Applications of Mathematics

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Applications of Mathematics, Vol. 59 (2014), No. 4, 473–487

Persistent URL: http://dml.cz/dmlcz/143875

ON CONVERGENCE FOR SEQUENCES OF PAIRWISE NEGATIVELY QUADRANT DEPENDENT RANDOM VARIABLES

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(Received October 17, 2012)

Abstract. In this paper, some new results on complete convergence and complete moment convergence for sequences of pairwise negatively quadrant dependent random variables are presented. These results improve the corresponding theorems of S. X. Gan, P. Y. Chen (2008) and H. Y. Liang, C. Su (1999).

Keywords: complete convergence; complete moment convergence; pairwise NQD random variables

MSC 2010: 60F15

1. Introduction

A sequence of random variables $\{U_n, n \geqslant 1\}$ is said to converge completely to a constant a if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|U_n - a| > \varepsilon) < \infty.$$

This concept of complete convergence was given for the first time by Hsu and Robbins [6].

Let $\{Z_n, n \geqslant 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, $q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty \quad \text{for some or all } \varepsilon > 0,$$

The research of Yongfeng Wu was partially supported by the Humanities and Social Sciences Foundation for the Youth Scholars of Ministry of Education of China (12YJCZH217), the Key NSF of Anhui Educational Committe (KJ2014A255) and the Natural Science Foundation of Anhui Province (1308085MA03). The research of Guangjun Shen is supported by the National Natural Science Foundation of China (11271020).

then the above result was called the complete moment convergence. This concept was introduced by Chow [3].

Definition 1.1. Two random variables X and Y are said to be *negatively quadrant dependent* (abbreviated to NQD) if

$$P(X \leqslant x, Y \leqslant y) \leqslant P(X \leqslant x)P(Y \leqslant y) \quad \text{for all } x \text{ and } y.$$

A sequence of random variables $\{X_n, n \geqslant 1\}$ is said to be *pairwise* NQD if every two random variables are NQD. This concept was introduced by Lehmann [8].

Definition 1.2. A finite family of random variables $\{X_k, 1 \leqslant k \leqslant n\}$ is said to be *negatively associated* (abbreviated to NA) if for any disjoint subsets A and B of $\{1, 2, \ldots, n\}$ and any real coordinate-wise nondecreasing functions f on $\mathbb{R}^A$ and g on $\mathbb{R}^B$,

$$\operatorname{Cov}(f(X_i, i \in A), g(X_j, j \in B)) \leqslant 0$$

whenever the covariance exists. An infinite family of random variables is NA if every finite subfamily is NA. This concept was introduced by Joag and Proschan [7].

As pointed out in Joag and Proschan [7], NA class is a special case of pairwise NQD sequences. NA has been applied to reliability theory, multivariate statistical analysis and percolation theory, and attracted extensive attentions. So it is very significant to study probabilistic properties of this wider pairwise NQD class. Since the paper of Lehmann [8] appeared, the convergence properties of pairwise NQD random sequences were studied in various aspects: the moment inequalities (Wu [16]), the strong convergence (Matula [12], Liang et al. [10], Li and Yang [9], Wu and Jiang [18]), the weak convergence (Meng and Lin [13], Gan and Chen [5]), the complete convergence (Wu [16], Wan [15], Gan and Chen [4], Baek et al. [1]), the mean convergence (Cabrera and Volodin [2], Sung et al. [14], Wu and Guan [17]).

Recently Gan and Chen [4] proved the following theorems.

Theorem A. *Let $1 \leqslant p < 2$, $\alpha p > 1$, and let $\{X_n, n \geqslant 1\}$ be a pairwise* NQD *sequence with $EX_n = 0$. Suppose that there exists a constant $C > 0$ and a nonnegative random variable X such that*

$$\sup_{n \geqslant 1} P(|X_n| > x) \leqslant CP(X > x)$$

for all $x > 0$ and $EX^p < \infty$. Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P\left(\left|\sum_{k=1}^{n} X_k\right| > \varepsilon n^{\alpha}\right) < \infty. \tag{1.1}$$

Theorem B. *Let* $\{X_n, n \geqslant 1\}$ *be a pairwise* NQD *sequence with mean zero,* $\{a_n, n \geqslant 1\}$ *a positive number sequence with* $a_n \uparrow \infty$ *and* $\{\Psi_n(t), n \geqslant 1\}$ *a sequence of nonnegative and even functions such that for each* $n \geqslant 1$, $\Psi_n(t) > 0$ *as* $t > 0$ *and*

$$\frac{\Psi_n(|t|)}{|t|} \uparrow \quad \text{and} \quad \frac{\Psi_n(|t|)}{t^2} \downarrow \quad \text{as } |t| \uparrow . \tag{1.2}$$

If

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{E\Psi_k(X_k)}{\Psi_k(a_n)} < \infty, \tag{1.3}$$

then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{k=1}^{n} X_k\right| > \varepsilon a_n\right) < \infty. \tag{1.4}$$

Liang and Sung [11] obtained the following complete convergence theorem.

Theorem C. *Suppose* $p \geqslant 2$. *Let* $\{X_k, k \geqslant 1\}$ *be a sequence of zero mean* NA *random variables, and let* $\{a_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ *be an array of real numbers satisfying the conditions*

$$\sum_{k=1}^{n} a_{nk}^2 = O(n^{\delta}) \quad \text{as } n \to \infty, \tag{1.5}$$
$$|a_{nk}| = O(1), \quad 1 \leqslant k \leqslant n, \ n \geqslant 1, \ \text{for some } 0 < \delta < 2/p.$$

If $\beta_p =: \sup_{k \geqslant 1} E|X_k|^p < \infty$, *then for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leqslant k \leqslant n}\left|\sum_{i=1}^{k} a_{ni} X_i\right| > \varepsilon n^{1/p}\right) < \infty. \tag{1.6}$$

In this work, we will improve Theorem A under some weaker conditions, and will improve Theorem B by obtaining a stronger conclusion under some weaker conditions. In addition, we will improve Theorem C under some similar conditions.

We will state the next lemmas (cf. Lehmann [8], Wu [16]), which are very important in our study.

Lemma 1.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of pairwise* NQD *random variables. Let $\{f_n, n \geqslant 1\}$ be a sequence of increasing functions. Then $\{f_n(X_n), n \geqslant 1\}$ is a sequence of pairwise* NQD *random variables.*

Lemma 1.2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of pairwise* NQD *random variables with mean zero and $EX_n^2 < \infty$, and $T_j(k) = \sum_{i=j+1}^{j+k} X_i$, $j \geqslant 0$. Then*

$$E(T_j(k))^2 \leqslant C \sum_{i=j+1}^{j+k} EX_i^2, \quad E \max_{1\leqslant k\leqslant n} (T_j(k))^2 \leqslant C \log^2 n \sum_{i=j+1}^{j+n} EX_i^2.$$

Below, C will denote generic positive constants, whose value may vary from one application to another, $I(A)$ will indicate the indicator function of A.

2. Complete convergence for pairwise NQD sequence

In this section we will give some complete convergence theorems for sequences of pairwise NQD random variables which improve Theorem A and B.

Theorem 2.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of pairwise* NQD *random variables, and $\{c_n, n \geqslant 1\}$ a sequence of positive constants. Suppose that for some $\delta > 0$*

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} P(|X_k| > \delta) < \infty \tag{2.1}$$

and

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} EX_k^2 I(|X_k| \leqslant \delta) < \infty. \tag{2.2}$$

Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n P\left(\left|\sum_{k=1}^{n}(X_k - EX_k I(|X_k| \leqslant \delta))\right| > \varepsilon\right) < \infty. \tag{2.3}$$

P r o o f. For any $1 \leqslant k \leqslant n$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n P\left(\left|\sum_{k=1}^{n}(X_k - EX_k I(|X_k| \leqslant \delta))\right| > \varepsilon\right) \\
&\leqslant \sum_{n=1}^{\infty} c_n P\left(\left|\sum_{k=1}^{n}(X_k - EX_k I(|X_k| \leqslant \delta))\right| > \varepsilon,\ \bigcup_{k=1}^{n}\{|X_k| > \delta\}\right) \\
&\quad + \sum_{n=1}^{\infty} c_n P\left(\left|\sum_{k=1}^{n}(X_k - EX_k I(|X_k| \leqslant \delta))\right| > \varepsilon,\ \bigcap_{k=1}^{n}\{|X_k| \leqslant \delta\}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} P(|X_k| > \delta) + \sum_{n=1}^{\infty} c_n P\left(\left|\sum_{k=1}^{n}(X_k I(|X_k| \leqslant \delta) - EX_k I(|X_k| \leqslant \delta))\right| > \varepsilon\right) \\
&\quad =: I_1 + I_2.
\end{aligned}
$$

By (2.1), we can get $I_1 < \infty$. Then we prove $I_2 < \infty$. Let

$$
\begin{aligned}
Y_k &= -\delta I(X_k < -\delta) + X_k I(|X_k| \leqslant \delta) + \delta I(X_k > \delta), \\
Z_k &= -\delta I(X_k < -\delta) + \delta I(X_k > \delta).
\end{aligned}
$$

Clearly $\{Y_k, 1 \leqslant k \leqslant n\}$ is a sequence of pairwise NQD random variables by Lemma 1.1. Then

$$
\begin{aligned}
I_2 &= \sum_{n=1}^{\infty} c_n P\left(\left|\sum_{k=1}^{n}(Y_k - EY_k - Z_k + EZ_k)\right| > \varepsilon\right) \\
&\leqslant \sum_{n=1}^{\infty} c_n P\left(\left|\sum_{k=1}^{n}(Z_k - EZ_k)\right| > \varepsilon/2\right) + \sum_{n=1}^{\infty} c_n P\left(\left|\sum_{k=1}^{n}(Y_k - EY_k)\right| > \varepsilon/2\right) \\
&=: I_3 + I_4.
\end{aligned}
$$

For I_3, by Markov inequality, the definition of Z_k and (2.1), we have

$$
\begin{aligned}
I_3 &\leqslant C \sum_{n=1}^{\infty} c_n E\left|\sum_{k=1}^{n}(Z_k - EZ_k)\right| \\
&\leqslant C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} E|Z_k| \leqslant C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} P(|X_k| > \delta) < \infty.
\end{aligned}
$$

For I_4, by Markov inequality, Lemma 1.2 and C_r-inequality, we have

$$
\begin{aligned}
I_4 &\leqslant C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} E(Y_k - EY_k)^2 \leqslant C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} EY_k^2 \\
&= C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} EX_k^2 I(|X_k| \leqslant \delta) + C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} P(|X_k| > \delta).
\end{aligned}
$$

By (2.1) and (2.2), we have $I_4 < \infty$.

The proof is complete. □

Corollary 2.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of pairwise* NQD *random variables with $EX_n = 0$ for all $n \geqslant 1$, and let $\{c_n, n \geqslant 1\}$ be a sequence of positive constants. Then* (2.1), (2.2) *and*

$$\left|\sum_{k=1}^{n} EX_k I(|X_k| \leqslant \delta)\right| \to 0 \quad \text{as } n \to \infty \tag{2.4}$$

imply

$$\sum_{n=1}^{\infty} c_n P\left(\left|\sum_{k=1}^{n} X_k\right| > \varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0. \tag{2.5}$$

With Theorem 2.1 in hand, the proof of Corollary 2.1 is obvious and hence is omitted.

Taking $c_n = n^{\alpha p-2}$, and replacing X_k by X_k/n^{α} in Corollary 2.1, we can get the following corollary.

Corollary 2.2. *Let $1 \leqslant p < 2$, $\alpha p > 1$, and let $\{X_n, n \geqslant 1\}$ be a sequence of pairwise* NQD *random variables with $EX_n = 0$. Suppose that for some $\delta > 0$*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{k=1}^{n} P(|X_k| > \delta n^{\alpha}) < \infty, \tag{2.6}$$

$$\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} \sum_{k=1}^{n} EX_k^2 I(|X_k| \leqslant \delta n^{\alpha}) < \infty, \tag{2.7}$$

and

$$n^{-\alpha}\left|\sum_{k=1}^{n} EX_k I(|X_k| \leqslant \delta n^{\alpha})\right| \to 0 \quad \text{as } n \to \infty. \tag{2.8}$$

Then (1.1) *holds.*

R e m a r k 2.1. The following statements show that the conditions of Corollary 2.2 are weaker than those of Theorem A.

From the conditions of Theorem A and Lemma 1.4 of Gan and Chen [4], we have

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{k=1}^{n} P(|X_k| > \delta n^\alpha) \leqslant C \sum_{n=1}^{\infty} n^{\alpha p-1} P(X > \delta n^\alpha) \leqslant CEX^p < \infty,$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} \sum_{k=1}^{n} EX_k^2 I(|X_k| \leqslant \delta n^\alpha) \\
&\quad \leqslant C \sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} EX^2 I(X \leqslant \delta n^\alpha) + C \sum_{n=1}^{\infty} n^{\alpha p-1} P(X > \delta n^\alpha) \\
&\quad \leqslant C \sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} \sum_{m=1}^{n} EX^2 I(\delta(m-1)^\alpha < X \leqslant \delta m^\alpha) + CEX^p \\
&\quad = C \sum_{m=1}^{\infty} EX^2 I(\delta(m-1)^\alpha < X \leqslant \delta m^\alpha) \sum_{n=m}^{\infty} n^{\alpha p-1-2\alpha} + CEX^p \\
&\quad \leqslant C \sum_{m=1}^{\infty} m^{\alpha p-2\alpha} EX^2 I(\delta(m-1)^\alpha < X \leqslant \delta m^\alpha) + CEX^p \leqslant CEX^p \leqslant \infty
\end{aligned}$$

and

$$\begin{aligned}
n^{-\alpha} \left| \sum_{k=1}^{n} EX_k I(|X_k| \leqslant \delta n^\alpha) \right| &= n^{-\alpha} \left| \sum_{k=1}^{n} EX_k I(|X_k| > \delta n^\alpha) \right| \\
&\leqslant n^{-\alpha} \sum_{k=1}^{n} E|X_k| I(|X_k| > \delta n^\alpha) \leqslant C n^{1-\alpha p} EX^p \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore, we know that Corollary 2.2 improves Theorem A.

Let $\{a_n, n \geqslant 1\}$ be a positive number sequence with $a_n \uparrow \infty$. Taking $c_n = 1$ and $\delta = 1$, and replacing X_k by X_k/a_n in Corollary 2.1, we can get the following corollary.

Corollary 2.3. *Let $\{X_n, n \geqslant 1\}$ be a sequence of pairwise* NQD *random variables with $EX_n = 0$ for all $n \geqslant 1$, and $\{a_n, n \geqslant 1\}$ a positive number sequence with $a_n \uparrow \infty$. Suppose*

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} P(|X_k| > a_n) < \infty, \tag{2.9}$$

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{EX_k^2}{a_n^2} I(|X_k| \leqslant a_n) < \infty, \tag{2.10}$$

and

$$a_n^{-1} \left| \sum_{k=1}^{n} EX_k I(|X_k| \leqslant a_n) \right| \to 0 \quad \text{as } n \to \infty. \tag{2.11}$$

Then for all $\varepsilon > 0$, (1.4) holds.

R e m a r k 2.2. The following statements show that the conditions of Corollary 2.3 are weaker than those of Theorem B.

From the conditions of Theorem B, we have

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} P(|X_k|>a_n) \leqslant \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{E|X_k|}{a_n} I(|X_k|>a_n) \leqslant \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{E\Psi_k(X_k)}{\Psi_k(a_n)} < \infty,$$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{EX_k^2}{a_n^2} I(|X_k|\leqslant a_n) \leqslant \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{E\Psi_k(X_k)}{\Psi_k(a_n)} < \infty,$$

and

$$\begin{aligned} a_n^{-1}\left|\sum_{k=1}^{n} EX_k I(|X_k|\leqslant a_n)\right| &= a_n^{-1}\left|\sum_{k=1}^{n} EX_k I(|X_k|>a_n)\right| \\ &\leqslant \sum_{k=1}^{n} \frac{E|X_k|}{a_n} I(|X_k|>a_n) \leqslant \sum_{k=1}^{n} \frac{E\Psi_k(X_k)}{\Psi_k(a_n)} \to 0 \quad \text{as } n\to\infty. \end{aligned}$$

Therefore, we know that Corollary 2.3 improves Theorem B.

Theorem 2.2. *Let $\{X_n, n\geqslant 1\}$ be a sequence of pairwise* NQD *random variables, and $\{c_n, n\geqslant 1\}$ a sequence of positive constants. Suppose that for some $\delta>0$*

$$\sum_{n=1}^{\infty} c_n \log^2 n \sum_{k=1}^{n} P(|X_k|>\delta) < \infty \tag{2.12}$$

and

$$\sum_{n=1}^{\infty} c_n \log^2 n \sum_{k=1}^{n} EX_k^2 I(|X_k|\leqslant\delta) < \infty. \tag{2.13}$$

Then for all $\varepsilon>0$,

$$\sum_{n=1}^{\infty} c_n P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}(X_k - EX_k I(|X_k|\leqslant\delta))\right| > \varepsilon\right) < \infty. \tag{2.14}$$

By means of Lemma 1.2 and an argument similar to that in the proof of Theorem 2.1, we can easily prove Theorem 2.2. Therefore, we omit the details of the proof.

The next corollary is similar to Theorem 1.1 of Liang and Su [11]. However, we consider pairwise NQD instead of NA, and our result does not require the moments of order $p>2$ of random variables $\{X_n, n\geqslant 1\}$ to exist.

Corollary 2.4. *Let $\{X_n, n \geqslant 1\}$ be a sequence of pairwise* NQD *random variables with $EX_k = 0$ and $E|X_k|^p < \infty$ for all $k \geqslant 1$ and $1 < p \leqslant 2$. Let $\{a_{nk},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ be an array of real numbers satisfying the condition*

$$\sum_{k=1}^{n} |a_{nk}|^p E|X_k|^p = O(n^{\delta}) \quad \text{as } n \to \infty, \tag{2.15}$$

for some $0 < \delta < 1$. Then for all $\varepsilon > 0$ and $\alpha p \geqslant 1$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\biggl(\max_{1\leqslant j\leqslant n}\biggl|\sum_{k=1}^{j} a_{nk}X_k\biggr| > \varepsilon n^{\alpha}\biggr) < \infty. \tag{2.16}$$

Proof. Taking $c_n = n^{\alpha p-2}$ and replacing X_k by $a_{nk}X_k/n^{\alpha}$ for $1 \leqslant k \leqslant n$, $n \geqslant 1$ in Theorem 2.2, by (2.15) and $0 < \delta < 1$ we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} \log^2 n \sum_{k=1}^{n} P(|a_{nk}X_k| > \delta n^{\alpha}) \\
&\quad \leqslant C \sum_{n=1}^{\infty} n^{-2} \log^2 n \sum_{k=1}^{n} |a_{nk}|^p E|X_k|^p I(|a_{nk}X_k| > \delta n^{\alpha}) \\
&\quad \leqslant C \sum_{n=1}^{\infty} n^{-2+\delta} \log^2 n < \infty
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} \log^2 n \sum_{k=1}^{n} a_{nk}^2 EX_k^2 I(|a_{nk}X_k| \leqslant \delta n^{\alpha}) \\
&\quad \leqslant C \sum_{n=1}^{\infty} n^{-2} \log^2 n \sum_{k=1}^{n} |a_{nk}|^p E|X_{nk}|^p I(|a_{nk}X_k| \leqslant \delta n^{\alpha}) \\
&\quad \leqslant C \sum_{n=1}^{\infty} n^{-2+\delta} \log^2 n < \infty.
\end{aligned}$$

To complete the proof, it suffices to note that by $EX_k = 0$ and (2.15) we get

$$\begin{aligned}
&n^{-\alpha} \max_{1\leqslant j\leqslant n}\biggl|\sum_{k=1}^{j} a_{nk} EX_k I(|a_{nk}X_k| \leqslant \delta n^{\alpha})\biggr| \\
&\quad \leqslant n^{-\alpha} \sum_{k=1}^{n} |a_{nk}| E|X_k| I(|a_{nk}X_k| > \delta n^{\alpha}) \\
&\quad \leqslant Cn^{-\alpha p} \sum_{k=1}^{n} |a_{nk}|^p E|X_k|^p I(|a_{nk}X_k| > \delta n^{\alpha}) \leqslant Cn^{\delta-\alpha p} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

□

3. Complete moment convergence for pairwise NQD sequence

In this section, we will give some moment complete convergence theorems for sequences of pairwise NQD random variables, which improve Theorem B.

Theorem 3.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of pairwise* NQD *random variables, and $\{c_n, n \geqslant 1\}$ a sequence of positive constants. Suppose that for some $\delta > 0$*

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} E|X_k| I(|X_k| > \delta) < \infty \tag{3.1}$$

and

$$\sum_{k=1}^{n} E|X_k| I(|X_k| > \delta) \to 0 \quad \text{as } n \to \infty. \tag{3.2}$$

Then for all $\varepsilon > 0$, (2.2), (3.1) *and* (3.2) *imply*

$$\sum_{n=1}^{\infty} c_n E\Big\{\Big|\sum_{k=1}^{n} (X_k - EX_k I(|X_k| \leqslant \delta))\Big| - \varepsilon\Big\}_+ < \infty. \tag{3.3}$$

Proof. Let $S_n = \sum_{k=1}^{n} (X_k - EX_k I(|X_k| \leqslant \delta))$. For any fixed $\varepsilon > 0$,

$$\begin{aligned}
\sum_{n=1}^{\infty} c_n E\{|S_n| - \varepsilon\}_+ &= \sum_{n=1}^{\infty} c_n \int_0^{\infty} P(|S_n| - \varepsilon > t)\,\mathrm{d}t \\
&= \sum_{n=1}^{\infty} c_n \Big\{ \int_0^{\delta} P(|S_n| > \varepsilon + t)\,\mathrm{d}t + \int_{\delta}^{\infty} P(|S_n| > \varepsilon + t)\,\mathrm{d}t \Big\} \\
&\leqslant \delta \sum_{n=1}^{\infty} c_n P(|S_n| > \varepsilon) + \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} P(|S_n| > t)\,\mathrm{d}t \\
&=: I_5 + I_6.
\end{aligned}$$

Noting that (3.1) implies (2.1), by Theorem 2.1 in this paper we have $I_5 < \infty$. Hence, we need only to show $I_6 < \infty$. Clearly,

$$\begin{aligned}
P(|S_n| > t) &= P\Big(|S_n| > t,\ \bigcup_{k=1}^{n} \{|X_k| > t\}\Big) + P\Big(|S_n| > t,\ \bigcap_{k=1}^{n} \{|X_k| \leqslant t\}\Big) \\
&\leqslant \sum_{k=1}^{n} P(|X_k| > t) + P\Big(\Big|\sum_{k=1}^{n} (X_k I(|X_k| \leqslant t) - EX_k I(|X_k| \leqslant \delta))\Big| > t\Big).
\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
I_6 &\leqslant \sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} \int_{\delta}^{\infty} P(|X_k| > t)\,\mathrm{d}t \\
&\quad + \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} P\left(\left|\sum_{k=1}^{n}(X_k I(|X_k| \leqslant t) - EX_k I(|X_k| \leqslant \delta))\right| > t\right)\mathrm{d}t \\
&=: I_7 + I_8.
\end{aligned}
$$

By (3.1), we have

$$
I_7 \leqslant \sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} E|X_k| I(|X_k| > \delta) < \infty.
$$

For I_8, we let

$$
\begin{aligned}
Y_k &= -tI(X_k < -t) + X_k I(|X_k| \leqslant t) + tI(X_k > t), \\
Z_k &= -tI(X_k < -t) + tI(X_k > t).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
I_8 &\leqslant \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} P\left(\left|\sum_{k=1}^{n}(Y_k - EY_k - Z_k + EZ_k + EX_k I(\delta < |X_k| \leqslant t))\right| > t\right)\mathrm{d}t \\
&\leqslant \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} P\left(\left|\sum_{k=1}^{n}(Y_k - EY_k - Z_k + EZ_k)\right| + \left|\sum_{k=1}^{n} EX_k I(\delta < |X_k| \leqslant t)\right| > t\right)\mathrm{d}t.
\end{aligned}
$$

From (3.2), we have

$$
\begin{aligned}
\max_{t \geqslant \delta} t^{-1}\left|\sum_{k=1}^{n} EX_k I(\delta < |X_k| \leqslant t)\right| &\leqslant \max_{t \geqslant \delta} t^{-1} \sum_{k=1}^{n} E|X_k| I(\delta < |X_k| \leqslant t) \\
&\leqslant \sum_{k=1}^{n} P(|X_k| > \delta) \leqslant \delta^{-1} \sum_{k=1}^{n} E|X_k| I(|X_k| > \delta) \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
I_8 &\leqslant \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} P\left(\left|\sum_{k=1}^{n}(Y_k - EY_k - Z_k + EZ_k)\right| > t/2\right)\mathrm{d}t \\
&\leqslant \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} P\left(\left|\sum_{k=1}^{n}(Z_k - EZ_k)\right| > t/4\right)\mathrm{d}t \\
&\quad + \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} P\left(\left|\sum_{k=1}^{n}(Y_k - EY_k)\right| > t/4\right)\mathrm{d}t \\
&=: I_9 + I_{10}.
\end{aligned}
$$

By the Markov inequality, the definition of Z_k and (3.1), we have

$$
\begin{aligned}
I_9 &\leqslant C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} \int_{\delta}^{\infty} t^{-1} E|Z_k| \,\mathrm{d}t \leqslant C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} \int_{\delta}^{\infty} P(|X_k|>t)\,\mathrm{d}t \\
&\leqslant C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} E|X_k| I(|X_k|>\delta) < \infty.
\end{aligned}
$$

Let $N=[\delta]+1$, then by Markov inequality, C_r-inequality and Lemma 1.2, we have

$$
\begin{aligned}
I_{10} &\leqslant C\sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} t^{-2} E\Big(\sum_{k=1}^{n}(Y_k-EY_k)\Big)^2 \,\mathrm{d}t \leqslant C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n}\int_{\delta}^{\infty} t^{-2} EY_k^2 \,\mathrm{d}t \\
&= C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n}\int_{\delta}^{\infty} t^{-2} EX_k^2 I(|X_k|\leqslant N)\,\mathrm{d}t \\
&\quad + C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n}\int_{\delta}^{\infty} t^{-2} EX_k^2 I(N<|X_k|\leqslant t)\,\mathrm{d}t \\
&\quad + C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n}\int_{\delta}^{\infty} P(|X_k|>t)\,\mathrm{d}t =: I_{10}'+I_{10}''+I_{10}'''.
\end{aligned}
$$

By (2.2) and (3.1), we have

$$
\begin{aligned}
I_{10}' &\leqslant C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} EX_k^2 I(|X_k|\leqslant N) \\
&= C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} EX_k^2 I(|X_k|\leqslant \delta) + CN^2\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} \frac{EX_k^2}{N^2} I(\delta<|X_k|\leqslant N) \\
&\leqslant C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} EX_k^2 I(|X_k|\leqslant \delta) + CN\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} E|X_k| I(|X_k|>\delta) < \infty.
\end{aligned}
$$

For I_{10}'', since

$$
C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n}\int_{\delta}^{N} t^{-2} EX_k^2 I(N<|X_k|\leqslant t)\,\mathrm{d}t = 0,
$$

we have

$$
\begin{aligned}
I_{10}'' &= C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n}\int_{N}^{\infty} t^{-2} EX_k^2 I(N<|X_k|\leqslant t)\,\mathrm{d}t \\
&= C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n}\sum_{m=N}^{\infty}\int_{m}^{m+1} t^{-2} EX_k^2 I(N<|X_k|\leqslant t)\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C\sum_{n=1}^{\infty}c_n\sum_{k=1}^{n}\sum_{m=N}^{\infty}m^{-2}EX_k^2I(N<|X_k|\leqslant m+1)\\
&= C\sum_{n=1}^{\infty}c_n\sum_{k=1}^{n}\sum_{m=N}^{\infty}m^{-2}\sum_{s=N}^{m}EX_k^2I(s<|X_k|\leqslant s+1)\\
&= C\sum_{n=1}^{\infty}c_n\sum_{k=1}^{n}\sum_{s=N}^{\infty}EX_k^2I(s<|X_k|\leqslant s+1)\sum_{m=s}^{\infty}m^{-2}\\
&\leqslant C\sum_{n=1}^{\infty}c_n\sum_{k=1}^{n}\sum_{s=N}^{\infty}s^{-1}EX_k^2I(s<|X_k|\leqslant s+1)\\
&\leqslant C\sum_{n=1}^{\infty}c_n\sum_{k=1}^{n}E|X_k|I(|X_k|>N)<\infty.
\end{aligned}
$$

By an argument similar to that in the proof of $I_7 < \infty$, we prove $I_{10}''' < \infty$. The proof is complete. □

Corollary 3.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of pairwise* NQD *random variables with $EX_n = 0$ for all $n \geqslant 1$, and $\{c_n, n \geqslant 1\}$ a sequence of positive constants. Then for all $\varepsilon > 0$, (2.2), (3.1) and (3.2) imply*

$$
\sum_{n=1}^{\infty}c_nE\left\{\left|\sum_{k=1}^{n}X_k\right|-\varepsilon\right\}_+<\infty. \tag{3.4}
$$

With Theorem 3.1 in hand, the proof of Corollary 3.1 is obvious and hence is omitted.

Let $\{a_n, n \geqslant 1\}$ be a positive number sequence with $a_n \uparrow \infty$. Taking $c_n = 1$ and $\delta = 1$, and replacing X_k by X_k/a_n in Corollary 3.1, we can get the following corollary.

Corollary 3.2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of pairwise* NQD *random variables with $EX_n = 0$ for all $n \geqslant 1$, and $\{a_n, n \geqslant 1\}$ a positive number sequence with $a_n \uparrow \infty$. Then for all $\varepsilon > 0$,*

$$
\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{E|X_k|}{a_n}I(|X_k|>a_n)<\infty \tag{3.5}
$$

and (2.10) imply

$$
\sum_{n=1}^{\infty}a_n^{-1}E\left\{\left|\sum_{k=1}^{n}X_k\right|-\varepsilon a_n\right\}_+<\infty. \tag{3.6}
$$

Moreover, (1.4) holds.

R e m a r k 3.1. Note that (3.5) implies

$$\sum_{k=1}^{n} \frac{E|X_k|}{a_n} I(|X_k| > a_n) \to 0 \quad \text{as } n \to \infty.$$

Hence we omit this condition in Corollary 3.2.

The following statements show that the conditions of Corollary 3.2 are weaker than those of Theorem B, but the conclusion of Corollary 3.2 is much stronger than that of Theorem B.

First, by an argument similar to that in Remark 2.2, we know that the conditions of Theorem B imply (3.5) and (2.10).

Secondly, we can get

$$\sum_{n=1}^{\infty} a_n^{-1} E\Big\{\Big|\sum_{k=1}^{n} X_k\Big| - \varepsilon a_n\Big\}_+ = \sum_{n=1}^{\infty} a_n^{-1} \int_0^{\infty} P\Big(\Big|\sum_{k=1}^{n} X_k\Big| > \varepsilon a_n + t\Big)\,\mathrm{d}t$$
$$\geqslant \sum_{n=1}^{\infty} a_n^{-1} \int_0^{\varepsilon a_n} P\Big(\Big|\sum_{k=1}^{n} X_k\Big| > \varepsilon a_n + t\Big)\,\mathrm{d}t \geqslant \varepsilon \sum_{n=1}^{\infty} P\Big(\Big|\sum_{k=1}^{n} X_k\Big| > 2\varepsilon a_n\Big).$$

To sum up, we know that Corollary 3.2 improves Theorem B.

Theorem 3.2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of pairwise* NQD *random variables, and $\{c_n, n \geqslant 1\}$ a sequence of positive constants. Suppose that for some $\delta > 0$*

$$\sum_{n=1}^{\infty} c_n \log^2 n \sum_{k=1}^{n} E|X_k| I(|X_k| > \delta) < \infty. \tag{3.7}$$

Then for all $\varepsilon > 0$, (2.13), (3.2) and (3.7) imply

$$\sum_{n=1}^{\infty} c_n E\Big\{\max_{1\leqslant j\leqslant n}\Big|\sum_{k=1}^{j}(X_k - EX_k I(|X_k| \leqslant \delta))\Big| - \varepsilon\Big\}_+ < \infty. \tag{3.8}$$

By means of Lemma 1.2 and an argument similar to that in the proof of Theorem 3.1, we can easily prove Theorem 3.2. Therefore, we omit the details of the proof.

A c k n o w l e d g e m e n t. The authors are grateful to the referee for carefully reading the manuscript and for providing some comments and suggestions which led to improvements in the paper.

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