Applications of Mathematics

Zhen Hai Liu; Qun Liu

Persistence and extinction of a stochastic delay predator-prey model under regime switching

Applications of Mathematics, Vol. 59 (2014), No. 3, 331–343

Persistent URL: http://dml.cz/dmlcz/143776

PERSISTENCE AND EXTINCTION OF A STOCHASTIC DELAY PREDATOR-PREY MODEL UNDER REGIME SWITCHING

Zhen Hai Liu, Qun Liu, Nanning

(Received July 17, 2012)

Abstract. The paper is concerned with a stochastic delay predator-prey model under regime switching. Sufficient conditions for extinction and non-persistence in the mean of the system are established. The threshold between persistence and extinction is also obtained for each population. Some numerical simulations are introduced to support our main results.

Keywords: persistence; extinction; Markov switching; delay; stochastic perturbations

MSC 2010: 34B16, 34C25

1. Introduction

The deterministic delay predator-prey model can be expressed as follows:

$$
\begin{aligned}
(1)\qquad & \mathrm{d}x(t)/\mathrm{d}t = x(t)[r_1(t) - a_{11}(t)x(t) - a_{12}(t)x^{\theta_{11}}(t-\tau_1) - a_{13}(t)y^{\theta_{12}}(t)],\\
& \mathrm{d}y(t)/\mathrm{d}t = y(t)[-r_2(t) + a_{21}(t)x(t) - a_{22}(t)y(t) - a_{23}(t)y^{\theta_{21}}(t-\tau_2)],
\end{aligned}
$$

where $x(t)$ and $y(t)$ are the prey population density and the predator population density at time t, respectively; $r_1(t)$ and $r_2(t)$ represent the intrinsic growth rates of the prey and the predator at time t, respectively; $a_{11}(t)$ and $a_{22}(t)$ denote the density-dependent coefficients of the prey and the predator, respectively; $a_{12}(t)$ is the capturing rate of the predator and $a_{21}(t)$ denotes the rate of conversion of nutrients into the reproduction of the predator; $a_{13}(t)$ provides a measure of intra-specific interference and $a_{23}(t)$ provides a measure of inter-specific interference; τ_1 and τ_2 are two positive constants which stand for the time delays; $\theta_{11}, \theta_{12}, \theta_{21} > 0$ and θ_{11}, θ_{21}

The research has been supported by NNSF of China Grant Nos. 11271087, 61263006.

provide nonlinear measures of intra-specific interference, θ_{21} provides a nonlinear measure of inter-specific interference; $r_i(t)$ and $a_{ij}(t)$ are positive continuous bounded functions on $R_+ = [0, +\infty)$, $i = 1, 2$; $j = 1, 2, 3$.

On the other hand, in the real world, population system is inevitably affected by the environmental noise (see e.g. [2], [13], [11], [12]). As we all know, there are various types of environmental noises. First, we shall consider a classical colored noise, say the telegraph noise. May [10] pointed out that due to the environmental noise, the birth rate, carrying capacity, competition coefficient and other parameters involved in the system are often affected by the telegraph noise. Several authors (see e.g. [9], [7], [14], [4]) have revealed that we can model the telegraph noise by a continuous-time Markov chain $\gamma(t), t \geqslant 0$ with a finite-state space $\mathscr{S} = \{1, 2, \dots, m\}$. Let $\gamma(t)$ be generated by $Q = (q_{ij})$, that is,

$$\mathbb{P}\{\gamma(t + \Delta t) = j | \gamma(t) = i\} = \begin{cases} q_{ij}\Delta t + o(\Delta t) & \text{if } j \neq i; \\ 1 + q_{ii}\Delta t + o(\Delta t) & \text{if } j = i, \end{cases}$$

where $q_{ij} \geqslant 0$ for $i, j = 1, 2, \dots, m$ with $j \neq i$ and $\sum_{j=1}^{m} q_{ij} = 0$ for $i = 1, 2, \dots, m$. Then model (1) will become

(2)
$$\begin{aligned} \mathrm{d}x(t)/\mathrm{d}t &= x(t)[r_1(\gamma(t)) - a_{11}(\gamma(t))x(t) - a_{12}(\gamma(t))x^{\theta_{11}}(t - \tau_1) - a_{13}(\gamma(t))y^{\theta_{12}}(t)], \\ \mathrm{d}y(t)/\mathrm{d}t &= y(t)[-r_2(\gamma(t)) + a_{21}(\gamma(t))x(t) - a_{22}(\gamma(t))y(t) - a_{23}(\gamma(t))y^{\theta_{21}}(t - \tau_2)]. \end{aligned}$$

The mechanism of system (2) is explained as follows. Assume that $\gamma(0) = \kappa \in \mathscr{S}$, then (2) satisfies

$$\begin{aligned} \mathrm{d}x(t)/\mathrm{d}t &= x(t)[r_1(\kappa) - a_{11}(\kappa)x(t) - a_{12}(\kappa)x^{\theta_{11}}(t - \tau_1) - a_{13}(\kappa)y^{\theta_{12}}(t)], \\ \mathrm{d}y(t)/\mathrm{d}t &= y(t)[-r_2(\kappa) + a_{21}(\kappa)x(t) - a_{22}(\kappa)y(t) - a_{23}(\kappa)y^{\theta_{21}}(t - \tau_2)] \end{aligned}$$

for a random amount of time until $\gamma(t)$ jumps to another state, say $\varsigma \in \mathscr{S}$. Then the system obeys

$$\begin{aligned} \mathrm{d}x(t)/\mathrm{d}t &= x(t)[r_1(\varsigma) - a_{11}(\varsigma)x(t) - a_{12}(\varsigma)x^{\theta_{11}}(t - \tau_1) - a_{13}(\varsigma)y^{\theta_{12}}(t)], \\ \mathrm{d}y(t)/\mathrm{d}t &= y(t)[-r_2(\varsigma) + a_{21}(\varsigma)x(t) - a_{22}(\varsigma)y(t) - a_{23}(\varsigma)y^{\theta_{21}}(t - \tau_2)] \end{aligned}$$

for a random amount of time until $\gamma(t)$ jumps to a new state again.

Further, let us consider the white noise. Recall that $r_1(i)$ represents the intrinsic growth rate in regime i ($i \in \mathscr{S}$). We estimate it by an error term plus an average value. Sometimes, the error term follows a normal distribution. Consequently, we can replace $r_1(i)$ by $r_1(i) + \sigma_{11}(i)\dot{B}_{11}(t)$ (see e.g. [7], [14], [4]), where $\dot{B}_{11}(t)$ is a white noise and $\sigma_{11}^2(i)$ stands for the intensity of the white noise. In the same way, $-a_{11}(i)$, $-a_{12}(i)$, $-a_{13}(i)$, $-r_2(i)$, $a_{21}(i)$, $-a_{22}(i)$ and $-a_{23}(i)$ will become $-a_{11}(i)+\sigma_{12}(i)\dot{B}_{12}(t)$, $-a_{12}(i)+\sigma_{13}(i)\dot{B}_{13}(t)$, $-a_{13}(i)+\sigma_{14}(i)\dot{B}_{14}(t)$, $-r_2(i)+\sigma_{21}(i)\dot{B}_{21}(t)$, $-a_{21}(i)+\sigma_{22}(i)\dot{B}_{22}(t)$, $-a_{22}(i)+\sigma_{23}(i)\dot{B}_{23}(t)$ and $-a_{23}(i)+\sigma_{24}(i)\dot{B}_{24}(t)$ (see e.g. [8]). Then we obtain the following stochastic delay predator-prey model under regime switching:

$$
\begin{aligned}
(3)\quad \mathrm{d}x(t) &= x(t)[r_1(\gamma(t)) - a_{11}(\gamma(t))x(t) - a_{12}(\gamma(t))x^{\theta_{11}}(t-\tau_1) \\
&\quad - a_{13}(\gamma(t))y^{\theta_{12}}(t)]\,\mathrm{d}t + \sigma_{11}(\gamma(t))x(t)\,\mathrm{d}B_{11}(t) + \sigma_{12}(\gamma(t))x^2(t)\,\mathrm{d}B_{12}(t) \\
&\quad + \sigma_{13}(\gamma(t))x(t)x^{\theta_{11}}(t-\tau_1)\,\mathrm{d}B_{13}(t) + \sigma_{14}(\gamma(t))x(t)y^{\theta_{12}}(t)\,\mathrm{d}B_{14}(t), \\
\mathrm{d}y(t) &= y(t)[-r_2(\gamma(t)) + a_{21}(\gamma(t))x(t) - a_{22}(\gamma(t))y(t) - a_{23}(\gamma(t))y^{\theta_{21}}(t-\tau_2)]\,\mathrm{d}t \\
&\quad + \sigma_{21}(\gamma(t))y(t)\,\mathrm{d}B_{21}(t) + \sigma_{22}(\gamma(t))x(t)y(t)\,\mathrm{d}B_{22}(t) \\
&\quad + \sigma_{23}(\gamma(t))y^2(t)\,\mathrm{d}B_{23}(t) + \sigma_{24}(\gamma(t))y(t)y^{\theta_{21}}(t-\tau_2)\,\mathrm{d}B_{24}(t),
\end{aligned}
$$

where $B(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) & B_{13}(t) & B_{14}(t) \\ B_{21}(t) & B_{22}(t) & B_{23}(t) & B_{24}(t) \end{pmatrix}$ is a given 2×4 dimensional Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a filtration $\{\mathscr{F}_t\}_{t\in\mathbb{R}_+}$ satisfying the usual conditions. Suppose that the Markov chain $\gamma(\cdot)$ is independent of $B(t)$. As the standing hypothesis, we assume that $\gamma(\cdot)$ has a unique stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ which can be obtained by solving the linear equation $\pi Q = 0$ subject to $\sum_{i=1}^{m} \pi_i = 1$ and $\pi_i > 0$, $i \in \mathscr{S}$. Throughout this article, we assume that $\min_{i\in\mathscr{S}} a_{jj}(i) > 0$, $\min_{i\in\mathscr{S}} a_{jk}(i) \geqslant 0$, $\min_{i\in\mathscr{S}} r_2(i) > 0$, $\min_{i\in\mathscr{S}} \sigma_{jl}^2(i) > 0$, $j \neq k$, $j = 1, 2$; $k = 1, 2, 3$; $l = 1, 2, 3, 4$ and define $\hat{\nu} = \max_{i\in\mathscr{S}} \nu(i)$, $\check{\nu} = \min_{i\in\mathscr{S}} \nu(i)$.

To begin with, we give the following useful definition.

Definition 1. 1. The population $x(t)$ is said to go to extinction if $\lim_{t\to+\infty} x(t) = 0$.

2. The population $x(t)$ is said to be nonpersistent in the mean if $\langle x(t)\rangle^* = 0$, where $\langle f(t)\rangle = \int_0^t f(s)\,\mathrm{d}s/t$, $f^* = \limsup_{t\to+\infty} f(t)$, $f_* = \liminf_{t\to+\infty} f(t)$.

The organization of this paper is as follows: In Section 2, we analyze the persistence and extinction of a stochastic delay predator-prey model under regime switching. Some simulation figures are provided to illustrate our main results in Section 3. Finally we give some conclusions and discussion.

2. Persistence and extinction

Since $x(t)$ and $y(t)$ in system (3) represent population sizes at time t, they should be nonnegative. For further study, we must first give some conditions under which system (3) has a unique positive solution.

Theorem 2.1. *Consider system (3). For any given positive initial value* $(x(t), y(t)) = (\xi(t), \eta(t))$ *on* $[-\tau, 0]$ *and* $(\gamma_1(0), \gamma_2(0))$*, there is a unique positive solution* $(x(t), y(t))$ *on* $t \geqslant -\tau$ *and the solution will remain in* $\mathbb{R}^2_+$ *with probability* 1*, where* $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2;\ x_i > 0, i = 1, 2\}$ *and* $\tau = \max\{\tau_1, \tau_2\}$*. In addition, the solution satisfies*

$$\limsup_{t\to+\infty} \ln x(t)/t \leqslant 0, \quad \limsup_{t\to+\infty} \ln y(t)/t \leqslant 0. \tag{4}$$

P r o o f. As the proof is similar to Cheng [1] we omit it here. □

Now we are in the position to give our main results.

Theorem 2.2. (A) *For the prey population* x *modeled by* (3)*, let* $b_1(\gamma(t)) = r_1(\gamma(t)) - 0.5\sigma_{11}^2(\gamma(t))$*. Then:*

(I) *If* $\langle b_1(\gamma(t))\rangle^* = \sum_{i=1}^{m} \pi_i b(i) < 0$*, then the population* x *goes to extinction almost surely.*

(II) *If* $\langle b_1(\gamma(t))\rangle^* = 0$*, then the population* x *is nonpersistent in the mean a.s.*

(B) *For the predator population* y *represented by* (3)*, let* $b_2(\gamma(t)) = r_2(\gamma(t)) + 0.5\sigma_{21}^2(\gamma(t))$*. Then:*

(i) *If* $(a_{11}(\gamma(t)))_*\langle -b_2(\gamma(t))\rangle^* + (a_{21}(\gamma(t)))^*\langle b_1(\gamma(t))\rangle^* < 0$*, then the population* y *goes to extinction a.s.*

(ii) *If* $(a_{11}(\gamma(t)))_*\langle -b_2(\gamma(t))\rangle^* + (a_{21}(\gamma(t)))^*\langle b_1(\gamma(t))\rangle^* = 0$*, then the population* y *is nonpersistent in the mean a.s.*

P r o o f. The proof is motivated by the methods of Liu and Wang [5], [6].

(A). Case (I). Making use of the generalized Itô's formula to the first equation of system (3) leads to

$$\begin{aligned} \mathrm{d}\ln x = \mathrm{d}x/x - (\mathrm{d}x)^2/2x^2 = [&b_1(\gamma(t)) - a_{11}(\gamma(t))x(t) - a_{12}(\gamma(t))x^{\theta_{11}}(t-\tau_1) \\ &- a_{13}(\gamma(t))y^{\theta_{12}}(t) - 0.5\sigma_{12}^2(\gamma(t))x^2(t) - 0.5\sigma_{13}^2(\gamma(t))x^{2\theta_{11}}(t-\tau_1) \\ &- 0.5\sigma_{14}^2(\gamma(t))y^{2\theta_{12}}(t)]\,\mathrm{d}t + \sigma_{11}(\gamma(t))\,\mathrm{d}B_{11}(t) + \sigma_{12}(\gamma(t))x(t)\,\mathrm{d}B_{12}(t) \\ &+ \sigma_{13}(\gamma(t))x^{\theta_{11}}(t-\tau_1)\,\mathrm{d}B_{13}(t) + \sigma_{14}(\gamma(t))y^{\theta_{12}}(t)\,\mathrm{d}B_{14}(t). \end{aligned}$$

Then we obtain that

$$\begin{aligned}(5)\quad \ln x(t)-\ln x(0)=\int_0^t&[b_1(\gamma(s))-a_{11}(\gamma(s))x(s)-a_{12}(\gamma(s))x^{\theta_{11}}(s-\tau_1)\\&-a_{13}(\gamma(s))y^{\theta_{12}}(s)-0.5\sigma_{12}^2(\gamma(s))x^2(s)\\&-0.5\sigma_{13}^2(\gamma(s))x^{2\theta_{11}}(s-\tau_1)-0.5\sigma_{14}^2(\gamma(s))y^{2\theta_{12}}(s)]\,\mathrm{d}s\\&+M_1(t)+M_2(t)+M_3(t)+M_4(t),\end{aligned}$$

where

$$M_1(t)=\int_0^t\sigma_{11}(\gamma(s))\,\mathrm{d}B_{11}(s),\quad M_2(t)=\int_0^t\sigma_{12}(\gamma(s))x(s)\,\mathrm{d}B_{12}(s),$$
$$M_3(t)=\int_0^t\sigma_{13}(\gamma(s))x^{\theta_{11}}(s-\tau_1)\,\mathrm{d}B_{13}(s),\quad M_4(t)=\int_0^t\sigma_{14}(\gamma(s))y^{\theta_{12}}(s)\,\mathrm{d}B_{14}(s).$$

In the same manner, we can show that

$$\begin{aligned}(6)\quad \ln y(t)-\ln y(0)=\int_0^t&[-b_2(\gamma(s))+a_{21}(\gamma(s))x(s)-a_{22}(\gamma(s))y(s)\\&-a_{23}(\gamma(s))y^{\theta_{21}}(s-\tau_2)-0.5\sigma_{22}^2(\gamma(s))x^2(s)\\&-0.5\sigma_{23}^2(\gamma(s))y^2(s)-0.5\sigma_{24}^2(\gamma(s))y^{2\theta_{21}}(s-\tau_2)]\,\mathrm{d}s+M_5(t)\\&+M_6(t)+M_7(t)+M_8(t),\end{aligned}$$

where

$$M_5(t)=\int_0^t\sigma_{21}(\gamma(s))\,\mathrm{d}B_{21}(s),\quad M_6(t)=\int_0^t\sigma_{22}(\gamma(s))x(s)\,\mathrm{d}B_{22}(s),$$
$$M_7(t)=\int_0^t\sigma_{23}(\gamma(s))y(s)\,\mathrm{d}B_{23}(s),\quad M_8(t)=\int_0^t\sigma_{24}(\gamma(s))y^{\theta_{21}}(s-\tau_2)\,\mathrm{d}B_{24}(s).$$

Note that $M_1(t)$ and $M_5(t)$ are local martingales, whose quadratic variations are $\langle M_1(t),M_1(t)\rangle=\int_0^t\sigma_{11}^2(\gamma(s))\,\mathrm{d}s\leqslant\widehat{\sigma_{11}^2}t$ and $\langle M_5(t),M_5(t)\rangle=\int_0^t\sigma_{21}^2(\gamma(s))\,\mathrm{d}s\leqslant\widehat{\sigma_{21}^2}t$. Applying the strong law of large numbers for local martingales (see e.g. [9] on p. 16) leads to

$$(7)\qquad \lim_{t\to+\infty}M_1(t)/t=0,\quad \lim_{t\to+\infty}M_5(t)/t=0\quad \text{a.s.}$$

On the other hand, we have

$$\begin{aligned}\langle M_2(t),M_2(t)\rangle&=\int_0^t\sigma_{12}^2(\gamma(s))x^2(s)\,\mathrm{d}s,\\ \langle M_3(t),M_3(t)\rangle&=\int_0^t\sigma_{13}^2(\gamma(s))x^{2\theta_{11}}(s-\tau_1)\,\mathrm{d}s,\\ \langle M_4(t),M_4(t)\rangle&=\int_0^t\sigma_{14}^2(\gamma(s))y^{2\theta_{12}}(s)\,\mathrm{d}s,\end{aligned}$$

$$\langle M_6(t), M_6(t)\rangle = \int_0^t \sigma_{22}^2(\gamma(s))x^2(s)\,\mathrm{d}s,$$
$$\langle M_7(t), M_7(t)\rangle = \int_0^t \sigma_{23}^2(\gamma(s))y^2(s)\,\mathrm{d}s,$$
$$\langle M_8(t), M_8(t)\rangle = \int_0^t \sigma_{24}^2(\gamma(s))y^{2\theta_{21}}(s-\tau_2)\,\mathrm{d}s.$$

By virtue of the exponential martingale inequality (see e.g. [9] on p. 74), for any positive constants T, α and β, we get

$$(8)\qquad \mathscr{P}\Big\{\sup_{0\leqslant t\leqslant T}[M_i(t) - 0.5\alpha\langle M_i(t), M_i(t)\rangle] > \beta\Big\} \leqslant \mathrm{e}^{-\alpha\beta}, \quad i = 2,3,4,6,7,8.$$

Choosing $T = k$, $\alpha = 1$, $\beta = 2\ln k$, we have

$$\mathscr{P}\Big\{\sup_{0\leqslant t\leqslant k}[M_i(t) - 0.5\langle M_i(t), M_i(t)\rangle] > 2\ln k\Big\} \leqslant 1/k^2, \quad i = 2,3,4,6,7,8.$$

Applying the Borel-Cantelli Lemma (see e.g. [9], p. 10) yields that for almost all $\omega \in \Omega$ there exists a random integer $k_0 = k_0(\omega)$ such that for $k \geqslant k_0$, $\sup_{0\leqslant t\leqslant k}[M_i(t) - 0.5\langle M_i(t), M_i(t)\rangle] \leqslant 2\ln k$, which means that

$$M_2(t) \leqslant 2\ln k + 0.5\langle M_2(t), M_2(t)\rangle = 2\ln k + 0.5\int_0^t \sigma_{12}^2(\gamma(s))x^2(s)\,\mathrm{d}s,$$
$$M_3(t) \leqslant 2\ln k + 0.5\langle M_3(t), M_3(t)\rangle = 2\ln k + 0.5\int_0^t \sigma_{13}^2(\gamma(s))x^{2\theta_{11}}(s-\tau_1)\,\mathrm{d}s,$$
$$M_4(t) \leqslant 2\ln k + 0.5\langle M_4(t), M_4(t)\rangle = 2\ln k + 0.5\int_0^t \sigma_{14}^2(\gamma(s))y^{2\theta_{12}}(s)\,\mathrm{d}s,$$
$$M_6(t) \leqslant 2\ln k + 0.5\langle M_6(t), M_6(t)\rangle = 2\ln k + 0.5\int_0^t \sigma_{22}^2(\gamma(s))x^2(s)\,\mathrm{d}s,$$
$$M_7(t) \leqslant 2\ln k + 0.5\langle M_7(t), M_7(t)\rangle = 2\ln k + 0.5\int_0^t \sigma_{23}^2(\gamma(s))y^2(s)\,\mathrm{d}s,$$
$$M_8(t) \leqslant 2\ln k + 0.5\langle M_8(t), M_8(t)\rangle = 2\ln k + 0.5\int_0^t \sigma_{24}^2(\gamma(s))y^{2\theta_{21}}(s-\tau_2)\,\mathrm{d}s$$

for all $0 \leqslant t \leqslant k$, $k \geqslant k_0$ a.s. Substituting the above inequalities into (5) yields

$$(9)\quad \ln x(t) - \ln x(0) \leqslant \int_0^t b_1(\gamma(s))\,\mathrm{d}s - \int_0^t a_{11}(\gamma(s))x(s)\,\mathrm{d}s - \int_0^t a_{12}(\gamma(s))x^{\theta_{11}}$$
$$- \int_0^t a_{13}(\gamma(s))y^{\theta_{12}}(s)\,\mathrm{d}s + M_1(t) + 6\ln k$$
$$\leqslant \int_0^t b_1(\gamma(s))\,\mathrm{d}s + M_1(t) + 6\ln k.$$

In the same way, we can also show

$$
\begin{aligned}
(10)\quad \ln y(t) - \ln y(0) &\leqslant -\int_0^t b_2(\gamma(s))\,\mathrm{d}s + \int_0^t a_{21}(\gamma(s))x(s)\,\mathrm{d}s - \int_0^t a_{22}(\gamma(s))y(s)\,\mathrm{d}s \\
&\quad - \int_0^t a_{23}(\gamma(s))y^{\theta_{21}}(s-\tau_2)\,\mathrm{d}s + M_5(t) + 6\ln k \\
&\leqslant -\int_0^t b_2(\gamma(s))\,\mathrm{d}s + \int_0^t a_{21}(\gamma(s))x(s)\,\mathrm{d}s + M_5(t) + 6\ln k
\end{aligned}
$$

for all $0 \leqslant t \leqslant k$, $k \geqslant k_0$ a.s. In other words, we have proved that for $0 < k-1 \leqslant t \leqslant k$,

$$
\begin{aligned}
t^{-1}[\ln x(t) - \ln x(0)] &\leqslant t^{-1}\int_0^t b_1(\gamma(s))\,\mathrm{d}s + 6\ln k/t + M_1(t)/t \\
&\leqslant t^{-1}\int_0^t b_1(\gamma(s))\,\mathrm{d}s + 6\ln k/(k-1) + M_1(t)/t.
\end{aligned}
$$

In view of (7) and the ergodicity of $\gamma(\cdot)$, we get

$$
\limsup_{t\to+\infty} t^{-1}\ln x(t) \leqslant \limsup_{t\to+\infty} t^{-1}\int_0^t b_1(\gamma(s))\,\mathrm{d}s = \langle b_1(\gamma(t))\rangle^* = \sum_{i=1}^m \pi_i b(i).
$$

Namely, if $\langle b_1(\gamma(t))\rangle^* = \sum\limits_{i=1}^m \pi_i b(i) < 0$, then $\lim\limits_{t\to+\infty} x(t) = 0$.

Case (II). For any given $\varepsilon > 0$, there is a constant $T_1 = T_1(\varepsilon)$ such that

$$
\begin{aligned}
t^{-1}\int_0^t b_1(\gamma(s))\,\mathrm{d}s &\leqslant \limsup_{t\to+\infty} t^{-1}\int_0^t b_1(\gamma(s))\,\mathrm{d}s + \varepsilon/2 \\
&= \langle b_1(\gamma(t))\rangle^* + \varepsilon/2 = \varepsilon/2, \quad t \geqslant T_1.
\end{aligned}
$$

Substituting the above inequality into (9), one can derive

$$
\begin{aligned}
\ln x(t) - \ln x(0) &\leqslant \int_0^t b_1(\gamma(s))\,\mathrm{d}s - \int_0^t a_{11}(\gamma(s))x(s)\,\mathrm{d}s + 6\ln k + M_1(t) \\
&\leqslant \varepsilon t/2 - \breve{a}_{11}\int_0^t x(s)\,\mathrm{d}s + 6\ln k + M_1(t)
\end{aligned}
$$

for all $T_1 \leqslant t \leqslant k$, $k \geqslant k_0$ almost surely. Notice that there is a $T > T_1$ such that for all $T \leqslant k-1 \leqslant t \leqslant k$ and $k \geqslant k_0$, we have that $6\ln k/t \leqslant \varepsilon/4$ and $M_1(t)/t \leqslant \varepsilon/4$ hold outside a $\mathbb{P}$-null set. In other words, we have shown that $\ln x(t) - \ln x(0) \leqslant$

$\varepsilon t - \breve{a}_{11}\int_0^t x(s)\,\mathrm{d}s$ holds outside a $\mathbb{P}$-null set for sufficiently large $t > T$. Set $h(t) = \int_0^t x(s)\,\mathrm{d}s$, then we get

$$\ln(\mathrm{d}h/\mathrm{d}t) \leqslant \varepsilon t - \breve{a}_{11}h(t) + \ln x(0), \quad t > T, \text{ a.s.}$$

That is, $\breve{a}_{11}^{-1}[\mathrm{e}^{\breve{a}_{11}h(t)} - \mathrm{e}^{\breve{a}_{11}h(T)}] \leqslant x(0)\varepsilon^{-1}[\mathrm{e}^{\varepsilon t} - \mathrm{e}^{\varepsilon T}]$ a.s. Rewriting the inequality, we have

$$\mathrm{e}^{\breve{a}_{11}h(t)} \leqslant \mathrm{e}^{\breve{a}_{11}h(T)} + x(0)\breve{a}_{11}\varepsilon^{-1}[\mathrm{e}^{\varepsilon t} - \mathrm{e}^{\varepsilon T}] \quad \text{a.s.}$$

Taking logarithm on both sides yields

$$h(t) \leqslant \breve{a}_{11}^{-1}\ln\{x(0)\breve{a}_{11}\varepsilon^{-1}\mathrm{e}^{\varepsilon t} + \mathrm{e}^{\breve{a}_{11}h(T)} - x(0)\breve{a}_{11}\varepsilon^{-1}\mathrm{e}^{\varepsilon T}\} \quad \text{a.s.}$$

In other words, we obtain

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x(s)\,\mathrm{d}s \leqslant \breve{a}_{11}^{-1}\limsup_{t\to+\infty} t^{-1}\ln[x(0)\breve{a}_{11}\varepsilon^{-1}\mathrm{e}^{\varepsilon t} + \mathrm{e}^{\breve{a}_{11}h(T)} - x(0)\breve{a}_{11}\varepsilon^{-1}\mathrm{e}^{\varepsilon T}].$$

Applying L'Hospital's rule, one can see that

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x(s)\,\mathrm{d}s \leqslant \breve{a}_{11}^{-1}\limsup_{t\to+\infty} t^{-1}\ln[x(0)\breve{a}_{11}\varepsilon^{-1}\mathrm{e}^{\varepsilon t}] = \varepsilon/\breve{a}_{11} \quad \text{a.s.}$$

Due to the arbitrariness of ε, we have $\limsup\limits_{t\to+\infty} t^{-1}\int_0^t x(s)\,\mathrm{d}s \leqslant 0$ a.s.

(B). Case (i). If $\langle b_1(\gamma(t))\rangle^* \leqslant 0$, then from (A) we see that $\langle x(t)\rangle^* = 0$. On the other hand, according to the specific property of the limit superior, we have that for arbitrarily given and sufficiently small $\varepsilon > 0$, there is a $T > 0$ such that $\langle -b_2(\gamma(t))\rangle < \langle -b_2(\gamma(t))\rangle^* + \varepsilon$ for all $t > T$. Let ε be small enough so that $\langle -b_2(\gamma(t))\rangle^* + \varepsilon < 0$. Applying (10) results in

$$\begin{aligned}[t^{-1}\ln y(t)]^* &\leqslant \langle -b_2(\gamma(t))\rangle^* + \varepsilon + \hat{a}_{21}\langle x(t)\rangle^* + \lim_{t\to+\infty} M_5(t)/t + \lim_{t\to+\infty} 6\ln k/t \\ &= \langle -b_2(\gamma(t))\rangle^* + \varepsilon < 0 \quad \text{a.s.}\end{aligned}$$

That is, $\lim\limits_{t\to+\infty} y(t) = 0$. Now if $\langle b_1(\gamma(t))\rangle^* > 0$, it follows from (5) that

$$t^{-1}\ln[x(t)/x(0)] \leqslant \langle b_1(\gamma(t))\rangle^* + \varepsilon/2 - [(a_{11}(\gamma(t)))_* - \varepsilon]\langle x(t)\rangle + \varepsilon/2 \quad \text{a.s.}$$

Making use of the same method as in (A) and of the arbitrariness of ε, we have

$$\langle x(t)\rangle^* \leqslant \langle b_1(\gamma(t))\rangle^*/(a_{11}(\gamma(t)))_*. \tag{11}$$

By the above inequality, we have

$$[t^{-1}\ln y(t)]^* \leqslant \langle -b_2(\gamma(t))\rangle^* + \langle a_{21}(\gamma(t))x(t)\rangle^* + \lim_{t\to+\infty} M_5(t)/t + \lim_{t\to+\infty} 6\ln k/t$$
$$\leqslant [(a_{11}(\gamma(t)))_* \langle -b_2(\gamma(t))\rangle^* + ((a_{21}(\gamma(t)))^* + \varepsilon)\langle b_1(\gamma(t))\rangle^*]/(a_{11})_* < 0,$$

from which we derive the desired statement.

Case (ii). In Case (i), we have proved that if $\langle b_1(\gamma(t))\rangle^* \leqslant 0$, then $\lim_{t\to+\infty} y(t) = 0$, that is to say, $\langle y(t)\rangle^* = 0$. Now assume that $\langle b_1(\gamma(t))\rangle^* > 0$; we shall verify Case (ii) by contradiction. If $\langle y(t)\rangle^* > 0$, then by (4) we see that $[t^{-1}\ln y(t)]^* = 0$. Then in view of (6) one can obtain that

$$0 = [t^{-1}\ln y(t)]^* \leqslant \langle -b_2(\gamma(t))\rangle^* + \langle a_{21}(\gamma(t))x(t)\rangle^* \leqslant \langle -b_2(\gamma(t))\rangle^* + (a_{21}(\gamma))^* \langle x(t)\rangle^*.$$

On the other hand, for any fixed $\varepsilon > 0$ there is a $T > 0$ such that

$$\langle -b_2(\gamma(t))\rangle < \langle -b_2(\gamma(t))\rangle^* + \varepsilon/4, \quad \langle a_{21}(\gamma(t))x(t)\rangle < (a_{21}(\gamma(t)))^* \langle x(t)\rangle^* + \varepsilon/4,$$
$$6\ln k/t < \varepsilon/4, \quad M_5(t)/t < \varepsilon/4$$

for all $t > T$. Substituting these inequalities into (10) results in

$$\ln(y(t)/y(0))/t \leqslant \langle -b_2(\gamma(t))\rangle^* + (a_{21}(\gamma(t)))^* \langle x(t)\rangle^* + \varepsilon - (a_{22}(\gamma(t)))_* \langle y(t)\rangle.$$

Then in the same way as in (A), we obtain

$$\langle y(t)\rangle^* \leqslant \frac{\langle -b_2(\gamma(t))\rangle^* + (a_{21}(\gamma(t)))^* \langle x(t)\rangle^* + \varepsilon}{(a_{22}(\gamma(t)))_*},$$

which shows that $\langle y(t)\rangle^* \leqslant \langle -b_2(\gamma(t))\rangle^* + (a_{21}(\gamma(t)))^* \langle x(t)\rangle^* / (a_{22}(\gamma(t)))_*$. By (14), we get that

$$\langle y(t)\rangle^* \leqslant \frac{(a_{11}(\gamma(t)))_* \langle -b_2(\gamma(t))\rangle^* + (a_{21}(\gamma(t)))^* \langle b_1(\gamma(t))\rangle^*}{(a_{11}(\gamma(t)))_* (a_{22}(\gamma(t)))_*} = 0,$$

which leads to a contradiction. Thus $\langle y(t)\rangle^* = 0$ a.s. □

3. Example and numerical simulations

In this section we will introduce an example and some figures to demonstrate our main results. In the following discussion, we first give the stationary distribution $\pi = (\pi_1, \ldots, \pi_m)$ of the Markov chain $\gamma(t)$ directly because π can be derived by solving the linear equations

$$\pi Q = 0, \quad \sum_{i=1}^{m} \pi_i = 1,$$

where Q is the generator of the Markov chain $\gamma(t)$.

Example 3.1. Consider the system

$$\begin{aligned}
\mathrm{d}x(t) &= x(t)[r_1(\gamma(t)) - a_{11}(\gamma(t))x(t) - a_{12}(\gamma(t))x^{0.5}(t-\tau_1) \\
&\quad - a_{13}(\gamma(t))y^{0.5}(t)]\,\mathrm{d}t + \sigma_{11}(\gamma(t))x(t)\,\mathrm{d}B_{11}(t) + \sigma_{12}(\gamma(t))x^2(t)\,\mathrm{d}B_{12}(t) \\
&\quad + \sigma_{13}(\gamma(t))x(t)x^{0.5}(t-\tau_1)\,\mathrm{d}B_{13}(t) + \sigma_{14}(\gamma(t))x(t)y^{0.5}(t)\,\mathrm{d}B_{14}(t), \\
\mathrm{d}y(t) &= y(t)[-r_2(\gamma(t)) + a_{21}(\gamma(t))x(t) - a_{22}(\gamma(t))y(t) - a_{23}(\gamma(t))y^{0.5}(t-\tau_2)]\,\mathrm{d}t \\
&\quad + \sigma_{21}(\gamma(t))y(t)\,\mathrm{d}B_{21}(t) + \sigma_{22}(\gamma(t))x(t)y(t)\,\mathrm{d}B_{22}(t) \\
&\quad + \sigma_{23}(\gamma(t))y^2(t)\,\mathrm{d}B_{23}(t) + \sigma_{24}(\gamma(t))y(t)y^{0.5}(t-\tau_2)\,\mathrm{d}B_{24}(t),
\end{aligned} \tag{12}$$

where $\gamma = \gamma(t)$ is a Markov chain with state space $\mathscr{S} = \{1, 2\}$.

Let $r_1(1) = 0.2$, $r_1(2) = 0.1$, $a_{11}(\gamma(t)) = a_{12}(\gamma(t)) = a_{13}(\gamma(t)) \equiv 0.2$, $\tau_1 = 1$, $\sigma_{11}^2(\gamma(t)) \equiv 0.24$, $\sigma_{12}(\gamma(t)) = \sigma_{13}(\gamma(t)) = \sigma_{14}(\gamma(t)) \equiv 5$. That is to say, $b_1(1) = r_1(1) - 0.5\sigma_{11}^2(1) = 0.08$, $b_1(2) = r_1(2) - 0.5\sigma_{11}^2(2) = -0.02$. Solving the equations

$$\begin{aligned}
\pi_{11} + \pi_{12} &= 1, \\
0.08\pi_{11} - 0.02\pi_{12} &= 0
\end{aligned}$$

yields $\pi_{11} = 0.2$, $\pi_{12} = 0.8$.

Now, let $r_2(1) = 0.4$, $r_2(2) = 0.2$, $a_{21}(\gamma(t)) = a_{22}(\gamma(t)) = a_{23}(\gamma(t)) \equiv 0.4$, $\tau_2 = 2$, $\sigma_{21}^2(\gamma(t)) \equiv 0.48$, $\sigma_{22}(\gamma(t)) = \sigma_{23}(\gamma(t)) = \sigma_{24}(\gamma(t)) \equiv 8$. By a simple calculation, we obtain $\pi_{21} = 0.2$, $\pi_{22} = 0.8$.

Let us simulate the above example. In the following simulations, for the sake of convenience, set $\xi(t) = 0.32\mathrm{e}^{0.5t}$, $\eta(t) = 0.16\mathrm{e}^{0.5t}$, $t \in [-\tau, 0]$. There are two kinds of random processes in Eq. (12). One is the Markov switching; the other is the Brownian motion. As for the Brownian motion, we will use the Milstein method (see e.g. [3]). Namely, if the state of the Markov chain $\gamma(t)$ is $i(i \in \mathscr{S})$, we will consider the following discretization equations:

$$
\begin{aligned}
x_{k+1} = x_k &+ x_k[r_1(i) - a_{11}(i)x_k - a_{12}(i)x_{k-\tau_1/\Delta t}^{0.5} - a_{13}(i)y_k^{0.5}]\Delta t \\
&+ \sigma_{11}(i)x_k\sqrt{\Delta t}\xi_{11k} + 0.5\sigma_{11}^2(i)x_k^2[(\xi_{11k})^2 - 1]\sqrt{\Delta t} + \sigma_{12}(i)x_k^2\sqrt{\Delta t}\xi_{12k} \\
&+ 0.5\sigma_{12}^2(i)x_k^4[(\xi_{12k})^2 - 1]\sqrt{\Delta t} + \sigma_{13}(i)x_k x_{k-\tau_1/\Delta t}^{0.5}\sqrt{\Delta t}\xi_{13k} \\
&+ 0.5\sigma_{13}^2(i)x_k^2 x_{k-\tau_1/\Delta t}[(\xi_{13k})^2 - 1]\sqrt{\Delta t} + \sigma_{14}(i)x_k y_k^{0.5}\sqrt{\Delta t}\xi_{14k} \\
&+ 0.5\sigma_{14}^2(i)x_k^2 y_k[(\xi_{14k})^2 - 1]\sqrt{\Delta t}, \\
y_{k+1} = y_k &+ y_k[-r_2(i) + a_{21}(i)x_k - a_{22}(i)y_k - a_{23}(i)y_{k-\tau_2/\Delta t}^{0.5}]\Delta t \\
&+ \sigma_{21}(i)y_k\sqrt{\Delta t}\xi_{21k} + 0.5\sigma_{21}^2(i)y_k^2[(\xi_{21k})^2 - 1]\sqrt{\Delta t} + \sigma_{22}(i)x_k y_k\sqrt{\Delta t}\xi_{22k} \\
&+ 0.5\sigma_{22}^2(i)x_k^2 y_k^2[(\xi_{22k})^2 - 1]\sqrt{\Delta t} + \sigma_{23}(i)y_k^2\sqrt{\Delta t}\xi_{23k} \\
&+ 0.5\sigma_{23}^2(i)y_k^4[(\xi_{23k})^2 - 1]\sqrt{\Delta t} + \sigma_{24}(i)y_k y_{k-\tau_2/\Delta t}^{0.5}\sqrt{\Delta t}\xi_{24k} \\
&+ 0.5\sigma_{24}^2(i)y_k^2 y_{k-\tau_2/\Delta t}[(\xi_{24k})^2 - 1]\sqrt{\Delta t},
\end{aligned}
$$

where $\xi_{jlk}(j = 1, 2; l = 1, 2, 3, 4; k = 1, 2, \ldots, n)$ is the Gaussian random variable which follows $N(0, 1)$.

In Fig. 1, we choose $r_1(1) = 0.2$, $r_1(2) = 0.1$, $r_2(1) = 0.4$, $r_2(2) = 0.2$, $a_{11}(\gamma(t)) = a_{12}(\gamma(t)) = a_{13}(\gamma(t)) \equiv 0.2$, $a_{21}(\gamma(t)) = a_{22}(\gamma(t)) = a_{23}(\gamma(t)) \equiv 0.4$, $\theta_{11} = \theta_{12} = \theta_{21} = 0.5$, $\tau_1 = 1$, $\tau_2 = 2$, $\sigma_{11}^2(\gamma(t)) \equiv 0.24$, $\sigma_{12}(\gamma(t)) = \sigma_{13}(\gamma(t)) = \sigma_{14}(\gamma(t)) \equiv 5$, $\sigma_{21}^2(\gamma(t)) \equiv 0.48$, $\sigma_{22}(\gamma(t)) = \sigma_{23}(\gamma(t)) = \sigma_{24}(\gamma(t)) \equiv 8$. The only distinction between conditions of Fig. 1(A)–(B) is that the values of π_{11} and π_{21} are different. In Fig. 1(A), we choose $\pi_{11} = 0.18$ and $\pi_{21} = 0.15$. By virtue of Theorem 2.2, we see that both the prey population x and the predator population y represented by (12) will go to extinction. In Fig. 1(B), we choose $\pi_{11} = \pi_{21} = 0.2$. In view of Theorem 2.2, one can obtain that both the prey population x and the predator population y will be nonpersistent in the mean.

4. Conclusions and discussion

In this paper, we investigate a stochastic delay predator-prey model under regime switching. Sufficient criteria for extinction and non-persistence in the mean of each population are established. Furthermore, we obtain the critical value between persistence and extinction of each population in many cases. These results are useful, because the persistence-extinction thresholds are very important for assessing the risk of extinction of populations in models.

Some interesting topics deserve further investigation. One may propose some more realistic models, such as considering the effects of impulsive perturbations on

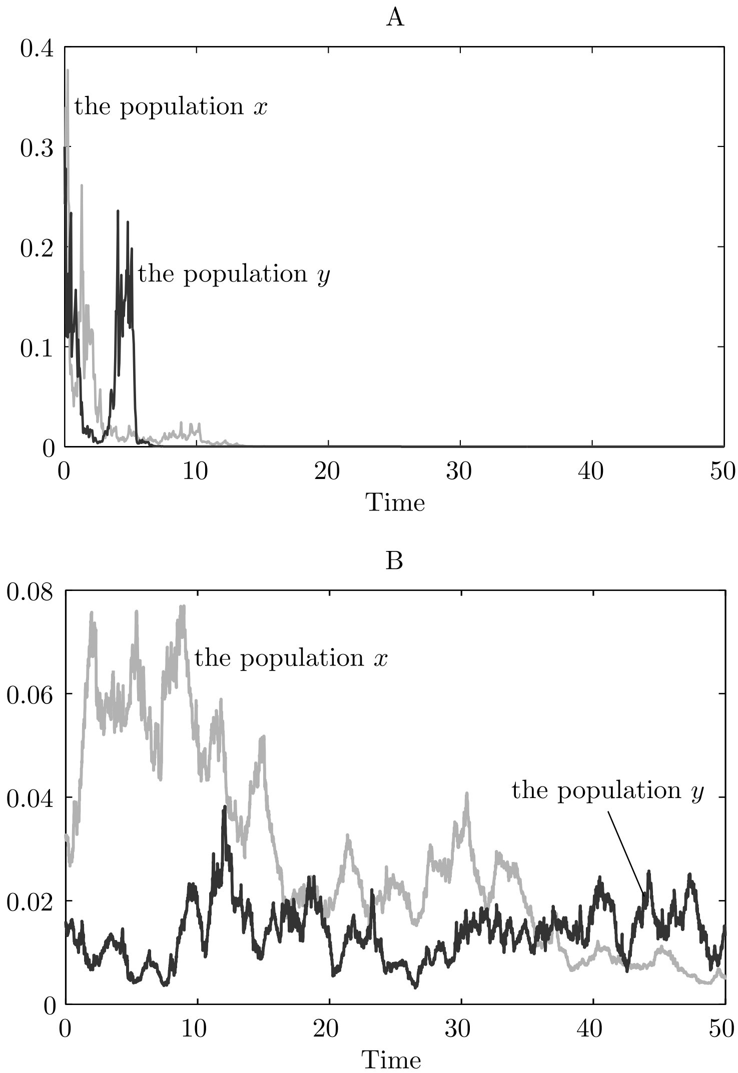

Fig. 1. Solutions of system (12) for $r_1(1) = 0.2$, $r_1(2) = 0.1$, $r_2(1) = 0.4$, $r_2(2) = 0.2$, $a_{11}(\gamma(t)) = a_{12}(\gamma(t)) = a_{13}(\gamma(t)) \equiv 0.2$, $a_{21}(\gamma(t)) = a_{22}(\gamma(t)) = a_{23}(\gamma(t)) \equiv 0.4$, $\theta_{11} = \theta_{12} = \theta_{21} = 0.5$, $\tau_1 = 1$, $\tau_2 = 2$, $\sigma_{11}^2(\gamma(t)) \equiv 0.24$, $\sigma_{12}(\gamma(t)) = \sigma_{13}(\gamma(t)) = \sigma_{14}(\gamma(t)) \equiv 5$, $\sigma_{21}^2(\gamma(t)) \equiv 0.48$, $\sigma_{22}(\gamma(t)) = \sigma_{23}(\gamma(t)) = \sigma_{24}(\gamma(t)) \equiv 8, \xi(t) \equiv 0.32\mathrm{e}^{0.5t}$, $\eta(t) \equiv 0.16\mathrm{e}^{0.5t}$, $t \in [-\tau, 0]$, step size $\Delta t = 0.05$. (A) is with $\pi_{11} = 0.18$, $\pi_{21} = 0.15$; (B) is with $\pi_{11} = \pi_{21} = 0.2$.

the systems. It is also interesting to investigate the systems with distributed delays. We will consider these problems in our future work.

References

[1] *S. R. Cheng*: Stochastic population systems. Stochastic Anal. Appl. *27* (2009), 854–874.

[2] *T. C. Gard*: Introduction to Stochastic Differential Equations. Pure and Applied Mathematics 114, Marcel Dekker, New York, 1988.

[3] *D. J. Higham*: An algorithmic introduction to numerical simulation of stochastic differential equations. SIAM Rev. *43* (2001), 525–546.

[4] *X. Li, A. Gray, D. Jiang, X. Mao*: Sufficient and necessary conditions of stochastic permanence and extinction for stochastic logistic populations under regime switching. J. Math. Anal. Appl. *376* (2011), 11–28.

[5] *M. Liu, W. Li, K. Wang*: Persistence and extinction of a stochastic delay logistic equation under regime switching. Appl. Math. Lett. *26* (2013), 140–144.

[6] *M. Liu, K. Wang*: Persistence, extinction and global asymptotical stability of a non-autonomous predator-prey model with random perturbation. Appl. Math. Modelling *36* (2012), 5344–5353.

[7] *Q. Luo, X. Mao*: Stochastic population dynamics under regime switching II. J. Math. Anal. Appl. *355* (2009), 577–593.

[8] *X. Mao, G. Marion, E. Renshaw*: Environmental Brownian noise suppresses explosions in population dynamics. Stochastic Processes Appl. *97* (2002), 95–110.

[9] *X. Mao, C. Yuan*: Stochastic Differential Equations with Markovian Switching. World Scientific, Hackensack, 2006; Imperial College Press, London, 2006.

[10] *R. M. May*: Stability and Complexity in Model Ecosystems. With a new introduction by the author. 2nd ed. Princeton Landmarks in Biology, Princeton University Press, Princeton, 2001.

[11] *G. P. Samanta*: Influence of environmental noise in Gompertzian growth model. J. Math. Phys. Sci. *26* (1992), 503–511.

[12] *G. P. Samanta*: Logistic growth under colored noise. Bull. Math. Soc. Sci. Math. Roum., Nouv. Sér. *37* (1993), 115–122.

[13] *G. P. Samanta, C. G. Chakrabarti*: On stability and fluctuation in Gompertzian and logistic growth models. Appl. Math. Lett. *3* (1990), 119–121.

[14] *C. Zhu, G. Yin*: On hybrid competitive Lotka-Volterra ecosystems. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *71* (2009), e1370–e1379.

Authors' address: *Zhenhai Liu*, *Qun Liu*, Guangxi University for Nationalities, Nanning, P. R. China, e-mails: `zhhliu@hotmail.com`, `liuqun151608@163.com`.