e_1, e_2, \ldots, e_6\}$ of $\mathcal{N}_{3,2}$, with $\mathcal{Z}_{3,2} = \operatorname{span}\{e_3 - e_4, e_2 - e_5, e_1 - e_6\}$, such that $\langle e_1, e_1\rangle = -\langle e_2, e_2\rangle = \langle e_3, e_3\rangle = -\langle e_4, e_4\rangle = \langle e_5, e_5\rangle = -\langle e_6, e_6\rangle = 1$ and all other products are zeros, we have

$$\begin{aligned}
[e_1, e_2] = [e_6, e_2] = [e_1, e_5] = [e_6, e_5] &= \mu\frac{e_3 - e_4}{2\sqrt{2}}\,, \\
[e_1, e_3] = [e_6, e_3] = [e_1, e_4] = [e_6, e_4] &= \mu\frac{e_2 - e_5}{2\sqrt{2}}\,, \\
[e_2, e_3] = [e_2, e_4] = [e_5, e_3] = [e_5, e_4] &= \mu\frac{e_1 - e_6}{2\sqrt{2}}\,,
\end{aligned}$$

and all the other brackets are zeros.

Proof. Let $\{e_1, e_2, \ldots, e_6\}$ be as in the statement of the lemma. By setting

$$\begin{aligned}
E_1 &= \frac{e_1 + e_6}{\sqrt{2}}\,, & E_2 &= \frac{e_2 + e_5}{\sqrt{2}}\,, & E_3 &= \frac{e_3 + e_4}{\sqrt{2}}\,, \\
E_4 &= \frac{e_3 - e_4}{\sqrt{2}}\,, & E_5 &= \frac{e_2 - e_5}{\sqrt{2}}\,, & E_6 &= \frac{e_1 - e_6}{\sqrt{2}}\,,
\end{aligned}$$

we see that all the vectors E_i are null, $\langle E_1, E_6\rangle = -\langle E_2, E_5\rangle = \langle E_3, E_4\rangle = 1$, and $\mathcal{Z}_{3,2} = \operatorname{span}\{E_4, E_5, E_6\}$. The hypotheses of Lemma 22 are therefore fulfilled, and this leads to the required result. □

Lemma 24. Let $N_{3,2}$ be the free 2-step nilpotent Lie group on 3 generators endowed with a bi-invariant metric $\langle\,,\rangle$. Then, for any pseudo-orthonormal basis $\{e_1, \ldots, e_6\}$ of $\mathcal{N}_{3,2}$, with $\mathcal{Z}_{3,2} = \operatorname{span}\{e_5 - e_6, e_3 - e_4, e_2\}$, such that $\langle e_1, e_2\rangle = -\langle e_3, e_3\rangle = \langle e_4, e_4\rangle = \langle e_5, e_5\rangle = -\langle e_6, e_6\rangle = 1$ and all products are zeros except, we have

$$[e_1, e_3] = [e_1, e_4] = \mu \frac{e_5 - e_6}{2} ,$$

$$[e_1, e_5] = [e_1, e_6] = \mu \frac{e_3 - e_4}{2} ,$$

$$[e_3, e_5] = [e_3, e_6] = \mu \frac{e_2}{2} ,$$

$$[e_4, e_5] = [e_4, e_6] = \mu \frac{e_2}{2} ,$$

and all the other brackets are zeros.

Proof. Let $\{e_1, e_2, \ldots, e_6\}$ be as in the statement of the lemma. By setting

$$E_1 = e_1 , \qquad E_2 = \frac{e_3 + e_4}{\sqrt{2}} , \qquad E_3 = \frac{e_5 + e_6}{\sqrt{2}} ,$$

$$E_4 = \frac{e_5 - e_6}{\sqrt{2}} , \qquad E_5 = \frac{e_3 - e_4}{\sqrt{2}} , \qquad E_6 = e_2 ,$$

we see that all the vectors E_i are null, $\langle E_1, E_6 \rangle = -\langle E_2, E_5 \rangle = \langle E_3, E_4 \rangle = 1$, and $\mathcal{Z}_{3,2} = \operatorname{span}\{E_4, E_5, E_6\}$. The hypotheses of Lemma 22 are therefore fulfilled, and this leads to the required result. □

Theorem 25. *Let $N_{3,2}$ be the free 2-step nilpotent Lie group on 3 generators, and $\langle\langle\, , \rangle\rangle$ a bi-invariant metric on $N_{3,2}$. Let $\langle\, , \rangle$ be a left-invariant Lorentzian metric on $N_{3,2}$, and ϕ the self-adjoint map $\phi\colon \mathcal{N}_{3,2} \to \mathcal{N}_{3,2}$ relative to $\langle\langle\, , \rangle\rangle$ determining $\langle\, , \rangle$ in the sense that $\langle X, Y \rangle = \langle\langle \phi(X), Y \rangle\rangle$ for all X, $Y \in \mathcal{N}_{3,2}$. Then, $(N_{3,2}, \langle\, , \rangle)$ is Ricci-flat if and only if ϕ has a matrix of exactly one of the following types:*

1. *Relative to an orthonormal basis $\{e_1, \ldots, e_6\}$ of $(\mathcal{N}_{3,2}, \langle\langle\, , \rangle\rangle)$ such that $\langle\langle e_1, e_1 \rangle\rangle = -\langle\langle e_2, e_2 \rangle\rangle = \langle\langle e_3, e_3 \rangle\rangle = -\langle\langle e_4, e_4 \rangle\rangle = \langle\langle e_5, e_5 \rangle\rangle = -\langle\langle e_6, e_6 \rangle\rangle = 1$ and all other scalar products are zeros, we have*

   $$\phi = \operatorname{diag}(\lambda_1, -\lambda_2, \lambda_3, -\lambda_4, \lambda_5, \lambda_1) ,$$

   with $\lambda_i > 0$, $1 \le i \le 5$, and $\lambda_1^2 = \lambda_2\lambda_5 + \lambda_3\lambda_4$.
2. *Relative to a pseudo-orthonormal basis $\{e_1, \ldots, e_6\}$ of $(\mathcal{N}_{3,2}, \langle\langle\, , \rangle\rangle)$ such that $\langle\langle e_1, e_2 \rangle\rangle = -\langle\langle e_3, e_3 \rangle\rangle = -\langle\langle e_6, e_6 \rangle\rangle = \langle\langle e_4, e_4 \rangle\rangle = \langle\langle e_5, e_5 \rangle\rangle = 1$ and all other scalar products are zeros, we have*

   $$\phi = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} \oplus \operatorname{diag}(-\lambda_3, \lambda_4, \lambda_5, -\lambda_6) ,$$

   with $\lambda \neq 0$, and $\lambda_i > 0$, $3 \le i \le 6$, and $\lambda^2 = \lambda_5\lambda_6 + \lambda_3\lambda_4$,.
3. *Relative to a pseudo-orthonormal basis $\{e_1, \ldots, e_6\}$ of $(\mathcal{N}_{3,2}, \langle\langle\, , \rangle\rangle)$ such that $\langle\langle e_1, e_2 \rangle\rangle = \langle\langle e_3, e_3 \rangle\rangle = \langle\langle e_5, e_5 \rangle\rangle = -\langle\langle e_4, e_4 \rangle\rangle = -\langle\langle e_6, e_6 \rangle\rangle = 1$ and all other scalar products are zeros, we have*

   $$\phi = \begin{pmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix} \oplus \operatorname{diag}(-\lambda_4, \lambda_5, -\lambda_6) ,$$

with $\lambda > 0,\ \lambda_i > 0,\ 4 \le i \le 6,$ *and* $\lambda^2 = \lambda_5\lambda_6 + \lambda\lambda_4.$

Proof. It is well known that a self-adjoint endomorphism in a indefinite vector space need not be diagonalizable. In the present situation, where the signatures of $\langle\langle\,,\rangle\rangle$ and $\langle\,,\rangle$ are respectively $(3,3)$ and $(1,2)$, we can easily show (by using a result of [21], pp 224) that ϕ can be one of the following four forms (see [2]).
<u>*Case 1.*</u> ϕ is diagonalizable. In this case, ϕ has the form

$$\phi = \operatorname{diag}(\lambda_1, -\lambda_2, \lambda_3, -\lambda_4, \lambda_5, \lambda_6)\,, \quad \text{with} \quad \lambda_i > 0 \quad \text{for all} \quad i,$$

relative to an orthonormal basis $\{e_1, e_2, \dots, e_6\}$ of $(\mathcal{N}_{3,2}, \langle\langle\,,\rangle\rangle)$ such that

$$\langle\langle e_1, e_1\rangle\rangle = -\langle\langle e_2, e_2\rangle\rangle = \langle\langle e_3, e_3\rangle\rangle = -\langle\langle e_4, e_4\rangle\rangle = \langle\langle e_5, e_5\rangle\rangle = -\langle\langle e_6, e_6\rangle\rangle = 1\,,$$

and all other scalar products are zeros.

We claim that we can suppose that $\mathcal{Z}_{3,2} = \operatorname{span}\{e_1 - e_6, e_2 - e_5, e_3 - e_4\}$. Indeed, since $\mathcal{Z}_{3,2}$ is three-dimensional and totally degenerate relative to $\langle\langle\,,\rangle\rangle$, then without loss of generality we can assume that $\mathcal{Z}_{3,2}$ is spanned by three null vectors u, v, w so that $u \in \operatorname{span}\{e_1, e_6\}$, $v \in \operatorname{span}\{e_2, e_5\}$, and $w \in \operatorname{span}\{e_3, e_4\}$. It follows that

$$u = \alpha(e_1 \pm e_6)\,, \quad v = \beta(e_2 \pm e_5)\,, \quad w = \gamma(e_3 \pm e_4)\,, \quad \text{where} \quad \alpha,\ \beta,\ \gamma \in \mathbb{R}\,,$$

and the claim is therefore proved.

According to Lemma 23, the nonzero brackets are

$$\begin{aligned}
[e_1, e_2] = [e_6, e_2] = [e_1, e_5] = [e_6, e_5] &= \frac{\mu(e_3 - e_4)}{2\sqrt{2}}\,,\\
[e_1, e_3] = [e_6, e_3] = [e_1, e_4] = [e_6, e_4] &= \frac{\mu(e_2 - e_5)}{2\sqrt{2}},\\
[e_2, e_3] = [e_2, e_4] = [e_5, e_3] = [e_5, e_4] &= \frac{\mu(e_1 - e_6)}{2\sqrt{2}}\,.
\end{aligned}$$

Since $\{e_1, \dots, e_6\}$ is orthogonal with respect to $\langle\,,\rangle$, we let $\{\overline{e}_1, \dots, \overline{e}_6\}$ with

$$\overline{e}_1 = \frac{e_1}{\sqrt{\lambda_1}}\,, \quad \overline{e}_2 = \frac{e_2}{\sqrt{\lambda_2}}\,, \quad \overline{e}_3 = \frac{e_3}{\sqrt{\lambda_3}}\,, \quad \overline{e}_4 = \frac{e_4}{\sqrt{\lambda_4}}\,, \quad \overline{e}_5 = \frac{e_5}{\sqrt{\lambda_5}}\,, \quad \overline{e}_6 = \frac{e_6}{\sqrt{\lambda_6}}\,,$$

which is an orthonormal basis. Next, we compute the Ricci curvatures:

$$\mathrm{Ric}\,(\overline{e}_1) = -\frac{\mu^2\lambda_6\,(\lambda_3+\lambda_4)\,(\lambda_2+\lambda_5)\left(\lambda_3\lambda_4+\lambda_2\lambda_5-\lambda_1^2\right)}{16\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6}\,,$$

$$\mathrm{Ric}\,(\overline{e}_2) = -\frac{\mu^2\lambda_5\,(\lambda_1-\lambda_6)\,(\lambda_3+\lambda_4)\left(\lambda_1\lambda_6-\lambda_3\lambda_4+\lambda_2^2\right)}{16\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6}\,,$$

$$\mathrm{Ric}\,(\overline{e}_3) = -\frac{\mu^2\lambda_4\,(\lambda_2+\lambda_5)\,(\lambda_1-\lambda_6)\left(\lambda_1\lambda_6-\lambda_2\lambda_5+\lambda_3^2\right)}{16\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6}\,,$$

$$\mathrm{Ric}\,(\overline{e}_4) = -\frac{\mu^2\lambda_3\,(\lambda_2+\lambda_5)\,(\lambda_1-\lambda_6)\left(\lambda_1\lambda_6-\lambda_2\lambda_5+\lambda_4^2\right)}{16\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6}\,,$$

$$\mathrm{Ric}\,(\overline{e}_5) = -\frac{\mu^2\lambda_2\,(\lambda_1-\lambda_6)\,(\lambda_3+\lambda_4)\left(\lambda_1\lambda_6-\lambda_3\lambda_4+\lambda_5^2\right)}{16\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6}\,,$$

$$\mathrm{Ric}\,(\overline{e}_6) = -\frac{\mu^2\lambda_1\,(\lambda_3+\lambda_4)\,(\lambda_2+\lambda_5)\left(\lambda_3\lambda_4+\lambda_2\lambda_5-\lambda_6^2\right)}{16\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6}\,,$$

$$\mathrm{Ric}\,(\overline{e}_1,\overline{e}_6) = -\frac{\mu^2\,(\lambda_3+\lambda_4)\,(\lambda_2+\lambda_5)\,(\lambda_2\lambda_5-\lambda_1\lambda_6+\lambda_3\lambda_4)}{16\lambda_2\lambda_5\lambda_3\lambda_4\sqrt{\lambda_1\lambda_6}}\,,$$

$$\mathrm{Ric}\,(\overline{e}_2,\overline{e}_5) = \frac{\mu^2\,(\lambda_3+\lambda_4)\,(\lambda_1-\lambda_6)\,(\lambda_2\lambda_5-\lambda_1\lambda_6+\lambda_3\lambda_4)}{16\lambda_1\lambda_6\lambda_3\lambda_4\sqrt{\lambda_2\lambda_5}}\,,$$

$$\mathrm{Ric}\,(\overline{e}_3,\overline{e}_4) = \frac{\mu^2\,(\lambda_2+\lambda_5)\,(\lambda_1-\lambda_6)\,(\lambda_2\lambda_5-\lambda_1\lambda_6+\lambda_3\lambda_4)}{16\lambda_1\lambda_2\lambda_5\lambda_6\sqrt{\lambda_3\lambda_4}}\,.$$

It follows that the metric $\langle\,,\rangle$ is Ricci-flat if and only if the following identities are fulfilled:

$$\lambda_3\lambda_4+\lambda_2\lambda_5-\lambda_1^2=0\,,$$

$$\lambda_3\lambda_4+\lambda_2\lambda_5-\lambda_6^2=0\,,$$

$$(\lambda_1-\lambda_6)\left(\lambda_1\lambda_6-\lambda_3\lambda_4+\lambda_2^2\right)=0\,,$$

$$(\lambda_1-\lambda_6)\left(\lambda_1\lambda_6-\lambda_3\lambda_4+\lambda_5^2\right)=0\,,$$

$$(\lambda_1-\lambda_6)\left(\lambda_1\lambda_6-\lambda_2\lambda_5+\lambda_3^2\right)=0\,,$$

$$(\lambda_1-\lambda_6)\left(\lambda_1\lambda_6-\lambda_2\lambda_5+\lambda_4^2\right)=0\,,$$

$$\lambda_3\lambda_4+\lambda_2\lambda_5-\lambda_1\lambda_6=0\,,$$

$$(\lambda_1-\lambda_6)\,(\lambda_3\lambda_4+\lambda_2\lambda_5-\lambda_1\lambda_6)=0\,.$$

It is now clear that all these identities reduce to $\lambda_1=\lambda_6$ and $\lambda_3\lambda_4+\lambda_2\lambda_5=\lambda_1^2$.

Case 2. ϕ admits complex eigenvalues. In this case, ϕ has the form

$$\phi = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \oplus \mathrm{diag}\,(\lambda_3,-\lambda_4,\lambda_5,-\lambda_6)\;,$$

with $b \neq 0$ and $\lambda_i > 0$ for all i, relative to an orthonormal basis $\{e_1, e_2, \dots, e_6\}$ of $(\mathcal{N}_{3,2}, \langle\langle\, ,\rangle\rangle)$ such that $\langle\langle e_1, e_1\rangle\rangle = -\langle\langle e_2, e_2\rangle\rangle = \langle\langle e_3, e_3\rangle\rangle = -\langle\langle e_4, e_4\rangle\rangle = \langle\langle e_5, e_5\rangle\rangle = -\langle\langle e_6, e_6\rangle\rangle = 1$, and all other scalar products are zeros.

As before, we can assume that $\mathcal{Z}_{3,2} = \operatorname{span}\{e_1 - e_6, e_2 - e_5, e_3 - e_4\}$. According to Lemma 23, the nonzero brackets are

$$[e_1, e_2] = [e_6, e_2] = [e_1, e_5] = [e_6, e_5] = \frac{\mu\,(e_3 - e_4)}{2\sqrt{2}}\,,$$

$$[e_1, e_3] = [e_6, e_3] = [e_1, e_4] = [e_6, e_4] = \frac{\mu\,(e_2 - e_5)}{2\sqrt{2}}\,,$$

$$[e_2, e_3] = [e_2, e_4] = [e_5, e_3] = [e_5, e_4] = \frac{\mu\,(e_1 - e_6)}{2\sqrt{2}}\,.$$

Since $\{e_1, e_2, \dots, e_6\}$ is not an orthonormal basis with respect to $\langle\, ,\rangle$, we consider the new orthonormal basis $\{\overline{e}_1, \overline{e}_2, \dots, \overline{e}_6\}$ such that

$$\overline{e}_1 = \frac{e_1}{\sqrt{a}}\,,\ \overline{e}_2 = \frac{a e_2 - b e_1}{\sqrt{a\,(a^2 + b^2)}}\,,\ \overline{e}_3 = \frac{e_3}{\sqrt{\lambda_3}}\,,\ \overline{e}_4 = \frac{e_4}{\sqrt{\lambda_4}}\,,\ \overline{e}_5 = \frac{e_5}{\sqrt{\lambda_5}}\,,\ \overline{e}_6 = \frac{e_6}{\sqrt{\lambda_6}}\,.$$

We compute

$$\operatorname{Ric}(\overline{e}_2, \overline{e}_6) = \frac{1}{2}\frac{ba\mu^2\,(\lambda_3 + \lambda_4)\left(\lambda_3\lambda_4 + \lambda_5\lambda_6 + \lambda_5^2 - 2a\lambda_5\right)}{8a\lambda_5\lambda_3\lambda_4\sqrt{a\lambda_6\,(a^2 + b^2)}}\,,$$

$$\operatorname{Ric}(\overline{e}_5, \overline{e}_6) = \frac{1}{2}\frac{b\mu^2\,(\lambda_3 + \lambda_4)\left(\lambda_3\lambda_4 - \left(a^2 + b^2\right) + \lambda_6\lambda_5\right)}{8\lambda_3\lambda_4\,(a^2 + b^2)\sqrt{\lambda_5\lambda_6}}\,.$$

It follows that the metric $\langle\, ,\rangle$ is Ricci-flat if and only if $\lambda_3\lambda_4 + \lambda_5\lambda_6 + \lambda_5^2 - 2a\lambda_5 = 0$ and $\lambda_3\lambda_4 + \lambda_5\lambda_6 = a^2 + b^2$. This implies that $(a - \lambda_5)^2 + b^2 = 0$, from which we deduce that $b = 0$, a contradiction. Hence, this case cannot occur.

<u>Case 3.</u> ϕ is not diagonalizable and has the form

$$\phi = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} \oplus \operatorname{diag}(-\lambda_3, \lambda_4, \lambda_5, -\lambda_6)\,,$$

with $\lambda \neq 0$ and $\lambda_i > 0$ for all i, relative to a pseudo-orthonormal basis $\{e_1, \dots, e_6\}$ of $(\mathcal{N}_{3,2}, \langle\langle\, ,\rangle\rangle)$ such that $\langle\langle e_1, e_2\rangle\rangle = -\langle\langle e_3, e_3\rangle\rangle = \langle\langle e_4, e_4\rangle\rangle = \langle\langle e_5, e_5\rangle\rangle = -\langle\langle e_6, e_6\rangle\rangle = 1$, and all other scalar products are zeros.

We claim that we can suppose that $\mathcal{Z}_{3,2} = \operatorname{span}\{e_2, e_3 - e_4, e_5 - e_6\}$. Indeed, since $\mathcal{Z}_{3,2}$ is three-dimensional and totally degenerate with respect to $\langle\langle\, ,\rangle\rangle$, then without loss of generality we can assume that $\mathcal{Z}_{3,2}$ is spanned by three null vectors u, v, w so that $u \in \mathbb{R}e_2$, $v \in \operatorname{span}\{e_3, e_4\}$, and $w \in \operatorname{span}\{e_5, e_6\}$. This implies that

$$u = \alpha e_2\,,\quad v = \beta\,(e_3 \pm e_4)\,,\quad w = \gamma\,(e_5 \pm e_6)\,,\quad \text{where}\quad \alpha,\ \beta,\ \gamma \in \mathbb{R}\,,$$

and the claim is therefore proved.

According to Lemma 24, the nonzero brackets are

$$[e_1, e_3] = [e_1, e_4] = \frac{\mu\,(e_5 - e_6)}{2}\,,$$
$$[e_1, e_5] = [e_1, e_6] = \frac{\mu\,(e_3 - e_4)}{2}\,,$$
$$[e_3, e_5] = [e_3, e_6] = [e_4, e_5] = [e_4, e_6] = \frac{e_2}{2}\,.$$

Since $\{e_1, \dots, e_6\}$ is not a pseudo-orthonormal basis with respect to $\langle\,,\rangle$, we let $\{\overline{e}_1, \dots, \overline{e}_6\}$ with

$$\overline{e}_1 = e_1 - \frac{1}{2\lambda} e_2\,, \quad \overline{e}_2 = \frac{e_2}{\lambda}\,, \quad \overline{e}_3 = \frac{e_3}{\sqrt{\lambda_3}}\,, \quad \overline{e}_4 = \frac{e_4}{\sqrt{\lambda_4}}\,, \quad \overline{e}_5 = \frac{e_5}{\sqrt{\lambda_5}}\,, \quad \overline{e}_6 = \frac{e_6}{\sqrt{\lambda_6}}\,,$$

to obtain a pseudo-orthonormal basis.

The only non trivial Ricci curvature is

$$\operatorname{Ric}(\overline{e}_1) = -\frac{\mu^2}{2}\left\{\frac{(\lambda_3 + \lambda_4)(\lambda_5 + \lambda_6)\left(\lambda_3\lambda_4 + \lambda_5\lambda_6 - \lambda^2\right)}{4\lambda_3\lambda_4\lambda_5\lambda_6}\right\}.$$

It follows that the metric $\langle\,,\rangle$ is Ricci-flat if and only if $\lambda_3\lambda_4 + \lambda_5\lambda_6 - \lambda^2 = 0$, as desired.

Case 4. ϕ is not diagonalizable and has the form

$$\phi = \begin{pmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix} \oplus \operatorname{diag}(-\lambda_4, \lambda_5, -\lambda_6)\,,$$

with λ, $\lambda_i > 0$ for all i, relative to a pseudo-orthonormal basis $\{e_1, e_2, \dots, e_6\}$ of $(\mathcal{N}_{3,2}, \langle\langle\,,\rangle\rangle)$ such that $\langle\langle e_1, e_2\rangle\rangle = \langle\langle e_3, e_3\rangle\rangle = -\langle\langle e_4, e_4\rangle\rangle = \langle\langle e_5, e_5\rangle\rangle = -\langle\langle e_6, e_6\rangle\rangle = 1$, and all other scalar products are zeros.

This case can be handed in a similar way as we did in Case 3, by choosing $u = \alpha e_1$ instead of $u = \alpha e_2$. This completes the proof of Theorem 25. □

The following result will show that the free 2-step nilpotent Lie group on m generators $N_{m,2}$ cannot admit a Ricci-flat left-invariant Lorentzian metric if $m \geq 4$.

Theorem 26. *The free 2-step nilpotent Lie group on m generators $N_{m,2}$ admits a Ricci-flat left-invariant Lorentzian metric if and only if $m = 2$ or $m = 3$.*

Proof. First we prove that both $N_{2,2}$ and $N_{3,2}$ admit Ricci-flat left-invariant Lorentzian metrics. We know that the free 2-step nilpotent Lie group on 2 generators $N_{2,2}$, which is nothing but the three-dimensional Heisenberg group H_3, admits a flat (hence a Ricci-flat) left-invariant Lorentzian metric (see [14] or [20]). We also know, by Theorem 25, that $N_{3,2}$ admits a lot of Ricci-flat left-invariant Lorentzian metrics.

Conversely, assume that $N_{m,2}$ admits a Ricci-flat left-invariant Lorentzian metric $\langle\,,\rangle$. Then, by Theorem 15, there exists a pseudo-orthonormal basis $\{b, z_1, \dots, z_p, c,$

$e_1, \dots, e_q\}$ of $\mathcal{N}_{m,2}$ so that b and c are null vectors with $\langle b, c\rangle = 1$ and the Lie brackets are

$$[c, e_i] = a_i b + \sum_{k=1}^{p} c_{ik} z_k \,, \qquad [e_i, e_j] = a_{ij} b \,,$$

with $1 \le i,\, j \le q$ and $\sum\limits_{i,j=1}^{q} a_{ij}^2 = 2 \sum\limits_{k=1}^{p} \sum\limits_{i=1}^{q} c_{ik}^2$.

From this we deduce that $\dim [\mathcal{N}_{m,2}, \mathcal{N}_{m,2}] \le \min(p,q) + 1$, and by Proposition 19, we have $\mathcal{Z}_{m,2} = [\mathcal{N}_{m,2}, \mathcal{N}_{m,2}]$ and $\dim \mathcal{Z}_{m,2} = \frac{m(m-1)}{2}$. Thus,

$$m\,(m-1) \le 2 \min(p,q) + 2 \,.$$

Since by Proposition 18, we have $\dim \mathcal{N}_{m,2} = \frac{m(m+1)}{2}$, it follows that $p + q + 2 = \frac{m(m+1)}{2}$. Hence,

$$m\,(m-1) \le 2 \min(p,q) + 2 \le p + q + 2 = \frac{m\,(m+1)}{2} \,,$$

which is equivalent to saying that $m \le 3$, as desired. □

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