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NONLINEAR BOUNDARY VALUE PROBLEMS INVOLVING THE EXTRINSIC MEAN CURVATURE OPERATOR

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Abstract. The paper surveys recent results obtained for the existence and multiplicity of radial solutions of Dirichlet problems of the type

$$\nabla \cdot \left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = f(|x|, v) \quad \text{in } B_R, \quad u = 0 \quad \text{on } \partial B_R,$$

where B_R is the open ball of center 0 and radius R in $\mathbb{R}^n$, and f is continuous. Comparison is made with similar results for the Laplacian. Topological and variational methods are used and the case of positive solutions is emphasized. The paper ends with the case of a general domain.

Keywords: extrinsic mean curvature operator; Dirichlet problem; radial solution; positive solution; Leray-Schauder degree; critical point theory

MSC 2010: 35J20, 35J60, 35J93, 35J87

1. The extrinsic mean curvature operator

During recent years, many results about the existence, non-existence and multiplicity of solutions of nonlinear perturbations of the Laplacian in a bounded domain Ω of $\mathbb{R}^N$, with various boundary conditions, have been extended to nonlinear perturbations of nonlinear operators like the *p-Laplacian operator* $(p > 1)$,

$$\Delta_p v := \nabla \cdot (|\nabla v|^{p-2} \nabla v),$$

the *nonlinear diffusion operator* Δv^m $(m > 0)$, the *mean curvature operator*

$$\mathcal{C}(v) := \nabla \cdot \left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right),$$

and, more generally, the *quasilinear elliptic operator*

$$\mathcal{Q}(v) = \nabla \cdot A(x, v, \nabla v),$$

where $A\colon \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies suitable conditions. The results for problems associated to those rivals of the Laplacian require in general stronger assumptions, more sophisticated underlying Banach spaces, possibly greater inspiration and surely greater technical efforts.

In this paper, we consider another variant of the Laplacian, the so-called extrinsic mean curvature operator. Let $\mathbb{L}^{N+1}$ denote the *flat Minkowski space*

$$\mathbb{L}^{N+1} = \{(x,t)\colon\ x \in \mathbb{R}^N,\ t \in \mathbb{R}\},$$

with metric $\sum\limits_{j=1}^{N} (\mathrm{d}x_j)^2 - (\mathrm{d}t)^2$. Let

$$\Omega \subset \{(x,t) \in \mathbb{L}^{N+1}\colon\ t = 0\} \simeq \mathbb{R}^N$$

be a bounded domain. A *spacelike hypersurface* is the graph of a smooth function $v\colon\ \Omega \to \mathbb{R}$, and the corresponding *extrinsic mean curvature operator* $\mathcal{M}$ is defined by

$$\mathcal{M}(v) := \nabla \cdot \left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \right).$$

This operator occurs for example in papers of Calabi, Choquet-Bruhat, Cheng, Yau, Flaherty, Bartnik, Simon, Marsden, Tipler, or Treibergs (see [3] for references), motivated by some questions of general relativity or differential geometry. Notice that $\mathcal{M}$ is not homogeneous, not everywhere defined, and that $-\int_\Omega v\mathcal{M}(v) \geqslant 0$ for sufficiently smooth functions v satisfying, for example, the Dirichlet or Neumann boundary conditions on $\partial\Omega$.

2. Radial solutions on a ball prescribed extrinsic mean curvature problems

In this paper, we will be mostly concerned with the radial solutions in

$$\Omega = \{(x,0) \in \mathbb{L}^{N+1}\colon\ |x| < R\} \simeq B_R := \{x \in \mathbb{R}^N\colon\ |x| < R\}$$

of equations of the form

$$-\mathcal{M}(v) = f(|x|, v, \partial_\nu v) \quad \text{in } B_R,$$

with Dirichlet boundary conditions $v = 0$ or Neumann boundary conditions $\partial_\nu v = 0$ on ∂B_R, where $f \in C([0,R] \times \mathbb{R}^2, \mathbb{R})$ and $\partial_\nu v$ denotes the normal derivative.

Letting $|x| = r$ and $v(x) = u(|x|) = u(r)$, those radial solutions correspond to the solutions of the ordinary differential equation

$$-\left(r^{N-1}\frac{u'(r)}{\sqrt{1-|u'(r)|^2}}\right)' = r^{N-1} f(r,u,u') \quad \text{in } (0,R),$$

such that, for Dirichlet boundary conditions, $u'(0) = 0 = u(R)$, and, for Neumann boundary conditions, $u'(0) = 0 = u'(R)$.

A *solution* of the problem above is any function $u \in C^1([0,R],\mathbb{R})$ such that

$$\max_{[0,R]} |u'| < 1, \quad r^{N-1}\frac{u'(r)}{\sqrt{1-|u'(r)|^2}} \in C^1([0,R],\mathbb{R}),$$

which satisfies the differential equation and the boundary conditions.

3. The Laplacian versus the extrinsic mean curvature operator

Before describing the methods used to study nonlinear perturbations of the extrinsic mean curvature operator, let us describe, without proofs, a few classical existence, multiplicity and non-existence results for the solutions of perturbations of the Laplacian, and compare them to the corresponding ones for similar perturbations of the extrinsic mean curvature operator.

3.1. Hammerstein-type results for the Dirichlet problem. If the function $f \in C([0,R] \times \mathbb{R}, \mathbb{R})$ and

$$F(r,v) := \int_0^v f(r,s)\,\mathrm{d}s, \tag{1}$$

one can use the direct method of the calculus of variations to prove the following result, which can be traced to Hammerstein [22] in 1930:

The Dirichlet problem

$$-\Delta v = f(|x|, v) \quad \textit{in } B_R, \quad v = 0 \quad \textit{on } \partial B_R$$

has at least one radial solution if

$$\limsup_{|v|\to\infty} \frac{2F(r,v)}{v^2} < \lambda_1,$$

uniformly in $r \in [0,R]$.

On the other hand, the Fredholm alternative for the Laplacian implies that no solution exists for $f(r,v) = \lambda_1 v + \varphi_1(r)$, with λ_1 the first eigenvalue of the Dirichlet problem for $-\Delta$ on B_R and φ_1 a corresponding eigenfunction. This shows that Hammerstein's condition is sharp.

For the corresponding problem associated to the extrinsic mean curvature operator, Bereanu, Jebelean and the author [5] have proved in 2009, by topological methods, the following statement:

The Dirichlet problem

$$-\mathcal{M}(v) = f(|x|, v) \quad \text{in } B_R, \quad v = 0 \quad \text{on } \partial B_R,$$

has at least one radial solution for any $f \in C([0,R] \times \mathbb{R}, \mathbb{R})$.

Indeed, the same is true for *any* continuous $f(|x|, v, \partial_\nu v)$.

In particular, the 'spectral' Dirichlet problem

$$-\mathcal{M}(v) = \lambda v + h(x) \quad \text{in } B_R, \quad v = 0 \quad \text{on } \partial B_R,$$

has a radial solution for *any* $\lambda \in \mathbb{R}$ and *any* $h \in C(\overline{\Omega})$, a result which strongly contrasts with the spectral theory of $-\Delta$ with Dirichlet boundary conditions.

3.2. Resonant Neumann boundary value problem. If $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$, a consequence of a result proved by the author [23] in 1987 goes as follows:

If $f(r,\cdot)$ *is nonincreasing for each* $r \in [0,R]$, *then the Neumann problem*

$$-\Delta v = f(|x|, v) \quad \text{in } B_R, \quad \partial_\nu v = 0 \quad \text{on } \partial B_R \tag{2}$$

has a radial solution if and only if there exists $w \in \mathbb{R}$ *such that*

$$\int_0^R f(r,w) r^{N-1}\, \mathrm{d}r = 0. \tag{3}$$

Furthermore, a simple adaptation of a theorem of Ward, Willem and the author [25] in 1986 implies the following statement, where ν_2 is the second eigenvalue of Neumann problem for the radial Laplacian on B_R and F is defined in (1):

If $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ *is such that*

$$|f(r,v)| \leqslant a|v| + b$$

for some nonnegative a *and* b *and all* $(r,v) \in [0,R] \times \mathbb{R}$, *and if* $f(r,\cdot)$ *is nondecreasing for each* $r \in [0,R]$ *and verifies the inequality*

$$\limsup_{|v|\to\infty} \frac{2F(r,v)}{v^2} < \nu_2$$

uniformly in $r \in [0, R]$, then the Neumann problem (2) *has a radial solution if and only if there exists $w \in \mathbb{R}$ satisfying equation* (3).

On the other hand, the Fredholm alternative implies that no solution exists if $f(r, v) = \nu_2 v + \psi_2(r)$, where ψ_2 is an eigenfunction associated to ν_2. This shows that the statements of [23] and [25] are sharp.

For the case of extrinsic mean curvature operator, Bereanu, Jebelean and the author have proved in 2010 in [6] the following result, by a technique of lower and upper solutions:

If $f(r, \cdot)$ is either nonincreasing for all $r \in [0, R]$, or nondecreasing for all $r \in [0, R]$, the Neumann problem

$$-\mathcal{M}(v) = f(|x|, v) \quad \textit{in } B_R, \quad \partial_\nu v = 0 \quad \textit{on } \partial B_R$$

is solvable if and only if there exists $w \in \mathbb{R}$ satisfying equation (3).

This implies in particular that the Neumann 'spectral' problem

$$-\mathcal{M}(v) = \lambda v + h(|x|) \quad \text{in } B_R, \quad \partial_\nu v = 0 \quad \text{on } \partial B_R$$

is solvable for any $h \in C([0, R], \mathbb{R})$ if and only if $\lambda \neq 0$, a result which strongly contrasts with the spectral theory of $-\Delta$ with Neumann boundary conditions.

3.3. Neumann boundary value problem with periodic nonlinearity. Following the variational approach introduced by Willem and the author [26] in 1984 or a category argument like Rabinowitz [28] in 1988, one can prove the following multiplicity theorem:

The Neumann problem

$$-\Delta v = a \sin v + h(|x|) \quad \textit{in } B_R, \quad \partial_\nu v = 0 \quad \textit{on } \partial B_R$$

has at least two radial solutions for any $a \in \mathbb{R}$ if and only if

$$\int_0^R h(r) r^{N-1} \, \mathrm{d}r = 0. \tag{4}$$

The same result holds in the case of the extrinsic mean curvature operator:

The Neumann problem

$$-\mathcal{M}(v) = a \sin v + h(|x|) \quad \textit{in } B_R, \quad \partial_\nu v = 0 \quad \textit{on } \partial B_R$$

has at least two radial solutions for any $a \in \mathbb{R}$ if and only if condition (4) *holds.*

The existence of one solution was proved using a variational approach in 2013 by Bereanu, Jebelean and the author [10], following an idea of Brezis and the author [17] for $N = 1$ and periodic boundary conditions. The existence of a second solution was obtained in 2013 by Bereanu and Torres [15] through an adaptation of the mountain pass argument of [26], and, in the more general frame of systems, by the author [24] in 2012 and Bereanu-Jebelean [4] in 2013 using different category arguments.

3.4. Positive solutions for linearly perturbed Dirichlet problems. A trivial consequence of classical spectral theory is the following statement, where again λ_1 is the first eigenvalue of the Dirichlet problem on B_R for $-\Delta$:

The Dirichlet problem

$$-\Delta v = \lambda v \quad \textit{in } B_R, \quad v = 0 \quad \textit{on } \partial B_R$$

has a positive solution if and only if $\lambda = \lambda_1$.

On the other hand, Bereanu, Jebelean and Torres [13] have proved in 2013, by variational arguments, the following corresponding statement:

The Dirichlet problem

$$-\mathcal{M}(v) = \lambda v \quad \textit{in } B_R, \quad v = 0 \quad \textit{on } \partial B_R$$

has a positive solution for any $\lambda > (N+1)(N+2)/R^2$.

The domain of existence of a positive solution in the (λ, N, R)-parameter space is much larger in the extrinsic mean curvature case.

3.5. Pohozaev-type results for positive solutions of Dirichlet problems. We now consider the case of a perturbation λv^p with $p > 1$. A combination of classical results of Pohozaev [27] in 1965 and Brezis [16] in 1982 implies the following existence result:

The Dirichlet problem

$$-\Delta v = \lambda v^p \quad \textit{in } B_R, \quad v = 0 \quad \textit{on } \partial B_R, \tag{5}$$

with $N \geqslant 3$, $\lambda > 0$ *has a positive radial solution if and only if* $1 < p < (N+2)/(N-2)$.

The nice presentation in [16] emphasizes the role of the homogeneity of $-\Delta$ and of v^p.

Bereanu, Jebelean and Torres [14] have proved in 2013, by the method of lower-upper solutions and topological degree arguments, the following multiplicity theorem:

If $N \geqslant 2$ and $p > 1$, there exists $\Lambda > 2N/R^{p+1}$ such that the Dirichlet problem

$$-\mathcal{M}(v) = \lambda v^p \quad \text{in } B_R, \quad v = 0 \quad \text{on } \partial B_R \tag{6}$$

has zero, at least one or at least two positive radial solutions according to $\lambda \in (0, \Lambda)$, $\lambda = \Lambda$ or $\lambda > \Lambda$.

One can observe that the domains of existence of a positive radial solution in the (λ, p, N)-parameter space are very different. Notice also that an integration by parts after multiplication by v of both members of the equations shows that problems (5) and (6) have no positive solutions when $\lambda \leqslant 0$.

3.6. Brezis-Nirenberg-type results for positive solutions of Dirichlet problems. In 1983, Brezis and Nirenberg [18] have proved that one can sometimes recover the existence by a small linear perturbation in (5):

The Dirichlet problem

$$-\Delta v = \lambda v + v^p \quad \text{on } B_R, \quad v = 0 \quad \text{on } \partial B_R$$

has at least one positive radial solution if and only if

(1) $\lambda < \lambda_1$, *when* $N \geqslant 3$, $1 < p < (N+2)/(N-2)$;
(2) $\lambda \in (\lambda_1/4, \lambda_1)$, *when* $N = 3$, $p = (N+2)/(N-2)$;
(3) $\lambda \in (0, \lambda_1)$, *when* $N \geqslant 4$, $p = (N+2)/(N-2)$.

See also the nice exposition in [16], [30].

Bereanu, Jebelean and Torres [13] have considered in 2013, using variational arguments, the same type of nonlinearity for the extrinsic mean curvature operator:

If $N \geqslant 2$, $p \geqslant 1$ there exists $\Lambda > 0$ such that the Dirichlet problem

$$-\mathcal{M}(v) = \lambda v + v^p \quad \text{in } B_R, \quad v = 0 \quad \text{on } \partial B_R$$

has at least one positive radial solution for any $\lambda \geqslant \Lambda$.

Observe that in the Laplacian case, the existence of a positive radial solution is ensured when λ is sufficiently small and, in the extrinsic mean curvature case, when λ is sufficiently large. Furthermore, in this last case, no critical exponent is involved in the statement.

3.7. Positive solutions of Dirichlet problems with concave-convex nonlinearities. Ambrosetti, Brezis and Cerami [1] have considered in 1994, using lower and upper solutions, the case of concave-convex nonlinearities:

If $0 < q < 1 < p$, *there exists* $\Lambda > 0$ *such that the Dirichlet problem*

$$-\Delta v = \lambda v^q + v^p \quad \text{in } B_R, \quad v = 0 \quad \text{on } \partial B_R$$

has at least one positive radial solution if and only if $\lambda \in (0, \Lambda]$.

In 2013, Bereanu, Jebelean and Torres [13] have proved, using variational arguments, the following existence result:

If $0 < q < 1 < p$, *the Dirichlet problem*

$$-\mathcal{M}(v) = \lambda v^q + v^p \quad \text{in } B_R, \quad v = 0 \quad \text{on } \partial B_R$$

has at least one positive radial solution for any $\lambda > 0$.

Observe that in the extrinsic mean curvature case, no upper bound is required upon λ.

4. The topological approach

We sketch in this section the topological approach for radial solutions with Dirichlet boundary conditions. It will explain why the Dirichlet problem for the perturbed extrinsic mean curvature operator is always solvable.

4.1. Fixed point formulation and solvability. For simplicity of notations, set

$$\phi\colon (-1,1) \to \mathbb{R}, \quad s \mapsto \frac{s}{\sqrt{1-s^2}}$$

so that

$$\phi^{-1}(s) = \frac{s}{\sqrt{1+s^2}}, \ |\phi^{-1}(s)| < 1 \quad \text{for all } s \in \mathbb{R}.$$

We denote by

$$C := C([0,R], \mathbb{R}), \quad C_0 = \{u \in C\colon\ u(R) = 0\},$$

the Banach space of continuous functions over $[0, R]$ and its useful subspace, with the usual norm $\|v\|_\infty$.

Elementary computations show that for any $h \in C$, the mixed problem

$$-(r^{N-1}\phi(u'(r)))' = r^{N-1}h(r) \quad \text{in } (0,R), \quad u'(0) = 0 = u(R) \tag{7}$$

has the unique solution

$$U_h(r) = \int_r^R \phi^{-1}\left(t^{1-N}\int_0^t s^{N-1}h(s)\,\mathrm{d}s\right)\mathrm{d}t. \tag{8}$$

It follows immediately from formula (7) that if $h \geqslant 0$, then $U_h \geqslant 0$, and is nonincreasing. Also, $U_h \not\equiv 0$ if $h \not\equiv 0$. So we have a *maximum principle.*

For $f \in C([0,R]\times\mathbb{R},\mathbb{R})$, let us consider the radial solutions of the Dirichlet problem

$$-\mathcal{M}(v) = f(|x|, v) \quad \text{in } B_R, \quad v = 0 \quad \text{on } \partial B_R. \tag{9}$$

As mentioned before, this is equivalent to the following boundary value problem

$$-(r^{N-1}\phi(u'(r)))' = r^{N-1} f(r,u) \quad \text{in } (0,R), \quad u'(0) = 0 = u(R). \tag{10}$$

Theorem 1. *The Dirichlet problem* (9) *always has at least one radial solution.*

P r o o f. It suffices to prove that problem (10) has at least one solution. It follows from formula (7) that u is a solution of problem (10) if and only if

$$u(r) = \int_r^R \phi^{-1}\Big(t^{1-N}\int_0^t s^{N-1} f(s,u(s))\,\mathrm{d}s\Big)\,\mathrm{d}t,$$

i.e., if and only if u is a fixed point of the operator $\mathcal{T}\colon C_0 \to C_0$ defined by

$$\mathcal{T}(u)(r) := \int_r^R \phi^{-1}\Big(t^{1-N}\int_0^t s^{N-1} f(s,u(s))\,\mathrm{d}s\Big)\,\mathrm{d}t.$$

It is standard to show, using Ascoli-Arzelà theorem, that $\mathcal{T}$ is completely continuous and, furthermore, because the inequality

$$|\mathcal{T}(u)(r)| \leqslant \int_0^R \Big|\phi^{-1}\Big(r^{1-N}\int_0^r s^{N-1} f(s,u(s))\,\mathrm{d}s\Big)\Big|\,\mathrm{d}r < R,$$

holds for all $r \in [0,R]$, $\mathcal{T}$ maps C_0 into $B_R \subset C_0$. Hence $\mathcal{T}$ has at least one fixed point by Schauder fixed point theorem. Using instead the Leray-Schauder degree, one see easily that the Leray-Schauder degree $d_{LS}[\mathcal{I}-\mathcal{T}, B_R, 0]$ is equal to one. □

It follows from this approach that if $f(r,s) \geqslant 0$ for all $r \in [0,R]$) and $s \in \mathbb{R}$ and if u is a solution of (10), then $u \geqslant 0$ and is nonincreasing on $[0,R)$. Also, using the maximum principle mentioned above, the method of lower and upper solutions can be extended to problem (10) (see [13]).

4.2. Positive radial solutions of the Dirichlet problem. The following interesting result has been obtained in 2013 by Bereanu, Jebelean and Torres [13]. Let $f \in C([0,R]\times[0,\infty),\mathbb{R})$.

Theorem 2. *The Dirichlet problem* (9) *has at least one positive radial solution if*

(1) $f(r,v) \geqslant 0$ *for all* $r \in [0,R]$ *and* $v \geqslant 0$

(2) $\lim\limits_{s\to 0^+} f(r,s)/s = \infty$ *uniformly in* $r \in [0,R]$.

Sketch of the proof. If one replaces f by $\overline{f} \in C((0,R] \times \mathbb{R}, \mathbb{R})$ defined by $\overline{f}(r,s) = f(r,s^+)$, it follows from the remark at the end of the previous subsection that any solution of the modified problem is nonnegative, and hence it is a solution of the original problem. By the remark made at the end of the proof of Theorem 1, $d_{LS}[\mathcal{I} - \mathcal{T}, B_R, 0] = 1$. One then shows that Assumption 2 implies that $d_{LS}[\mathcal{I} - \mathcal{T}, B_\varrho, 0] = 0$ on a small ball B_ϱ (this is the technical part of the proof), and uses the excision property of the Leray-Schauder degree to obtain that $d_{LS}[\mathcal{I} - \mathcal{T}, B_R \setminus \overline{B_\varrho}, 0] = 1 - 0 = 1$. The conclusion comes from the existence property of the Leray-Schauder degree. □

For other applications of the topological approach to radial solutions, see [6], [7], [14], [19].

5. The variational approach

The following variational approach for positive solutions of the Dirichlet problem, used by Bereanu, Jebelean and Torres [14] in 2013, is modeled on the one introduced by Brezis and the author [17] in 2010 and Bereanu, Jebelean and the author [9] in 2011 for other boundary conditions.

5.1. Variational setting. For simplicity of notation, we denote by

$$W^{1,\infty} := W^{1,\infty}(0,R), \quad W_0^{1,\infty} := \{u \in W^{1,\infty}\colon\ u(R) = 0\}$$

the Banach space of Lipschitzian functions over $[0,R]$ and its useful subspace, with their respective usual norms $\|v\|_\infty + \|v'\|_\infty$ and $\|v'\|_\infty$.

It is not difficult to guess that the natural energy integral associated to problem (10) is given by

$$\widetilde{\mathcal{I}}(u) := \int_0^R [1 - \sqrt{1 - |u'(r)|^2} - F(r, u(r))] r^{N-1}\, \mathrm{d}r,$$

where F is defined in (1). It is easy to see that $\widetilde{\mathcal{I}}$ is well defined on

$$K_0 := \{u \in W_0^{1,\infty}\colon\ \|u'\|_\infty \leqslant 1\}$$

and that K_0 is compact and convex in C_0. For this reason, we will consider an equivalent functional on C_0. Define

$$\Psi\colon\ C_0 \to (-\infty, \infty], \quad u \mapsto \begin{cases} \displaystyle\int_0^R \big[1 - \sqrt{1 - u'(r)^2}\big] r^{N-1}\, \mathrm{d}r & \text{if } u \in K_0, \\ \infty & \text{otherwise.} \end{cases}$$

One can show that Ψ is convex, proper and lower semicontinuous on C_0. Furthermore, define

$$\mathcal{F}\colon\ C_0 \to \mathbb{R}, \quad u \mapsto -\int_0^R F(r, u(r)) r^{N-1}\, \mathrm{d}r.$$

It is standard to show that $\mathcal{F} \in C^1(C_0, \mathbb{R})$ and that, for all $u, v \in C_0$,

$$\langle \mathcal{F}'(u), v\rangle = -\int_0^R f(r, u(r)) v(r) r^{N-1}\, \mathrm{d}r.$$

Define finally $\mathcal{I} := \mathcal{F} + \Psi$. It is clearly an extension of $\widetilde{\mathcal{I}}$ to C_0, but it is not of class C^1. Hence, its critical points have to be defined and will not *a priori* correspond to the solutions of (10).

5.2. Szulkin's critical point theory. To deal with other applications, Szulkin [29] has developed in 1986 a critical point theory in a real Banach space X for real functions on X of the type $I = \mathcal{F} + \psi$, where $\psi\colon\ X \to (-\infty, \infty]$ is proper, convex, lower semi-continuous, and $\mathcal{F} \in C^1(X, \mathbb{R})$. He defines a *critical point* of I as a point $u \in X$ such that, for all $v \in X$, one has

$$\langle \mathcal{F}'(u), v - u\rangle + \psi(v) - \psi(u) \geqslant 0,$$

and proves that *any local minimum of I is a critical point of I.*

He then introduces the following *Palais-Smale condition: any sequence $\{u_n\}$ in X such that $I(u_n) \to c \in \mathbb{R}$ and*

$$\langle \mathcal{F}'(u_n), v - u_n\rangle + \psi(v) - \psi(u_n) \geqslant -\varepsilon_n \|v - u_n\|$$

for all $v \in X$ and some sequence $\{\varepsilon_n\}$ converging to 0^+ has a convergent subsequence, and proves the following extension of *mountain pass lemma* of Ambrosetti and Rabinowitz [2]: *if I satisfies the Palais-Smale condition and has the geometry of Ambrosetti-Rabinowitz mountain pass lemma, then I has a critical point.*

The functional $\mathcal{I}$ fits within the frame of Szulkin's critical point theory. It remains therefore to be sure that its critical points in Szulkin's sense provide solutions of problem (10).

5.3. Critical points of $\mathcal{I}$ and solutions of (10). Using the approach introduced by Brezis and the author [17] in 2011 for $N = 1$ and periodic boundary conditions, and by Bereanu, Jebelean and the author [9] for radial solutions with Neumann boundary conditions, Bereanu, Jebelean and Torres [13] have proved in 2013 the following result:

Lemma 1. *Any critical point of $\mathcal{I}$ is a radial solution of problem* (9).

P r o o f. It suffices to show that u is a solution of (10). So, let u be a critical point of $\mathcal{I}$. For $h \in C$, let U_h be the unique solution of problem (7) given by (8). It is not difficult to show that U_h is also the unique solution of the variational inequality

$$\int_0^R [\sqrt{1-|u'|^2} - \sqrt{1-|v'|^2}] r^{N-1}\,\mathrm{d}r \geqslant \int_0^R h(v-u) r^{N-1}\,\mathrm{d}r \quad \text{for all } v \in K_0.$$

By the unique solvability of this variational inequality, it follows that $u = U_{f(\cdot,u)}$. Therefore, $\|u'\|_\infty = \|U'_{f(\cdot,u)}\|_\infty < 1$ and u is a solution of (10). $\square$

For other applications of this approach to radial solutions, see [8], [9], [12], [13], [20].

6. The case of an arbitrary domain Ω

Let us end the paper by mentioning that both topological and variational methods can be extended to the solvability of the general problem

$$-\mathcal{M}(v) = f(x,v) \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega, \tag{11}$$

with $\Omega \subset \mathbb{R}^N$ bounded, sufficiently smooth and $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$.

The essential ingredient for this generalization is the following result, essentially proved by Bartnik and Simon [3] in 1982, and given in a more detailed and precise formulation by Corsato, Obersnel, Omari and Rivetti [21] in 2013.

Lemma 2. *For any $h \in C(\overline{\Omega}, \mathbb{R})$, the Dirichlet problem*

$$-\mathcal{M}(v) = h(x) \quad \textit{in } \Omega, \quad v = 0 \quad \textit{on } \partial\Omega$$

has a unique solution V_h. Furthermore, if $h \geqslant 0$, then $V_h \geqslant 0$.

In the topological approach, Lemma 2 can replace the explicit formula (8) in constructing the equivalent fixed point formulation. More technicalities are of course required to show that this fixed point operator is completely continuous. Many interesting applications to the existence of positive solutions to (11) can be found in [21].

In the variational approach, the reduction to Szulkin's frame in $C_0(\overline{\Omega})$ is similar and Lemma 2 can replace the explicit formula (8) in the argument used to show that the critical points of the energy functional in Szulkin's sense are solutions of (11). Of course, the technical background is more sophisticated than in the radial case.

Applications to the existence and multiplicity of nontrivial and of positive solutions to (11) can be found in [11], where, in particular, the following result is proved. Let $g\colon \Omega \times [0,\infty) \to \mathbb{R}$ be a L^∞-Carathéodory function,

$$\overline{g}(x,v) := g(x,v^+), \quad G(x,v) := \int_0^v g(x,s)\,\mathrm{d}s, \quad \overline{G}(x,v) := \int_0^v \overline{g}(x,s)\,\mathrm{d}s.$$

Theorem 3. *Assume that $q > 0$, $g(x,0) = 0$ for a.e. $x \in \Omega$, $\mu \in L^\infty(\Omega)$ is non-negative a.e. in Ω and that there exists $R > 0$ such that $g(x,s) > 0$ for a.e. $x \in \Omega$ and for all $s \in (0,R)$. Then there exists $\Lambda > 0$ such that the Dirichlet problem*

$$-\mathcal{M}(v) = \lambda g(x,v) - \mu(x)v^q \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega \tag{12}$$

has at least one nontrivial non-negative solution for all $\lambda > \Lambda$.
If, in addition, either

$$\lim_{s\to 0+} \frac{G(x,s)}{s^2} = 0 \quad \text{uniformly in } x \in \Omega,$$

or

$$\mu_m := \operatorname*{ess\,inf}_{\Omega} \mu > 0 \quad \text{and} \quad \lim_{s\to 0+} \frac{G(x,s)}{s^{q+1}} = 0 \quad \text{uniformly in } x \in \Omega,$$

there exists $\Lambda > 0$ such that problem (12) has at least two nontrivial non-negative solutions for all $\lambda > \Lambda$.

The first solution is obtained by finding a negative minimum of the energy functional on $C(\overline{\Omega})$

$$\overline{I}_\lambda(v) = \Psi(v) + \frac{1}{q+1}\int_\Omega \mu(x)|v(x)|^{q+1}\,\mathrm{d}x - \lambda\int_\Omega \overline{G}(x,v(x))\,\mathrm{d}x,$$

where

$$\Psi(v) = \begin{cases} \displaystyle\int_\Omega [1-\sqrt{1-|\nabla v(x)|^2}]\,\mathrm{d}x & (v \in K_0), \\ \infty & (v \in C(\overline{\Omega}) \setminus K_0), \end{cases}$$

and the second one by a mountain pass argument.

For example, *the Dirichlet problem*

$$-\mathcal{M}(v) = \lambda v \quad in\ \Omega, \quad v = 0 \quad on\ \partial\Omega$$

has at least one nontrivial non-negative solution for all $\lambda > 0$ *sufficiently large.* This result is also proved by topological arguments in [21].

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