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LINEARIZATION RELATIONS FOR THE GENERALIZED BEDIENT POLYNOMIALS OF THE FIRST AND SECOND KINDS VIA THEIR INTEGRAL REPRESENTATIONS

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Abstract. The main object of this paper is to investigate several general families of hypergeometric polynomials and their associated multiple integral representations. By suitably specializing our main results, the corresponding integral representations are deduced for such familiar classes of hypergeometric polynomials as (for example) the generalized Bedient polynomials of the first and second kinds. Each of the integral representations, which are derived in this paper, may be viewed also as a linearization relationship for the product of two different members of the associated family of hypergeometric polynomials.

Keywords: hypergeometric function; hypergeometric polynomial; Srivastava polynomial; Bedient polynomial; generalized Bedient polynomial of the first and second kinds; multiple integral representation; Gamma function; Eulerian beta integral linearization relationship; Pochhammer symbol; shifted factorial

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1. Introduction and definitions

Let $\{A_{m,n}\}_{m,n=0}^{\infty}$ be a suitably bounded double sequence of essentially arbitrary (real or complex) parameters. Almost four decades ago, Srivastava [12] introduced and investigated the following general family of polynomials:

$$S_n^N(z) := \sum_{k=0}^{[n/N]} \frac{(-n)_{Nk}}{k!} A_{n,k} z^k \quad (n \in \mathbb{N}_0;\ N \in \mathbb{N}), \tag{1}$$

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where $[\kappa]$ denotes the greatest integer not exceeding $\kappa \in \mathbb{R}$ and $(\lambda)_\nu$ denotes the Pochhammer symbol or the *shifted factorial*, in particular

$$(1)_n = n! \quad (n \in \mathbb{N}_0),$$

which is defined, in terms of the familiar Gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda+1)\dots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases}$$

it being understood conventionally that $(0)_0 := 1$ and assumed tacitly that the Γ-quotient exists.

The Srivastava polynomials $S_n^N(z)$ and their variants as well as special cases have been considered, in recent years, by numerous other workers on the subject (see, for details, González et al. [4, p. 145 et seq.], and Lin et al. [6], [9, p. 448 et seq.], [8] and [7], Liu et al. [10] and Srivastava et al. [14] and [16]; see also [5]). Here, in our present investigation, we consider the polynomial family defined by

$$\begin{aligned}(2)\quad & \Xi_{n;(M_r)}^{N;(L_s)}(z;(\alpha_r),(\beta_r);(\gamma_s),(\omega_s)) \\ &= \Xi_{n;M_1,\dots,M_r}^{N;L_1,\dots,L_s}(z;\alpha_1,\dots,\alpha_r,\beta_1,\dots,\beta_r;\gamma_1,\dots,\gamma_s,\omega_1,\dots,\omega_s) \\ &:= \sum_{k=0}^{[n/N]} \frac{(-n)_{Nk}(\gamma_1+\omega_1 n)_{L_1 k}\dots(\gamma_s+\omega_s n)_{L_s k}}{(\alpha_1+\beta_1 n)_{M_1 k}\dots(\alpha_r+\beta_r n)_{M_r k}} \frac{z^k}{k!} \\ &= \sum_{k=0}^{[n/N]} \frac{(-n)_{Nk}((\gamma_s+\omega_s n)_{L_s k})}{((\alpha_r+\beta_r n)_{M_r k})} \frac{z^k}{k!} \quad \big(N,(L_s),(M_r) \in \mathbb{N};(\beta_r),(\omega_s) \in \mathbb{C}\big),\end{aligned}$$

where (and throughout this paper) (L_s) abbreviates the array of s parameters

$$L_1,\dots,L_s$$

with similar interpretations for

$$(M_r), \quad ((\alpha_r+\beta_r n)_{M_r k}) \quad \text{and} \quad ((\gamma_s+\omega_s n)_{L_s k}).$$

Recently, Lin et al. [8] defined the following class of polynomials:

$$\mathcal{R}_{n,M_1,M_2}^{N,L_1,L_2}(z;\lambda_1,\lambda_2,\alpha_1,\alpha_2) := \sum_{k=0}^{[n/N]} \frac{(-n)_{Nk}(\lambda_1+n)_{L_1 k}(\lambda_2)_{L_2 k}}{(\alpha_1+1)_{M_1 k}(\alpha_2+1)_{M_2 k}} \frac{z^k}{k!}, \tag{3}$$
$$(L_1,L_2,M_1,M_2,n \in \mathbb{N}_0; N \in \mathbb{N}).$$

It is easily seen from the definitions (2) and (3) that

$$\Xi_{n;M_1,M_2}^{N;L_1,L_2}(z;\alpha_1+1,\alpha_2+1,0,0;\lambda_1,\lambda_2,1,0)=\mathcal{R}_{n,M_1,M_2}^{N,L_1,L_2}(z;\lambda_1,\lambda_2,\alpha_1,\alpha_2) \tag{4}$$
$$(n\in\mathbb{N}_0;\ L_1,L_2,M_1,M_2,N\in\mathbb{N};\ \alpha_1,\alpha_2,\lambda_1,\lambda_2\in\mathbb{C}),$$

$$\Xi_{n;M}^{N;L}(z;\alpha+1,0;\lambda,1):=\sum_{k=0}^{[n/N]}\frac{(-n)_{Nk}(\lambda+n)_{Lk}}{(\alpha+1)_{Mk}}\frac{z^k}{k!}=:\mathcal{S}_{n,N}^{L,M}(z;\lambda,\alpha) \tag{5}$$
$$(n\in\mathbb{N}_0;\ L,M,N\in\mathbb{N};\ \alpha,\lambda\in\mathbb{C}),$$

$$\Xi_{n;0}^{N;L}(z;\lambda,1):=\sum_{k=0}^{[n/N]}(-n)_{Nk}(\lambda+n)_{Lk}\frac{z^k}{k!}=:\mathcal{P}_{n,N}^{L}(z;\lambda) \tag{6}$$
$$(n\in\mathbb{N}_0;\ L,N\in\mathbb{N};\ \lambda\in\mathbb{C}),$$

$$\Xi_{n;M}^{N;0}(z;\alpha+1,0):=\sum_{k=0}^{[n/N]}\frac{(-n)_{Nk}}{(\alpha+1)_{Mk}}\frac{z^k}{k!}=:\mathcal{Q}_{n,N}^{M}(z;\alpha) \tag{7}$$
$$(n\in\mathbb{N}_0;\ M,N\in\mathbb{N};\ \alpha\in\mathbb{C})$$

and

$$\Xi_n^N(z):=\sum_{k=0}^{[n/N]}(-n)_{Nk}\frac{z^k}{k!}=:\mathcal{R}_{n,N}(z)\quad(n\in\mathbb{N}_0;\ N\in\mathbb{N}), \tag{8}$$

which exhibit the fact that each of the polynomials

$$\mathcal{R}_{n,M_1,M_2}^{N,L_1,L_2}(z;\lambda_1,\lambda_2,\alpha_1,\alpha_2),\ \mathcal{S}_{n,N}^{L,M}(z;\lambda,\alpha),\ \mathcal{P}_{n,N}^{L}(z;\lambda),\ \mathcal{Q}_{n,N}^{M}(z;\alpha)\text{ and }\mathcal{R}_{n,N}(z)$$

is a special case of the following polynomials:

$$\Xi_{n;(M_r)}^{N;(L_s)}\big(z;(\alpha_r),(\beta_r);(\gamma_s),(\omega_s)\big),$$

which, in turn, are a special case of the Srivastava polynomial system $S_n^N(z)$ defined by (1) when

$$A_{n,k}=\frac{(\gamma_1+\omega_1 n)_{L_1k}\dots(\gamma_s+\omega_s n)_{L_sk}}{(\alpha_1+\beta_1 n)_{M_1k}\dots(\alpha_r+\beta_r n)_{M_rk}}\quad\big((L_s),(M_r)\in\mathbb{N};\ n,k\in\mathbb{N}_0\big). \tag{9}$$

Our principal objective in this investigation is first to derive some general multiple integral representations associated with the polynomials defined by (2). We also consider several special cases and consequences of our main results.

2. Multiple integral representations

For the polynomials

$$\Xi_{n;(M_r)}^{N;(L_s)}\big(z;(\alpha_r),(\beta_r);(\gamma_s),(\omega_s)\big)$$

defined by (2), we begin by considering the following product:

$$\begin{aligned}
(10)\quad & \Xi_{m;(M_r)}^{N;(L_s)}\big(x;(\alpha_r),(\beta_r);(\gamma_s),(\omega_s)\big)\Xi_{n;(M_r)}^{N;(L_s)}\big(y;(\zeta_r),(\xi_r);(\delta_s),(\sigma_s)\big)\\
&= \sum_{k=0}^{[m/N]}\sum_{l=0}^{[n/N]} \frac{(-m)_{Nk}(-n)_{Nl}(\gamma_1+\omega_1 m)_{L_1k}(\delta_1+\sigma_1 n)_{L_1l}\dots(\gamma_s+\omega_s m)_{L_sk}}{(\alpha_1+\beta_1 m)_{M_1k}(\zeta_1+\xi_1 n)_{M_1l}\dots(\alpha_r+\beta_r m)_{M_rk}}\\
&\quad\times\frac{(\delta_s+\sigma_s n)_{L_sl}}{(\zeta_r+\xi_r n)_{M_rl}}\,\frac{x^k}{k!}\,\frac{y^l}{l!}\\
&= \sum_{k=0}^{[m/N]}\sum_{l=0}^{[n/N]} \frac{(-1)^{(k+l)N}m!n!}{\Gamma(m-Nk+1)\Gamma(n-Nl+1)}\\
&\quad\times\prod_{i=1}^{r}\Big\{\frac{\Gamma(\alpha_i+\beta_i m)}{\Gamma(\alpha_i+\beta_i m+M_ik)}\,\frac{\Gamma(\zeta_i+\xi_i n)}{\Gamma(\zeta_i+\xi_i n+M_il)}\Big\}\\
&\quad\times\prod_{j=1}^{s}\Big\{\frac{\Gamma(\gamma_j+\omega_j m+L_jk)}{\Gamma(\gamma_j+\omega_j m)}\,\frac{\Gamma(\delta_j+\sigma_j n+L_jl)}{\Gamma(\delta_j+\sigma_j n)}\Big\}\frac{x^k}{k!}\,\frac{y^l}{l!}\\
&= \frac{m!n!}{(m+n)!}\prod_{i,j=1}^{r,s}\Big\{\frac{\Gamma(\alpha_i+\beta_i m)\Gamma(\zeta_i+\xi_i n)}{\Gamma(\gamma_j+\omega_j m)\Gamma(\delta_j+\sigma_j n)}\Big\}\sum_{k,l=0}^{k+l\leqslant[(m+n)/N]}(-m-n)_{(k+l)N}\\
&\quad\times\frac{\Gamma(m+n-(k+l)N+1)}{\Gamma(m-Nk+1)\Gamma(n-Nl+1)}\\
&\quad\times\prod_{i,j=1}^{r,s}\Big\{\frac{\Gamma(\gamma_j+\omega_j m+L_jk)\Gamma(\delta_j+\sigma_j n+L_jl)}{\Gamma(\alpha_i+\beta_i m+M_ik)\Gamma(\zeta_i+\xi_i n+M_il)}\Big\}\frac{x^k}{k!}\,\frac{y^l}{l!}\\
&= \frac{m!n!}{(m+n)!}\prod_{i,j=1}^{r,s}\Big\{\frac{\Gamma(\alpha_i+\beta_i m)\Gamma(\zeta_i+\xi_i n)}{\Gamma(\alpha_i+\zeta_i+\beta_i m+\xi_i n-1)}\,\frac{\Gamma(\gamma_j+\delta_j+\omega_j m+\sigma_j n)}{\Gamma(\gamma_j+\omega_j m)\Gamma(\delta_j+\sigma_j n)}\Big\}\\
&\quad\times\sum_{k,l=0}^{k+l\leqslant[(m+n)/N]}(-m-n)_{(k+l)N}\frac{\Gamma(m+n-(k+l)N+1)}{\Gamma(m-Nk+1)\Gamma(n-Nl+1)}\\
&\quad\times\prod_{i,j=1}^{r,s}\Big\{\frac{\Gamma(\alpha_i+\zeta_i+\beta_i m+\xi_i n+(k+l)M_i-1)}{\Gamma(\alpha_i+\beta_i m+M_ik)\Gamma(\zeta_i+\xi_i n+M_il)}\\
&\quad\times\frac{\Gamma(\gamma_j+\omega_j m+L_jk)\Gamma(\delta_j+\sigma_j n+L_jl)}{\Gamma(\gamma_j+\delta_j+\omega_j m+\sigma_j n+(k+l)L_j)}\\
&\quad\times\frac{(\gamma_j+\delta_j+\omega_j m+\sigma_j n)_{(k+l)L_j}}{(\alpha_i+\zeta_i+\beta_i m+\xi_i n-1)_{(k+l)M_i}}\Big\}\frac{x^k}{k!}\,\frac{y^l}{l!},
\end{aligned}$$

where we have made repeated use of the following elementary identity:

$$(-n)_{Nk} = (-1)^{Nk} \frac{n!}{(n-Nk)!} \qquad \Big(0 \leqslant k \leqslant \Big[\frac{n}{N}\Big];\ N \in \mathbb{N};\ n \in \mathbb{N}_0\Big) \tag{11}$$

and (for convenience) we write

$$\prod_{i,j=1}^{r,s} \{a_i b_j\} := \prod_{i=1}^{r} \{a_i\} \prod_{j=1}^{s} \{b_j\}.$$

We now recall the following integral formula (cf., e.g., [19, Example 39, p. 263]; see also [11, p. 9]):

$$\int_0^{\pi/2} \cos^{\alpha+\beta}\theta \cos(\alpha-\beta)\theta \,\mathrm{d}\theta = \frac{\pi}{2^{\alpha+\beta+1}} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \tag{12}$$
$$\big(\min\{\Re(\alpha), \Re(\beta), \Re(\alpha+\beta)\} > -1\big),$$

which can readily be rewritten in the following (more convenient) form:

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} = \frac{2^{\alpha+\beta}}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{e}^{\mathrm{i}(\alpha-\beta)\theta} \cos^{\alpha+\beta}\theta \,\mathrm{d}\theta \tag{13}$$
$$\big(\min\{\Re(\alpha), \Re(\beta), \Re(\alpha+\beta)\} > -1;\ \mathrm{i} := \sqrt{-1}\big).$$

By appealing appropriately to (13) as well as to the familiar Eulerian beta integral in the form:

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\,\mathrm{d}t = (b-a)^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{14}$$
$$\big(\min\{\Re(\alpha), \Re(\beta)\} > 0;\ a \neq b\big),$$

we find from (10) that

$$\begin{aligned} (15)\ \ &\Xi_{m;(M_r)}^{N;(L_s)}\big(x; (\alpha_r), (\beta_r); (\gamma_s), (\omega_s)\big) \Xi_{n;(M_r)}^{N;(L_s)}\big(y; (\zeta_r), (\xi_r); (\delta_s), (\sigma_s)\big) \\ &= \frac{2^{m+n-2r+\sum_{i=1}^{r}(\alpha_i+\zeta_i+\beta_i m+\xi_i n)}}{\pi^{r+1}(b-a)^{\sum_{j=1}^{s}(\gamma_j+\delta_j+\omega_j m+\sigma_j n)-s}} \frac{m!n!}{(m+n)!} \\ &\quad \times \prod_{i,j=1}^{r,s} \Big\{ \frac{\Gamma(\alpha_i+\beta_i m)\Gamma(\zeta_i+\xi_i n)}{\Gamma(\alpha_i+\zeta_i+\beta_i m+\xi_i n-1)} \frac{\Gamma(\gamma_j+\delta_j+\omega_j m+\sigma_j n)}{\Gamma(\gamma_j+\omega_j m)\Gamma(\delta_j+\sigma_j n)} \Big\} \\ &\quad \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_a^b \cdots \int_a^b \prod_{j=1}^{s} \{(t_j-a)^{\gamma_j+\omega_j m-1}(b-t_j)^{\delta_j+\sigma_j n-1}\} \end{aligned}$$

$$\times\, \Theta_{m,n}^{((\alpha_r),(\beta_r),(\zeta_r),(\xi_r))}\big(\theta,(\varphi_r)\big) \sum_{k,l=0}^{k+l\leqslant[(m+n)/N]} (-m-n)_{(k+l)N}$$

$$\times \prod_{i,j=1}^{r,s} \left\{ \frac{\Gamma(\gamma_j+\delta_j+\omega_j m+\sigma_j n)_{(k+l)L_j}}{\Gamma(\alpha_i+\zeta_i+\beta_i m+\xi_i n-1)_{(k+l)M_i}} \right\}$$

$$\times\, \frac{\{f(x;(t_s),\theta,(\varphi_r))\}^k}{k!} \frac{\{g(y;(t_s),\theta,(\varphi_r))\}^l}{l!}\, \mathrm{d}t_1 \ldots \mathrm{d}t_s\, \mathrm{d}\varphi_1 \ldots \mathrm{d}\varphi_r\, \mathrm{d}\theta$$

$$\Big(\min_{1\leqslant i\leqslant r} \{\Re(\alpha_i+\zeta_i+\beta_i m+\xi_i n-1), \Re(\alpha_i+\beta_i m), \Re(\zeta_i+\xi_i n)\} > 0;$$
$$\min_{1\leqslant j\leqslant s} \{\Re(\gamma_j+\omega_j m), \Re(\delta_j+\sigma_j n)\} > 0;\ a\neq b;\ N,(M_r),(L_s)\in\mathbb{N};$$
$$m,n,r,s\in\mathbb{N}_0\Big),$$

where, for convenience, we have denoted

(16) $$\Theta_{m,n}^{((\alpha_r),(\beta_r),(\zeta_r),(\xi_r))}\big(\theta,(\varphi_r)\big)$$
$$:= \mathrm{e}^{\mathrm{i}[(m-n)\theta+\sum_{i=1}^{r}(\alpha_i-\zeta_i+\beta_i m-\xi_i n)\varphi_i]} \cos^{m+n}\theta \bigg(\prod_{i=1}^{r} \{\cos^{\alpha_i+\zeta_i+\beta_i m+\xi_i n-2}\varphi_i\} \bigg),$$

$$f\big(x;(t_s),\theta,(\varphi_r)\big)$$
$$:= (2\cos\theta)^{-N} \prod_{i=1}^{r} \{(2\cos\varphi_i)^{M_i}\} \prod_{j=1}^{s} \Big\{ \Big(\frac{t_j-a}{b-a}\Big)^{L_j} \Big\} x \mathrm{e}^{-\mathrm{i}(N\theta-\sum_{i=1}^{r} M_i\varphi_i)}$$

and

$$g\big(y;(t_s),\theta,(\varphi_r)\big)$$
$$:= (2\cos\theta)^{-N} \prod_{i=1}^{r} \{(2\cos\varphi_i)^{M_i}\} \prod_{j=1}^{s} \Big\{ \Big(\frac{b-t_j}{b-a}\Big)^{L_j} \Big\} y \mathrm{e}^{\mathrm{i}(N\theta-\sum_{i=1}^{r} M_i\varphi_i)},$$

it being understood (as indicated above and in what follows) that

$$-\frac{\pi}{2} \leqslant \theta,\ \varphi_i \leqslant \frac{\pi}{2} \quad (i=1,\ldots,r) \quad \text{and} \quad a\leqslant t_j\leqslant b \quad (j=1,\ldots,s).$$

Finally, since (see, for example, [15, p. 52, Equation 1.6(2)])

(17) $$\sum_{m,n=0}^{\infty} f(m+n)\frac{x^m}{m!}\frac{y^n}{n!} = \sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!},$$

provided that each of the series involved is absolutely convergent, the double sum in (15) can be reduced to a single sum which, in turn, is capable of being interpreted by the definition (2). We are thus led to the following multiple integral representation for the product of two polynomials of the class defined by (2):

$$
\begin{aligned}
(18)\qquad & \Xi^{N;(L_s)}_{m;(M_r)}\big(x;(\alpha_r),(\beta_r);(\gamma_s),(\omega_s)\big)\Xi^{N;(L_s)}_{n;(M_r)}\big(y;(\zeta_r),(\xi_r);(\delta_s),(\sigma_s)\big)\\
&= \frac{2^{m+n-2r+\sum_{i=1}^{r}(\alpha_i+\zeta_i+\beta_i m+\xi_i n)}}{\pi^{r+1}(b-a)^{\sum_{j=1}^{s}(\gamma_j+\delta_j+\omega_j m+\sigma_j n)-s}}\,\frac{m!\,n!}{(m+n)!}\\
&\quad\times \prod_{i,j=1}^{r,s}\Big\{\frac{\Gamma(\alpha_i+\beta_i m)\Gamma(\zeta_i+\xi_i n)}{\Gamma(\alpha_i+\zeta_i+\beta_i m+\xi_i n-1)}\,\frac{\Gamma(\gamma_j+\delta_j+\omega_j m+\sigma_j n)}{\Gamma(\gamma_j+\omega_j m)\Gamma(\delta_j+\sigma_j n)}\Big\}\\
&\quad\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_a^b\cdots\int_a^b \prod_{j=1}^{s}\{(t_j-a)^{\gamma_j+\omega_j m-1}(b-t_j)^{\delta_j+\sigma_j n-1}\}\\
&\quad\times \Theta^{((\alpha_r),(\beta_r),(\zeta_r),(\xi_r))}_{m,n}\big(\theta,(\varphi_r)\big)\cdot \Xi^{N;(L_s)}_{m+n;(M_r)}\Bigg(\Phi_{(M_r),N}\Bigg[x\prod_{j=1}^{s}\Big\{\Big(\frac{t_j-a}{b-a}\Big)^{L_j}\Big\},\\
&\quad y\prod_{j=1}^{s}\Big\{\Big(\frac{b-t_j}{b-a}\Big)^{L_j}\Big\};\ \theta,(\varphi_r)\Bigg];\ (\alpha_r+\zeta_r-1),\Big(\frac{\beta_r m+\xi_r n}{m+n}\Big);\\
&\quad (\gamma_s+\delta_s),\Big(\frac{\omega_s m+\sigma_s n}{m+n}\Big)\Bigg)\,\mathrm{d}t_1\ldots\mathrm{d}t_s\,\mathrm{d}\varphi_1\ldots\mathrm{d}\varphi_r\,\mathrm{d}\theta\\
&\quad \Big(\min_{1\leqslant i\leqslant r}\{\Re(\alpha_i+\zeta_i+\beta_i m+\xi_i n-1),\Re(\alpha_i+\beta_i m),\Re(\zeta_i+\xi_i n)\}>0;\\
&\quad \min_{1\leqslant j\leqslant s}\{\Re(\gamma_j+\omega_j m),\Re(\delta_j+\sigma_j n)\}>0;\ a\neq b;\ N,(M_r),(L_s)\in\mathbb{N};\\
&\quad m,n,r,s\in\mathbb{N}_0\Big),
\end{aligned}
$$

where $(\alpha_r+\zeta_r-1)$ abbreviates the array of r parameters

$$\alpha_1+\zeta_1-1,\ \ldots,\ \alpha_r+\zeta_r-1$$

with similar interpretations for

$$(\gamma_s+\delta_s),\quad \Big(\frac{\beta_r m+\xi_r n}{m+n}\Big)\quad\text{and}\quad\Big(\frac{\omega_s m+\sigma_s n}{m+n}\Big),$$

$\Theta^{((\alpha_r),(\beta_r),(\zeta_r),(\xi_r))}_{m,n}\big(\theta,(\varphi_r)\big)$ is given by (16) and

$$
\begin{aligned}
(19)\quad & \Phi_{(M_r),N}[x,y;\theta,(\varphi_r)]\\
&:= (2\cos\theta)^{-N}\prod_{i=1}^{r}\{(2\cos\varphi_i)^{M_i}\}\Big[x\mathrm{e}^{-\mathrm{i}(N\theta-\sum_{i=1}^{r}M_i\varphi_i)}+y\mathrm{e}^{\mathrm{i}(N\theta-\sum_{i=1}^{r}M_i\varphi_i)}\Big].
\end{aligned}
$$

Remark 1. By letting

$$\beta_j=\xi_j\ (j=1,\ldots,r)\quad\text{and}\quad\omega_j=\sigma_j\ (j=1,\ldots,s),$$

the parameters of the polynomials

$$\Xi_{m+n;(M_r)}^{N;(L_s)}(\ldots)$$

occurring on the right-hand side of the integral representation (18) would obviously become independent of m and n. Also, in the special case when $m = n = 0$, we find from the definition (2) that

$$\Xi_{m+n;(M_r)}^{N;(L_s)}(\ldots) := 0 \quad (m = n = 0)$$

on the right-hand side of the integral representation (18).

3. Applications to hypergeometric polynomials

Since

$$(20) \qquad (\lambda)_{Nk} = N^{Nk} \prod_{j=1}^{N} \Big(\frac{\lambda + j - 1}{N}\Big)_k \quad (N \in \mathbb{N};\ k \in \mathbb{N}_0;\ \lambda \in \mathbb{C}),$$

the polynomial $\Xi_{n;(M_r)}^{N;(L_s)}\big(z; (\alpha_r), (\beta_r); (\gamma_s), (\omega_s)\big)$ can easily be rewritten as a generalized hypergeometric polynomial as follows:

$$(21) \qquad \Xi_{n;(M_r)}^{N;(L_s)}\big(z; (\alpha_r), (\beta_r); (\gamma_s), (\omega_s)\big) = {}_{N+L_1+\ldots+L_s}F_{M_1+\ldots+M_r} \left[\begin{matrix} \Delta(N; -n), \Delta(L_1; \gamma_1 + \omega_1 n), \ldots, \Delta(L_s; \gamma_s + \omega_s n); \\ \Delta(M_1; \alpha_1 + \beta_1 n), \ldots, \Delta(M_r; \alpha_r + \beta_r n); \end{matrix} \Big(\frac{N^N {L_1}^{L_1} \ldots {L_s}^{L_s}}{{M_1}^{M_1} \ldots {M_r}^{M_r}}\Big) z\right],$$

where $\Delta(N; \lambda)$ abbreviates the array of N parameters

$$\frac{\lambda}{N}, \frac{\lambda+1}{N}, \ldots, \frac{\lambda + N - 1}{N} \quad (N \in \mathbb{N})$$

and ${}_pF_q$ denotes a generalized hypergeometric function with p numerator parameters $\alpha_1, \ldots, \alpha_p$ and q denominator parameters $\beta_1, \ldots, \beta_q$, defined by [3, Chapter 4]

$${}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = {}_pF_q\left[\begin{matrix} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{matrix} z\right] := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}$$

$$\big(p, q \in \mathbb{N}_0;\ p \leqslant q + 1;\ p \leqslant q \quad \text{and} \quad |z| < \infty;\ p = q + 1 \quad \text{and} \quad |z| < 1; \\ p = q + 1,\ |z| = 1, \quad \text{and} \quad \Re(\Lambda) > 0\big),$$

where

$$\Lambda := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j$$

$$\big(\alpha_j \in \mathbb{C}\ (j = 1, \ldots, p);\ \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-\ (j = 1, \ldots, q);\ \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}\big).$$

In terms of these families of generalized hypergeometric functions and polynomials, the integral representations (18) can be rewritten as follows:

(22)

$$\begin{aligned}
&{}_{N+L_1+\ldots+L_s}F_{M_1+\ldots+M_r}\left[\begin{matrix}\Delta(N;-m), \Delta(L_1;\gamma_1+\omega_1 m), \ldots, \Delta(L_s;\gamma_s+\omega_s m);\\ \Delta(M_1;\alpha_1+\beta_1 m), \ldots, \Delta(M_r;\alpha_r+\beta_r m);\end{matrix}\ x\right]\\
&\times {}_{N+L_1+\ldots+L_s}F_{M_1+\ldots+M_r}\left[\begin{matrix}\Delta(N;-n), \Delta(L_1;\delta_1+\sigma_1 n), \ldots, \Delta(L_s;\delta_s+\sigma_s n);\\ \Delta(M_1;\zeta_1+\xi_1 n), \ldots, \Delta(M_r;\zeta_r+\xi_r n);\end{matrix}\ y\right]\\
&= \frac{2^{m+n-2r+\sum_{i=1}^{r}(\alpha_i+\zeta_i+\beta_i m+\xi_i n)}}{\pi^{r+1}(b-a)^{\sum_{j=1}^{s}(\gamma_j+\delta_j+\omega_j m+\sigma_j n)-s}} \frac{m!\, n!}{(m+n)!}\\
&\times \prod_{i,j=1}^{r,s} \left\{\frac{\Gamma(\alpha_i+\beta_i m)\Gamma(\zeta_i+\xi_i n)}{\Gamma(\alpha_i+\zeta_i+\beta_i m+\xi_i n-1)} \frac{\Gamma(\gamma_j+\delta_j+\omega_j m+\sigma_j n)}{\Gamma(\gamma_j+\omega_j m)\Gamma(\delta_j+\sigma_j n)}\right\}\\
&\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_a^b \cdots \int_a^b \prod_{j=1}^{s} \{(t_j-a)^{\gamma_j+\omega_j m-1}(b-t_j)^{\delta_j+\sigma_j n-1}\}\\
&\times \Theta_{m,n}^{((\alpha_r),(\beta_r),(\zeta_r),(\xi_r))}\big(\theta,(\varphi_r)\big) \cdot {}_{N+L_1+\ldots+L_s}F_{M_1+\ldots+M_r}\\
&\left[\begin{matrix}\Delta(N;-m-n), \Delta(L_1;\gamma_1+\delta_1+\omega_1 m+\sigma_1 n), \ldots, \Delta(L_s;\gamma_s+\delta_s+\omega_s m+\sigma_s n);\\ \Delta(M_1;\alpha_1+\zeta_1+\beta_1 m+\xi_1 n-1), \ldots, \Delta(M_r;\alpha_r+\zeta_r+\beta_r m+\xi_r n-1);\end{matrix}\right.\\
&\left.\Phi_{(M_r),N}\Big[x \prod_{j=1}^{s}\Big(\frac{t_j-a}{b-a}\Big)^{L_j}, y \prod_{j=1}^{s}\Big(\frac{b-t_j}{b-a}\Big)^{L_j};\ \theta,(\varphi_r)\Big]\right] \mathrm{d}t_1 \ldots \mathrm{d}t_s\, \mathrm{d}\varphi_1 \ldots \mathrm{d}\varphi_r\, \mathrm{d}\theta\\
&\Big(\min_{1\leqslant i\leqslant r}\{\Re(\alpha_i+\zeta_i+\beta_i m+\xi_i n-1), \Re(\alpha_i+\beta_i m), \Re(\zeta_i+\xi_i n)\} > 0;\\
&\min_{1\leqslant j\leqslant s}\{\Re(\gamma_j+\omega_j m), \Re(\delta_j+\sigma_j n)\} > 0;\ a \neq b;\ N, (M_r), (L_s) \in \mathbb{N};\ m, n, r, s \in \mathbb{N}_0\Big),
\end{aligned}$$

where

$$\Theta_{m,n}^{((\alpha_r),(\beta_r),(\zeta_r),(\xi_r))}\big(\theta,(\varphi_r)\big) \quad \text{and} \quad \Phi_{(M_r),N}[x,y;\theta,(\varphi_r)]$$

are given by (16) and (19), respectively.

4. Integral representations for the generalized Bedient polynomials of the first and second kinds

In his study of some polynomials associated with Appell's double hypergeometric functions F_2 and F_3 (see, for details, [1] and [3, p. 224 et seq.]), Bedient ([2, p. 15, Equation (2.5); p. 44, Equation (3.4)]; see also [15, p. 186, Problem 48] and [7, p. 1336]) introduced the special hypergeometric polynomials

$$R_n(\beta,\gamma;x) \quad \text{and} \quad G_n(\alpha,\beta;x)$$

defined by

$$R_n(\beta,\gamma;x) = \frac{(\beta)_n}{n!}(2x)^n {}_3F_2\left[\begin{matrix}\Delta(2;-n),\gamma-\beta; \\ \gamma, 1-\beta-n;\end{matrix} \; \frac{1}{x^2}\right] \tag{23}$$

and

$$G_n(\alpha,\beta;x) = \frac{(\alpha)_n(\beta)_n}{n!(\alpha+\beta)_n}(2x)^n {}_3F_2\left[\begin{matrix}\Delta(2;-n), 1-\alpha-\beta-n; \\ 1-\alpha-n, 1-\beta-n;\end{matrix} \; \frac{1}{x^2}\right], \tag{24}$$

respectively. Here, in this section, we introduce and investigate the generalized Bedient polynomials $\mathcal{R}_n^{(N)}(\beta,\gamma,\delta;x)$ of the first kind and the generalized Bedient polynomials $\mathcal{G}_n^{(N)}(\alpha,\beta,\zeta;x)$ of the second kind, which are defined as follows:

$$\mathcal{R}_n^{(N)}(\beta,\gamma,\delta;x) = \frac{(\beta)_n}{n!}(Nx)^n {}_{N+1}F_2\left[\begin{matrix}\Delta(N;-n),\gamma-\beta; \\ \delta, 1-\beta-n;\end{matrix} \; \frac{1}{x^N}\right] \tag{25}$$

and

$$\mathcal{G}_n^{(N)}(\alpha,\beta,\zeta;x) = \frac{(\alpha)_n(\beta)_n}{n!(\zeta)_n}(Nx)^n {}_{N+1}F_2\left[\begin{matrix}\Delta(N;-n), 1-\zeta-n; \\ 1-\alpha-n, 1-\beta-n;\end{matrix} \; \frac{1}{x^N}\right], \tag{26}$$

respectively. Then, clearly, we have

$$\mathcal{G}_n^{(2)}(\alpha,\beta,\alpha+\beta;x) = G_n(\alpha,\beta;x) \tag{27}$$

and

$$\mathcal{G}_n^{(1)}(\alpha,\beta,\zeta;x) = \frac{1}{n!x^n} {}_3F_1\left[\begin{matrix}-n,\alpha,\beta; \\ \zeta;\end{matrix} \; -x^2\right]. \tag{28}$$

The special case of the polynomials $\mathcal{R}_n^{(N)}(\beta,\gamma,\delta;x)$ in (25) when $N=2$ was considered earlier by Lin et al. [7, p. 1336, Equation (33)].

By letting

$$r = 2,\ L_1 = M_1 = M_2 = s = 1,\ \gamma_1 \mapsto \gamma - \alpha,\ \delta_1 \mapsto \zeta - \beta,\ \alpha_1 \mapsto \delta,\ \zeta_1 \mapsto \xi$$

and

$$\alpha_2 \mapsto 1 - \alpha,\ \zeta_2 \mapsto 1 - \beta,\ \beta_1 = \xi_1 = \omega_1 = \sigma_1 = 0,$$
$$\beta_2 = \xi_2 = -1,\ x \mapsto \frac{1}{x^N} \quad \text{and} \quad y \mapsto \frac{1}{y^N}$$

in (22), we get

$$\begin{aligned}
(29) \quad & {}_{N+1}F_2\left[\begin{matrix}\Delta(N;-m), \gamma-\alpha; \\ \delta, 1-\alpha-m;\end{matrix}\ \frac{1}{x^N}\right] {}_{N+1}F_2\left[\begin{matrix}\Delta(N;-n), \zeta-\beta; \\ \xi, 1-\beta-n;\end{matrix}\ \frac{1}{y^N}\right] \\
&= \frac{2^{\delta+\xi-\alpha-\beta-2}}{\pi^3 (b-a)^{\gamma+\zeta-\alpha-\beta-1}} \frac{m!n!}{(m+n)!} \frac{\Gamma(\delta)\Gamma(\xi)}{\Gamma(\delta+\xi-1)} \frac{\Gamma(1-\alpha-m)\Gamma(1-\beta-n)}{\Gamma(1-\alpha-\beta-m-n)} \\
&\quad \times \frac{\Gamma(\gamma+\zeta-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\zeta-\beta)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_a^b (t-a)^{\gamma-\alpha-1} (b-t)^{\zeta-\beta-1} \\
&\quad \times \Theta_{m,n}^{(\alpha,\beta,\delta,\xi)}(\theta, \varphi_1, \varphi_2) \cdot {}_{N+1}F_2\Bigg[\begin{matrix}\Delta(N;-m-n), \gamma+\zeta-\alpha-\beta; \\ \delta+\xi-1, 1-\alpha-\beta-m-n;\end{matrix} \\
&\quad \Phi_{1,1,N}\left[\frac{1}{x^N}\left(\frac{t-a}{b-a}\right), \frac{1}{y^N}\left(\frac{b-t}{b-a}\right); \theta, \varphi_1, \varphi_2\right]\Bigg] \mathrm{d}t \, \mathrm{d}\theta \, \mathrm{d}\varphi_1 \, \mathrm{d}\varphi_2 \\
&\quad \big(\min\{\Re(\delta), \Re(\xi), \Re(\delta+\xi-1), \Re(1-\alpha-m), \Re(1-\beta-n), \\
&\quad \Re(1-\alpha-\beta-m-n), \Re(\gamma-\alpha), \Re(\zeta-\beta)\} > 0;\ a \neq b;\ m, n \in \mathbb{N}_0\big),
\end{aligned}$$

where

$$\begin{aligned}
(30) \quad \Theta_{m,n}^{(\alpha,\beta,\delta,\xi)}(\theta, \varphi_1, \varphi_2) &:= \mathrm{e}^{\mathrm{i}[(m-n)\theta + (\delta-\xi)\varphi_1 - (\alpha-\beta+m-n)\varphi_2]} \\
&\quad \times \cos^{m+n}\theta \cos^{\delta+\xi-2}\varphi_1 \sec^{\alpha+\beta+m+n}\varphi_2
\end{aligned}$$

and

$$\begin{aligned}
(31) \quad & \Phi_{1,1,N}[x, y; \theta, \varphi_1, \varphi_2] \\
&:= \frac{(2\cos\varphi_1)(2\cos\varphi_2)}{(2\cos\theta)^N} [x\mathrm{e}^{-\mathrm{i}(N\theta-\varphi_1-\varphi_2)} + y\mathrm{e}^{\mathrm{i}(N\theta-\varphi_1-\varphi_2)}].
\end{aligned}$$

Thus, by applying (25) and (29), we obtain the following integral representation:
(32)

$$
\begin{aligned}
&\mathcal{R}_m^{(N)}(\alpha,\gamma,\delta;x)\mathcal{R}_n^{(N)}(\beta,\zeta,\xi;y)\\
&= \frac{(\alpha)_m(\beta)_n}{(\alpha+\beta)_{m+n}}\,\frac{x^m y^n 2^{\delta+\xi-\alpha-\beta-2}}{\pi^3(b-a)^{\gamma+\zeta-\alpha-\beta-1}}\,\frac{\Gamma(\delta)\Gamma(\xi)}{\Gamma(\delta+\xi-1)}\,\frac{\Gamma(1-\alpha-m)\Gamma(1-\beta-n)}{\Gamma(1-\alpha-\beta-m-n)}\\
&\quad\times \frac{\Gamma(\gamma+\zeta-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\zeta-\beta)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_a^b (t-a)^{\gamma-\alpha-1}(b-t)^{\zeta-\beta-1}\\
&\quad\times \Theta_{m,n}^{(\alpha,\beta,\delta,\xi)}(\theta,\varphi_1,\varphi_2)\cdot\left(\Phi_{1,1,N}\left[\frac{1}{x^N}\left(\frac{t-a}{b-a}\right),\frac{1}{y^N}\left(\frac{b-t}{b-a}\right);\theta,\varphi_1,\varphi_2\right]\right)^{(m+n)/N}\\
&\quad\times \mathcal{R}_{m+n}^{(N)}\Big(\alpha+\beta,\gamma+\zeta,\delta+\xi-1;\\
&\qquad\qquad \left(\Phi_{1,1,N}\left[\frac{1}{x^N}\left(\frac{t-a}{b-a}\right),\frac{1}{y^N}\left(\frac{b-t}{b-a}\right);\theta,\varphi_1,\varphi_2\right]\right)^{-1/N}\Big)\,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\\
&= \frac{(\alpha)_m(\beta)_n}{(\alpha+\beta)_{m+n}}\,\frac{1}{y^m}\,\frac{1}{x^n}\,\frac{2^{\delta+\xi-\alpha-\beta-2}}{\pi^3(b-a)^{\gamma+\zeta-\alpha-\beta-1}}\,\frac{\Gamma(\delta)\Gamma(\xi)}{\Gamma(\delta+\xi-1)}\\
&\quad\times \frac{\Gamma(1-\alpha-m)\Gamma(1-\beta-n)}{\Gamma(1-\alpha-\beta-m-n)}\,\frac{\Gamma(\gamma+\zeta-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\zeta-\beta)}\\
&\quad\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_a^b (t-a)^{\gamma-\alpha-1}(b-t)^{\zeta-\beta-1}\Theta_{m,n}^{(\alpha,\beta,\delta,\xi)}(\theta,\varphi_1,\varphi_2)\\
&\quad\times \left(\Phi_{1,1,N}\left[y^N\left(\frac{t-a}{b-a}\right),x^N\left(\frac{b-t}{b-a}\right);\theta,\varphi_1,\varphi_2\right]\right)^{(m+n)/N}\\
&\quad\times \mathcal{R}_{m+n}^{(N)}\Bigg(\alpha+\beta,\gamma+\zeta,\delta+\xi-1;xy\Big(\Phi_{1,1,N}\Big[y^N\left(\frac{t-a}{b-a}\right),\\
&\quad\times x^N\left(\frac{b-t}{b-a}\right);\theta,\varphi_1,\varphi_2\Big]\Big)^{-1/N}\Bigg)\,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\\
&\quad\big(\min\{\Re(\delta),\Re(\xi),\Re(\delta+\xi-1),\Re(1-\alpha-m),\Re(1-\beta-n),\\
&\quad\Re(1-\alpha-\beta-m-n),\Re(\gamma-\alpha),\Re(\zeta-\beta)\}>0;\ a\neq b;\ m,n\in\mathbb{N}_0\big),
\end{aligned}
$$

where

$$\Theta_{m,n}^{(\alpha,\beta,\delta,\xi)}(\theta,\varphi_1,\varphi_2)\quad\text{and}\quad\Phi_{1,1,N}[x,y;\theta,\varphi_1,\varphi_2]$$

are given by (30) and (31), respectively.

Next, by setting

$$r=2,\ L_1=M_1=M_2=s=1,\ \gamma_1\mapsto 1-\zeta,\ \delta_1\mapsto 1-\xi,\ \alpha_1\mapsto 1-\alpha,\ \zeta_1\mapsto 1-\gamma$$

and

$$\alpha_2\mapsto 1-\beta,\ \zeta_2\mapsto 1-\delta,\ \beta_1=\beta_2=\xi_1=\xi_2=\omega_1=\sigma_1=-1,$$
$$x\mapsto\frac{1}{x^N}\quad\text{and}\quad y\mapsto\frac{1}{y^N}$$

in (22), we obtain

$$
{}_{N+1}F_2\left[\begin{matrix}\Delta(N;-m),1-\zeta-m; \\ 1-\alpha-m,1-\beta-m;\end{matrix}\ \frac{1}{x^N}\right]{}_{N+1}F_2\left[\begin{matrix}\Delta(N;-n),1-\xi-n; \\ 1-\gamma-n,1-\delta-n;\end{matrix}\ \frac{1}{y^N}\right] \tag{33}
$$

$$
\begin{aligned}
&= \frac{(b-a)^{\zeta+\xi+m+n-1}}{\pi^3 2^{\alpha+\beta+\gamma+\delta+m+n}}\,\frac{m!n!}{(m+n)!}\,\frac{\Gamma(1-\alpha-m)\Gamma(1-\gamma-n)}{\Gamma(1-\alpha-\gamma-m-n)} \\
&\times \frac{\Gamma(2-\zeta-\xi-m-n)}{\Gamma(1-\zeta-m)\Gamma(1-\xi-n)}\,\frac{\Gamma(1-\beta-m)\Gamma(1-\delta-n)}{\Gamma(1-\beta-\delta-m-n)} \\
&\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_a^b (t-a)^{-\zeta-m}(b-t)^{-\xi-n}\Theta_{m,n}^{(\alpha,\beta,\gamma,\delta)}(\theta,\varphi_1,\varphi_2) \\
&\times {}_{N+1}F_2\left[\begin{matrix}\Delta(N;-m-n),2-\zeta-\xi-m-n; \\ 1-\alpha-\gamma-m-n,1-\beta-\delta-m-n;\end{matrix}\right. \\
&\qquad \left.\Phi_{1,1,N}\left[\frac{1}{x^N}\left(\frac{t-a}{b-a}\right),\frac{1}{y^N}\left(\frac{b-t}{b-a}\right);\theta,\varphi_1,\varphi_2\right]\right]\,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2 \\
&\big(\min\{\Re(1-\alpha),\Re(1-\beta),\Re(1-\zeta)\}>m;\ \min\{\Re(1-\gamma),\Re(1-\delta), \\
&\Re(1-\xi)\}>n;\ \min\{\Re(1-\beta-\delta),\Re(1-\alpha-\gamma)\}>m+n;\ a\neq b; \\
&m,n\in\mathbb{N}_0\big),
\end{aligned}
$$

where

$$
\begin{aligned}
\Theta_{m,n}^{(\alpha,\beta,\gamma,\delta)}(\theta,\varphi_1,\varphi_2) &:= \mathrm{e}^{\mathrm{i}[(m-n)\theta-(\alpha-\gamma+m-n)\varphi_1-(\beta-\delta+m-n)\varphi_2]} \\
&\quad\times \cos^{m+n}\theta\,\sec^{\alpha+\gamma+m+n}\varphi_1\,\sec^{\beta+\delta+m+n}\varphi_2
\end{aligned} \tag{34}
$$

and $\Phi_{1,1,N}[x,y;\theta,\varphi_1,\varphi_2]$ is given by (31).

By means of (26) and (33), we obtain the following integral representation:

$$
\begin{aligned}
&\mathcal{G}_m^{(N)}(\alpha,\beta,\zeta;x)\mathcal{G}_n^{(N)}(\gamma,\delta,\xi;y) \\
&= \frac{(\alpha)_m(\beta)_m}{(\zeta)_m}\,\frac{(\gamma)_n(\delta)_n}{(\xi)_n}\,\frac{(Nx)^m(Ny)^n}{(m+n)!}\,\frac{(b-a)^{\zeta+\xi+m+n-1}}{\pi^3 2^{\alpha+\beta+\gamma+\delta+m+n}} \\
&\times \frac{\Gamma(1-\alpha-m)\Gamma(1-\gamma-n)}{\Gamma(1-\alpha-\gamma-m-n)}\,\frac{\Gamma(2-\zeta-\xi-m-n)}{\Gamma(1-\zeta-m)\Gamma(1-\xi-n)} \\
&\times \frac{\Gamma(1-\beta-m)\Gamma(1-\delta-n)}{\Gamma(1-\beta-\delta-m-n)} \\
&\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_a^b (t-a)^{-\zeta-m}(b-t)^{-\xi-n}\Theta_{m,n}^{(\alpha,\beta,\gamma,\delta)}(\theta,\varphi_1,\varphi_2) \\
&\times {}_{N+1}F_2\left[\begin{matrix}\Delta(N;-m-n),2-\zeta-\xi-m-n; \\ 1-\alpha-\gamma-m-n,1-\beta-\delta-m-n;\end{matrix}\right. \\
&\qquad \left.\Phi_{1,1,N}\left[\frac{1}{x^N}\left(\frac{t-a}{b-a}\right),\frac{1}{y^N}\left(\frac{b-t}{b-a}\right);\theta,\varphi_1,\varphi_2\right]\right]\,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
&= \frac{(\alpha)_m(\beta)_m}{(\zeta)_m}\,\frac{(\gamma)_n(\delta)_n}{(\xi)_n}\,\frac{(\zeta+\xi-1)_{m+n}}{(\alpha+\gamma)_{m+n}(\beta+\delta)_{m+n}}\,\frac{1}{y^m}\,\frac{1}{x^n}\,\frac{(b-a)^{\zeta+\xi+m+n-1}}{\pi^3 2^{\alpha+\beta+\gamma+\delta+m+n}}\\
&\quad\times \frac{\Gamma(1-\alpha-m)\Gamma(1-\gamma-n)}{\Gamma(1-\alpha-\gamma-m-n)}\,\frac{\Gamma(1-\beta-m)\Gamma(1-\delta-n)}{\Gamma(1-\beta-\delta-m-n)}\\
&\quad\times \frac{\Gamma(2-\zeta-\xi-m-n)}{\Gamma(1-\zeta-m)\Gamma(1-\xi-n)}\\
&\quad\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_a^b (t-a)^{-\zeta-m}(b-t)^{-\xi-n}\Theta_{m,n}^{(\alpha,\beta,\delta,\xi)}(\theta,\varphi_1,\varphi_2)\\
&\quad\times \Big(\Phi_{1,1,N}\Big[y^N\Big(\frac{t-a}{b-a}\Big),x^N\Big(\frac{b-t}{b-a}\Big);\theta,\varphi_1,\varphi_2\Big]\Big)^{(m+n)/N}\\
&\quad\times \mathcal{G}_{m+n}^{(N)}\Big(\alpha+\gamma,\ \beta+\delta,\ \zeta+\xi-1;\ xy\Big(\Phi_{1,1,N}\Big[y^N\Big(\frac{t-a}{b-a}\Big),\\
&\quad\times x^N\Big(\frac{b-t}{b-a}\Big);\theta,\varphi_1,\varphi_2\Big]\Big)^{-1/N}\Big)\,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\\
&\big(\min\{\Re(1-\alpha),\Re(1-\beta),\Re(1-\zeta)\}>m;\ \min\{\Re(1-\gamma),\Re(1-\delta),\\
&\Re(1-\xi)\}>n;\ \min\{\Re(1-\beta-\delta),\Re(1-\alpha-\gamma)\}>m+n;\\
&a\neq b;\ m,n\in\mathbb{N}_0\big),
\end{aligned}
$$

where

$$\Theta_{m,n}^{(\alpha,\beta,\delta,\xi)}(\theta,\varphi_1,\varphi_2)$$

and

$$\Phi_{1,1,N}[x,y;\theta,\varphi_1,\varphi_2]$$

are given by (34) and (31), respectively.

The integral representation for the Bedient polynomials $R_n(\beta,\gamma;x)$ of the first kind defined by (23) was given earlier by Lin et al. [7, p. 1338, Equation (42)]. Here we deduce the corresponding (presumably new) integral representation for the Bedient polynomials $G_n(\alpha,\beta;x)$ of the second kind defined by (24). Indeed, by setting

$$N=r=2,\quad \zeta\mapsto\alpha+\beta\quad\text{and}\quad \xi\mapsto\gamma+\delta$$

in the first and second members of (35), we obtain

$$
\begin{aligned}
(36)\quad & G_m(\alpha,\beta;x)G_n(\gamma,\delta;y)\\
&= \frac{(\alpha)_m(\beta)_m}{(\alpha+\beta)_m}\,\frac{(\gamma)_n(\delta)_n}{(\gamma+\delta)_n}\,\frac{x^m y^n}{(m+n)!}\,\frac{(b-a)^{\alpha+\beta+\gamma+\delta+m+n-1}}{\pi^3 2^{\alpha+\beta+\gamma+\delta}}\\
&\quad\times \frac{\Gamma(1-\alpha-m)\Gamma(1-\gamma-n)}{\Gamma(1-\alpha-\gamma-m-n)}\,\frac{\Gamma(2-\alpha-\beta-\gamma-\delta-m-n)}{\Gamma(1-\alpha-\beta-m)\Gamma(1-\gamma-\delta-n)}\\
&\quad\times \frac{\Gamma(1-\beta-m)\Gamma(1-\delta-n)}{\Gamma(1-\beta-\delta-m-n)}
\end{aligned}
$$

$$\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{a}^{b} (t-a)^{-\alpha-\beta-m}(b-t)^{-\gamma-\delta-n}\Theta_{m,n}^{(\alpha,\beta,\gamma,\delta)}(\theta,\varphi_1,\varphi_2)$$
$$\times \Big(\Phi_{1,1,2}\Big[y^2\Big(\frac{t-a}{b-a}\Big), x^2\Big(\frac{b-t}{b-a}\Big); \theta, \varphi_1, \varphi_2\Big]\Big)^{(m+n)/2}$$
$$\times \mathcal{G}_{m+n}^{(2)}\Big(\alpha+\gamma,\ \beta+\delta,\ \alpha+\beta+\gamma+\delta-1;$$
$$xy\Big(\Phi_{1,1,2}\Big[y^2\Big(\frac{t-a}{b-a}\Big), x^2\Big(\frac{b-t}{b-a}\Big); \theta, \varphi_1, \varphi_2\Big]\Big)^{-1/2}\Big)\,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2$$
$$\big(\min\{\Re(1-\alpha), \Re(1-\beta), \Re(1-\zeta)\} > m;\ \min\{\Re(1-\gamma), \Re(1-\delta), \Re(1-\xi)\} > n;\ \min\{\Re(1-\beta-\delta), \Re(1-\alpha-\gamma)\} > m+n;\ a \neq b;\ m, n \in \mathbb{N}_0\big),$$

where $\Theta_{m,n}^{(\alpha,\beta,\gamma,\delta)}(\theta,\varphi_1,\varphi_2)$ is given by (34) and

$$\Phi_{1,1,2}[x, y; \theta, \varphi_1, \varphi_2] := \frac{\cos\varphi_1 \cos\varphi_2}{\cos^2\theta}[x\mathrm{e}^{-\mathrm{i}(2\theta-\varphi_1-\varphi_2)} + y\mathrm{e}^{\mathrm{i}(2\theta-\varphi_1-\varphi_2)}].$$

Remark 2. The expression $\mathcal{G}_{m+n}^{(2)}(\ldots)$ on the right-hand side of (36) is not the original Bedient polynomial of the second kind with free parameters. We would, therefore, like to transform this expression $\mathcal{G}_{m+n}^{(2)}(\ldots)$ into the corresponding original Bedient polynomial of the second kind as follows:

(37)

$$G_m(\alpha, \beta; x)G_n(\gamma, \delta; y)$$
$$= \frac{(\alpha)_m(\beta)_m}{(\alpha+\beta)_m}\frac{(\gamma)_n(\delta)_n}{(\gamma+\delta)_n}\frac{x^m y^n}{(m+n)!}\frac{(b-a)^{\alpha+\beta+\gamma+\delta+m+n-1}}{\pi^3 2^{\alpha+\beta+\gamma+\delta}}$$
$$\times \frac{\Gamma(1-\alpha-m)\Gamma(1-\gamma-n)}{\Gamma(1-\alpha-\gamma-m-n)}\frac{\Gamma(1-\beta-m)\Gamma(1-\delta-n)}{\Gamma(1-\beta-\delta-m-n)}$$
$$\times \frac{\Gamma(1-\alpha-\beta-\gamma-\delta-m-n)}{\Gamma(1-\alpha-\beta-m)\Gamma(1-\gamma-\delta-n)}$$
$$\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{a}^{b} (t-a)^{-\alpha-\beta-m}(b-t)^{-\gamma-\delta-n}\Theta_{m,n}^{(\alpha,\beta,\gamma,\delta)}(\theta,\varphi_1,\varphi_2)$$
$$\times \sum_{k=0}^{[(m+n)/2]} \frac{(-\frac{1}{2}(m+n))_k(-\frac{1}{2}(m+n)+\frac{1}{2})_k(1-\alpha-\beta-\gamma-\delta-m-n)_k}{(1-\alpha-\gamma-m-n)_k(1-\beta-\delta-m-n)_k}$$
$$\times \Big[\Big(\Phi_{1,1,2}\Big[\frac{1}{x^2}\Big(\frac{t-a}{b-a}\Big), \frac{1}{y^2}\Big(\frac{b-t}{b-a}\Big); \theta, \varphi_1, \varphi_2\Big]\Big)^k \Big/ k!\Big]$$
$$\times \frac{\Gamma(2-\alpha-\beta-\gamma-\delta-m-n+k)}{\Gamma(1-\alpha-\beta-\gamma-\delta-m-n+k)\Gamma(2)}\,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2$$

$$
\begin{aligned}
&= \frac{(\alpha)_m(\beta)_m}{(\alpha+\beta)_m}\,\frac{(\gamma)_n(\delta)_n}{(\gamma+\delta)_n}\,\frac{x^m y^n}{(m+n)!}\,\frac{(b-a)^{\alpha+\beta+\gamma+\delta+m+n-1}}{\pi^4 2^{\alpha+\beta+\gamma+\delta}} \\
&\quad\times \frac{\Gamma(1-\alpha-m)\Gamma(1-\gamma-n)}{\Gamma(1-\alpha-\gamma-m-n)}\,\frac{\Gamma(1-\beta-m)\Gamma(1-\delta-n)}{\Gamma(1-\beta-\delta-m-n)} \\
&\quad\times \frac{\Gamma(1-\alpha-\beta-\gamma-\delta-m-n)}{\Gamma(1-\alpha-\beta-m)\Gamma(1-\gamma-\delta-n)} \\
&\quad\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_a^b (t-a)^{-\alpha-\beta-m}(b-t)^{-\gamma-\delta-n}\Theta_{m,n}^{(\alpha,\beta,\gamma,\delta)}(\theta,\varphi_1,\varphi_2,\varphi_3) \\
&\quad\times {}_3F_2\Bigg[\begin{array}{l}\Delta(2;-m-n), 1-\alpha-\beta-\gamma-\delta-m-n; \\ \qquad 1-\alpha-\gamma-m-n, 1-\beta-\delta-m-n;\end{array} \\
&\qquad\qquad \Phi_{1,1,2}\Big[\frac{1}{x^2}\Big(\frac{t-a}{b-a}\Big),\frac{1}{y^2}\Big(\frac{b-t}{b-a}\Big);\theta,\varphi_1,\varphi_2,\varphi_3\Big]\Bigg]\,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\,\mathrm{d}\varphi_3 \\
&= \frac{(\alpha)_m(\beta)_m}{(\alpha+\beta)_m}\,\frac{(\gamma)_n(\delta)_n}{(\gamma+\delta)_n}\,\frac{(\alpha+\beta+\gamma+\delta)_{m+n}}{(\alpha+\gamma)_{m+n}(\beta+\delta)_{m+n}}\,\frac{2(b-a)^{\alpha+\beta+\gamma+\delta+m+n-1}}{x^n y^m \pi^4 4^{\alpha+\beta+\gamma+\delta+m+n}} \\
&\quad\times \frac{\Gamma(1-\alpha-m)\Gamma(1-\gamma-n)}{\Gamma(1-\alpha-\gamma-m-n)}\,\frac{\Gamma(1-\beta-m)\Gamma(1-\delta-n)}{\Gamma(1-\beta-\delta-m-n)} \\
&\quad\times \frac{\Gamma(1-\alpha-\beta-\gamma-\delta-m-n)}{\Gamma(1-\alpha-\beta-m)\Gamma(1-\gamma-\delta-n)} \\
&\quad\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_a^b (t-a)^{-\alpha-\beta-m}(b-t)^{-\gamma-\delta-n}\Theta_{m,n}^{(\alpha,\beta,\gamma,\delta)}(\theta,\varphi_1,\varphi_2,\varphi_3) \\
&\quad\times \Big(\Phi_{1,1,2}\Big[y^2\Big(\frac{t-a}{b-a}\Big),x^2\Big(\frac{b-t}{b-a}\Big);\theta,\varphi_1,\varphi_2,\varphi_3\Big]\Big)^{(m+n)/2} \\
&\quad\times G_{m+n}\Big(\alpha+\gamma,\ \beta+\delta;\ xy\Big(\Phi_{1,1,2}\Big[y^2\Big(\frac{t-a}{b-a}\Big), \\
&\quad\times x^2\Big(\frac{b-t}{b-a}\Big);\theta,\varphi_1,\varphi_2,\varphi_3\Big]\Big)^{-1/2}\Big)\,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\,\mathrm{d}\varphi_3
\end{aligned}
$$

$$
\big(\min\{\Re(1-\alpha),\Re(1-\beta),\Re(1-\zeta)\}>m;\ \min\{\Re(1-\gamma),\Re(1-\delta),\Re(1-\xi)\}>n;\ \min\{\Re(1-\beta-\delta),\Re(1-\alpha-\gamma)\}>m+n;\ a\neq b;\ m,n\in\mathbb{N}_0\big),
$$

where

$$
\begin{aligned}
\Theta_{m,n}^{(\alpha,\beta,\gamma,\delta)}&(\theta,\varphi_1,\varphi_2,\varphi_3) \\
&:= \mathrm{e}^{\mathrm{i}[(m-n)\theta-(\alpha-\gamma+m-n)\varphi_1-(\beta-\delta+m-n)\varphi_2-(\alpha+\beta+\gamma+\delta+m+n+1)\varphi_3]} \\
&\qquad\times \cos^{m+n}\theta\,\sec^{\alpha+\gamma+m+n}\varphi_1\,\sec^{\beta+\delta+m+n}\varphi_2\,\sec^{\alpha+\beta+\gamma+\delta+m+n-1}\varphi_3
\end{aligned}
\tag{38}
$$

and

$$
\begin{aligned}
\Phi_{1,1,2}[x,y;\theta,\varphi_1,\varphi_2,\varphi_3] &:= 2\sec^2\theta\cos\varphi_1\cos\varphi_2\cos\varphi_3 \\
&\qquad\times[x\mathrm{e}^{-\mathrm{i}(2\theta-\varphi_1-\varphi_2-\varphi_3)}+y\mathrm{e}^{\mathrm{i}(2\theta-\varphi_1-\varphi_2+\varphi_3)}].
\end{aligned}
\tag{39}
$$

Remark 3. The methodology and techniques employed in this paper (as well as in some of the aforecited closely-related earlier works) can be applied *mutatis mutandis* with a view to obtaining the corresponding multiple integral representations of many other general families of polynomials which are special cases the Srivastava polynomials $S_n^N(z)$ and their known variants.

5. Concluding remarks and observations

First of all, in view of the generality of the definition (2), our main linearization formula (18) as well as its such special cases as the integral representations (22), (29), (32), (33), (35), (36) and (37) are new for all $N \in \mathbb{N}$. Moreover, in their special case when $N = 1$, if we further choose the involved parameters appropriately, some of these integral representations can be seen to correspond to earlier results investigated by (for example) Srivastava and Joshi (cf. [13]; see also the references cited therein).

Next, for the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$, it is known that (see, for details, [18, Chapter 4])

$$P_n^{(\alpha,\beta)}(z) := \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{z-1}{2}\right)^k\left(\frac{z+1}{2}\right)^{n-k} = \binom{n+\alpha}{n}\,{}_2F_1\left[\begin{matrix}-n, n+\alpha+\beta+1; \\ \alpha+1;\end{matrix}\ \frac{1-z}{2}\right]. \tag{40}$$

Thus, upon comparing the second member of (40) with the hypergeometric polynomials occurring on the left-hand side of the linearization formula (22), it is not difficult to observe that, by first setting

$$N = 1, \quad r = s = 1 \quad (L_1 = M_1 = 1), \quad \omega_1 = \beta_1 = 1 \quad \text{and} \quad \sigma_1 = \xi_1 = 1 \tag{41}$$

and then choosing

$$\alpha_1 = \alpha, \quad \gamma_1 = \alpha+\beta+1, \quad \zeta_1 = \gamma, \quad \delta_1 = \gamma+\delta+1, \quad x \mapsto \frac{1-x}{2} \quad \text{and} \quad y \mapsto \frac{1-y}{2}, \tag{42}$$

our result (22) reduces easily to an integral representation for the following product of two Jacobi polynomials (of different arguments and different indices):

$$P_m^{(\alpha,\beta)}(x)P_n^{(\gamma,\delta)}(y).$$

This will lead us eventually to one of the results which were investigated extensively by Srivastava and Panda (cf. [17]; see also the references to closely-related earlier works cited therein).

Finally, we conclude this paper by observing that, if we appropriately further specialize some of the integral representations presented in this paper, we can derive the corresponding (known or new) integral representations (or linearization relationships) for the products of two different members of such families of classical orthogonal polynomials as the Laguerre polynomials, the Hermite polynomials, the Bessel polynomials and so on (see, for details, [15, Chapter 1, Sections 1.8 and 1.9] and [18, Chapter 5]; see also [13] and [17]; see also the recent work [5]).

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