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WEAK SOLUTIONS FOR ELLIPTIC SYSTEMS WITH VARIABLE GROWTH IN CLIFFORD ANALYSIS

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Abstract. In this paper we consider the following Dirichlet problem for elliptic systems:

$$\begin{gathered}\overline{DA(x,u(x),Du(x))} = B(x,u(x),Du(x)), \quad x \in \Omega,\\ u(x) = 0, \quad x \in \partial\Omega,\end{gathered}$$

where D is a Dirac operator in Euclidean space, $u(x)$ is defined in a bounded Lipschitz domain Ω in $\mathbb{R}^n$ and takes value in Clifford algebras. We first introduce variable exponent Sobolev spaces of Clifford-valued functions, then discuss the properties of these spaces and the related operator theory in these spaces. Using the Galerkin method, we obtain the existence of weak solutions to the scalar part of the above-mentioned systems in the space $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ under appropriate assumptions.

Keywords: elliptic system; Clifford analysis; variable exponent; Dirichlet problem

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1. Introduction

Since O. Kováčik and J. Rákosník first discussed the $L^{p(x)}$ space and $W^{k,p(x)}$ space in [24], many results have been obtained concerning these kinds of variable exponent spaces, see for example [7], [14], [11], [12] and references therein. In [30] M. Růžička presented the mathematical theory for the application of variable exponent spaces in electrorheological fluids. For an overview of variable exponent spaces with various applications to differential equations we refer to [22] and the references quoted there.

Clifford algebras were introduced by W. K. Clifford as geometric algebras in 1878, which are a generalization of the complex numbers, the quaternions, and the exterior

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algebras, see [17]. Clifford algebras are playing a major role in quantum computing and the design of quantum computers, see [1]. As an active branch of mathematics over the past 40 years, Clifford analysis has usually studied the solutions of the Dirac equation for functions defined on domains in the Euclidean space and taking value in Clifford algebras, see [6], [18]–[21]. In [8] the authors gave in detail an overview of the intrinsic value and usefulness of Clifford algebras and Clifford analysis for mathematical physics.

In [27], [28] C. A. Nolder first introduced A-Dirac equations and developed tools for the study of solutions to nonlinear A-Dirac equations in the space $W^{1,p}_{\mathrm{loc}}(\Omega, \mathrm{C}\ell_n)$. Inspired by his papers, we are working to study the existence of weak solutions for A-Dirac equations. Also motivated by [15], we are interested in the following Dirichlet problem in the setting of Clifford algebra:

$$\overline{DA(x,u(x),Du(x))} = B(x,u(x),Du(x)), \quad x \in \Omega \tag{1.1}$$

$$u(x) = 0, \quad x \in \partial\Omega \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $u \in \mathrm{C}\ell_n$ and $A\colon \Omega\times\mathrm{C}\ell_n\times\mathrm{C}\ell_n \to \mathrm{C}\ell_n$, $B\colon \Omega\times\mathrm{C}\ell_n\times\mathrm{C}\ell_n \to \mathrm{C}\ell_n$ satisfy the following conditions:

(H1) $A(x,s,\xi)$ and $B(x,s,\xi)$ are measurable with respect to $x \in \Omega$ for all $(s,\xi) \in \mathrm{C}\ell_n \times \mathrm{C}\ell_n$ and continuous with respect to (s,ξ) for a.e. $x \in \Omega$.

(H2) $|A(x,s,\xi)| \leqslant C_0|\xi|^{p(x)-1} + C_1|s|^{p(x)-1} + G(x)$, where $G \in L^{p'(x)}(\Omega)$, $C_0, C_1 \geqslant 0$.

(H3) $|B(x,s,\xi)| \leqslant \widetilde{C}_0|\xi|^{p(x)-1} + \widetilde{C}_1|s|^{p(x)-1} + \widetilde{G}(x)$, where $\widetilde{G} \in L^{p'(x)}(\Omega)$, $\widetilde{C}_0, \widetilde{C}_1 \geqslant 0$ and small.

(H4) $[\overline{A(x,s,\xi)}\xi]_0 \geqslant C_2|\xi|^{p(x)} + C_3|s|^{p(x)} + h(x)$, where $h \in L^1(\Omega)$, $C_2, C_3 > 0$.

(H5) For almost every $x_0 \in \Omega$, $s_0 \in \mathrm{C}\ell_n$, the mapping $\xi \mapsto A(x_0,s_0,\xi)$ satisfies

$$\int_{\widetilde{\Omega}} [\overline{A(x_0,s_0,\xi_0+Dz(x))}Dz(x)]_0 \geqslant C_4 \int_{\widetilde{\Omega}} |Dz(x)|^{p(x)}\,\mathrm{d}x$$

for each $\xi_0 \in \mathrm{C}\ell_n$, $\widetilde{\Omega} \subset \Omega$, $z \in C^1_0(\widetilde{\Omega}, \mathrm{C}\ell_n)$, where $C_4 > 0$ is a constant. Here $p'(x)$ is the conjugate function of $p(x)$.

Throughout this paper we suppose

$$p \in P^{\log}(\Omega) \text{ and } 1 < p_- =: \inf_{x\in\overline{\Omega}} p(x) \leqslant p(x) \leqslant \sup_{x\in\overline{\Omega}} p(x) := p_+ < \infty. \tag{1.3}$$

This paper is organized as follows. In Section 2, we will recall some basic knowledge of Clifford algebras and variable exponent spaces of Clifford valued functions, then discuss the properties of such spaces, which will be needed later. In Section 3, we will prove the existence of weak solutions to the scalar part of the above equations in the space $W^{1,p(x)}_0(\Omega, \mathrm{C}\ell_n)$.

2. Preliminaries

First, we recall some related notions and results from Clifford algebras. For a detailed account we refer to [1], [2], [6], [18]–[21], [27]–[29], [31].

Let $\mathrm{C}\ell_n$ be the real universal Clifford algebra over $\mathbb{R}^n$, then

$$\mathrm{C}\ell_n = \operatorname{span}\{e_0, e_1, e_2, \ldots, e_n, e_1e_2, \ldots, e_{n-1}e_n, \ldots, e_1e_2\ldots e_n\}$$

where $e_0 = 1$ (the identity element in $\mathbb{R}^n$), $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$ with the relation $e_ie_j + e_je_i = -2\delta_{ij}$. Thus the dimension of $\mathrm{C}\ell_n$ is 2^n. For $I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$ with $1 \leqslant i_1 < i_2 < \ldots < i_n \leqslant n$, put $e_I = e_{i_1}e_{i_2}\ldots e_{i_r}$, while for $I = \emptyset$, $e_\emptyset = e_0$. For $0 \leqslant r \leqslant n$ fixed, the space $\mathrm{C}\ell_n^r$ is defined by

$$\mathrm{C}\ell_n^r = \operatorname{span}\{e_I \colon \ |I| := \operatorname{card}(I) = r\}.$$

The Clifford algebra $\mathrm{C}\ell_n$ is a graded algebra

$$\mathrm{C}\ell_n = \bigoplus_r \mathrm{C}\ell_n^r.$$

Any element $a \in \mathrm{C}\ell_n$ may thus be written in a unique way as

$$a = [a]_0 + [a]_1 + \ldots + [a]_n$$

where $[\,]_r \colon \mathrm{C}\ell_n \to \mathrm{C}\ell_n^r$ denotes the projection of $\mathrm{C}\ell_n$ onto $\mathrm{C}\ell_n^r$. It is customary to identify $\mathbb{R}$ with $\mathrm{C}\ell_n^0$ and identify $\mathbb{R}^n$ with $\mathrm{C}\ell_n^1$. For $u \in \mathrm{C}\ell_n$, we know that $[u]_0$ denotes the scalar part of u, that is the coefficient of the element e_0. We define the Clifford conjugation as follows:

$$\overline{(e_{i_1}e_{i_2}\ldots e_{i_r})} = (-1)^{r(r+1)/2}e_{i_1}e_{i_2}\ldots e_{i_r}.$$

For $A \in \mathrm{C}\ell_n$, $B \in \mathrm{C}\ell_n$, we have

$$\overline{AB} = \bar{B}\bar{A}, \quad \bar{\bar{A}} = \bar{A}.$$

We denote

$$(A, B) = [\bar{A}B]_0.$$

Then an inner product is thus obtained, giving rise to the norm $|\cdot|$ on $\mathrm{C}\ell_n$ given by

$$|A|^2 = [\bar{A}A]_0.$$

From [19], we know that this norm is submultiplicative:

$$|AB| \leqslant C_5|A||B| \tag{2.1}$$

where C_5 is a positive constant depending only on n and not greater than $2^{n/2}$.

Throughout, let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. A Clifford-valued function $u\colon \Omega \to C\ell_n$ can be written as $u = \sum_I u_I e_I$, where the coefficients $u_I\colon \Omega \to \mathbb{R}$ are real valued functions.

The Dirac operator on the Euclidean space used here is

$$D = \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j} = \sum_{j=1}^{n} e_j \partial_j.$$

If u is a C^1 real-valued function defined on a domain Ω in $\mathbb{R}^n$, then $Du = \partial u = (\partial_1 u, \partial_2 u, \dots, \partial_n u)$, where ∂ is the generalized derivative operator. A function is left monogenic if it satisfies the equation $Du(x) = 0$ for each $x \in \Omega$. A similar definition can be given for right monogenic functions. An important example of a left monogenic function is the generalized Cauchy kernel

$$G(x) = -\frac{1}{\omega_n} \frac{x}{|x|^n},$$

where ω_n denotes the surface area of the unit ball in $\mathbb{R}^n$. This function is a fundamental solution of the Dirac operator. Basic properties of left monogenic functions one can find in [18], [19].

Next we recall some basic properties of variable exponent spaces which will be used later. For the details see [7], [24].

Let $P(\Omega)$ be the set of all Lebesgue measurable functions $p\colon \Omega \to (1,\infty)$. Given $p \in P(\Omega)$ we define the conjugate function $p'(x) \in P(\Omega)$ by

$$p'(x) = \frac{p(x)}{p(x)-1}, \forall x \in \Omega.$$

Definition 2.1 (see [7]). A function $a\colon \Omega \to \mathbb{R}$ is globally log-Hölder continuous in Ω if there exist $C_i > 0$ $(i = 1, 2)$ and $a_\infty \in \mathbb{R}^n$ such that

$$|a(x) - a(y)| \leqslant \frac{C_1}{\log(\mathrm{e} + 1/|x-y|)}, \quad |a(x) - a_\infty| \leqslant \frac{C_2}{\log(\mathrm{e} + |x|)}$$

hold for all $x, y \in \Omega$. We define the class of variable exponents

$$P^{\log}(\Omega) = \Big\{p \in P(\Omega)\colon \frac{1}{p} \text{ is globally log-Hölder continuous}\Big\}.$$

Definition 2.2 (see [11], [24]). We define the variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega)=\left\{u\in P(\Omega)\colon \int_\Omega |u|^{p(x)}\,\mathrm{d}x<\infty\right\}$$

with the norm

$$\|u\|_{L^{p(x)}(\Omega)}=\inf\left\{t>0\colon \int_\Omega \left|\frac{u}{t}\right|^{p(x)}\mathrm{d}x\leqslant 1\right\}.$$

We define the Sobolev space $W^{k,p(x)}(\Omega)$ by

$$W^{k,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega)\colon\ |\partial^\alpha u|\in L^{p(x)}(\Omega),|\alpha|\leqslant k\}$$

with the norm

$$\|u\|_{W^{k,p(x)}(\Omega)}=\sum_{|\alpha|\leqslant k}\|\partial^\alpha u\|_{L^{p(x)}(\Omega)}. \tag{2.2}$$

Denote by $W_0^{k,p(x)}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$ with respect to the norm (2.2).

Remark 2.1. We say that $u\in L^{p(x)}(\Omega,\mathrm{C}\ell_n)$ can be understood coordinatewise. For example, $u\in L^{p(x)}(\Omega,\mathrm{C}\ell_n)$ means that $\{u_I\}\subset L^{p(x)}(\Omega)$ for $u=\Sigma_I u_I e_I\in\mathrm{C}\ell_n$ with the norm $\|u\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}=\sum\limits_I\|u_I\|_{L^{p(x)}(\Omega)}$. A simple computation shows that $\|u\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}$ is equivalent to $\||u|\|_{L^{p(x)}(\Omega)}$. In the same way, the spaces $W^{k,p(x)}(\Omega,\mathrm{C}\ell_n)$, $W_0^{k,p(x)}(\Omega,\mathrm{C}\ell_n)$, $C^k(\Omega,\mathrm{C}\ell_n)$ and $C_0^k(\Omega,\mathrm{C}\ell_n)$ $(k\in\mathbb{N}\cup\{0\})$ can be understood similarly.

Theorem 2.1. *$C_0^\infty(\Omega,\mathrm{C}\ell_n)$ is dense in $L^{p(x)}(\Omega,\mathrm{C}\ell_n)$.*

Proof. For any $u(x)=\sum\limits_I u_I(x)e_I\in L^{p(x)}(\Omega,\mathrm{C}\ell_n)$ we have $u_I(x)\in L^{p(x)}(\Omega)$ for each I. Since $C_0^\infty(\Omega)$ is dense in $L^{p(x)}(\Omega)$, there exists a sequence $\{u_{Ik}\}_{k=1}^\infty\subset C_0^\infty(\Omega)$ which converges to $u_I(x)$ in $L^{p(x)}(\Omega)$ for each I. Let $u_k(x)=\sum\limits_I u_{Ik}e_I$, then the sequence $\{u_k(x)\}\subset C_0^\infty(\Omega,\mathrm{C}\ell_n)$ converges to $u(x)$ in $L^{p(x)}(\Omega,\mathrm{C}\ell_n)$, since

$$\begin{aligned}\int_\Omega |u(x)-u_k(x)|^{p(x)}\,\mathrm{d}x &\leqslant \int_\Omega\left(\sum_I |u_I(x)-u_{Ik}(x)|\right)^{p(x)}\mathrm{d}x\\ &\leqslant 2^{np_+}\sum_I\int_\Omega |u_I(x)-u_{Ik}(x)|^{p(x)}\,\mathrm{d}x.\end{aligned}$$

This completes the proof of Theorem 2.1. $\square$

Theorem 2.2. *$L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ is a separable and reflexive Banach space.*

Proof. We first show that the dual of $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ is $L^{p'(x)}(\Omega, \mathrm{C}\ell_n)$ in several steps (see [16]).

(i) For fixed $v = \sum_I v_I e_I \in L^{p'(x)}(\Omega, \mathrm{C}\ell_n)$, we define a linear functional

$$L_v(u) = \int_\Omega [\bar{u}v]_0 \,\mathrm{d}x = \int_\Omega \sum_I u_I(x) v_I(x) \,\mathrm{d}x.$$

Then $L_v(u)$ is a bounded linear functional on $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$.

(ii) Let $L \in (L^{p(x)}(\Omega, \mathrm{C}\ell_n))'$, for any I and any $w \in L^{p(x)}(\Omega)$ we define a functional L_I as follows:

$$L_I\colon\ L^{p(x)}(\Omega) \to \mathbb{R}, \quad L_I(w) = L(we_I).$$

Then L_I is a continuous linear functional on $L^{p(x)}(\Omega)$. Let $u = \sum_I u_I e_I \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$, then there exists $v_I \in L^{p'(x)}(\Omega)$ such that L_I can be represented uniquely as follows:

$$L_I(u_I) = \int_\Omega u_I(x) v_I(x) \,\mathrm{d}x.$$

Let $v = \sum_I v_I e_I$, then $v \in L^{p'(x)}(\Omega, \mathrm{C}\ell_n)$ and $L(u) = \int_\Omega [\bar{u}v]_0 \,\mathrm{d}x$.

(iii) We shall show $\|v\|_{L^{p'(x)}(\Omega,\mathrm{C}\ell_n)} \leqslant C\|L_v\|$. Supposing $\|v_I\|_{L^{p'(x)}(\Omega)} \neq 0$, we take

$$u = \sum_I \Big(\Big(\frac{|v_I|}{\|v_I\|_{L^{p'(x)}(\Omega)}}\Big)^{1/(p(x)-1)} \operatorname{sgn} v_I\Big) e_I.$$

Then $\|u\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} = 2^n$ and $L_v(u) \geqslant 2^{p_-/(1-p_-)}\|v\|_{L^{p'(x)}(\Omega,\mathrm{C}\ell_n)}$. Therefore

$$\|v\|_{L^{p'(x)}(\Omega,\mathrm{C}\ell_n)} \leqslant 2^{p_-/(p_- -1)+n}\|L_v\|.$$

Now we reach the conclusion $(L^{p(x)}(\Omega, \mathrm{C}\ell_n))^* = L^{p'(x)}(\Omega, \mathrm{C}\ell_n)$ and, moreover, $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ is reflexive.

In the following, we will prove that $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ is separable. Let $u = \sum_I u_I e_I \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$. Since $L^{p(x)}(\Omega)$ is separable, there exists a dense, countable subset $\mathscr{F}$ of $L^{p(x)}(\Omega)$. Then for any $u_I(x)$ above we can extract a sequence $\{u_{Ik}(x)\}$ in $\mathscr{F}$ which converges to $u_I(x)$ in $L^{p(x)}(\Omega)$. Similarly to the proof of Theorem 2.1, the sequence $\big\{u_k\colon\ u_k = \sum_I u_{kI} e_I\big\}$ converges to $u(x)$ in $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$. $\square$

Theorem 2.3. *$W^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ is a separable and reflexive Banach space.*

Proof. We treat $W^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ as a subspace of the product space $\prod\limits_{m=1}^{n} L^{p(x)}$ $(\Omega, \mathrm{C}\ell_n)$. Then by Theorem 2.2, we need only to show that $W^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ is a closed subspace of the product space $\prod\limits_{m=1}^{n} L^{p(x)}(\Omega, \mathrm{C}\ell_n)$. Let $\{u_k\colon\ u_k = \sum\limits_I u_{kI} e_I\}$ be convergent in $W^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$, then $u_{kI}(x)$ is a convergent sequence in $L^{p(x)}(\Omega, Cl_n)$. By Theorem 2.2 in [14], there exists $u_I(x) \in L^{p(x)}(\Omega)$ such that $u_{kI}(x) \to u_I(x)$ in $L^{p(x)}(\Omega)$. Then we obtain $u_k(x) \to u(x)$ in $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$. Then, similarly to the proof of Theorem 2.4 in [14], we can get the desired conclusion. □

Theorem 2.4. *The embedding $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n) \to L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ is compact.*

Proof. First, we should show that $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n) \subset L^{p(x)}(\Omega, \mathrm{C}\ell_n)$. Let $u(x) = \sum\limits_I u_I(x) e_I \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. Then there exists a constant $C > 0$ such that $\|u_I\|_{L^{p(x)}(\Omega)} \leqslant C\|\partial u_I\|_{L^{p(x)}(\Omega)}$. Therefore, we have $\|u\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)} \leqslant C\|\partial u\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)}$.

Secondly, we should show that the embedding is compact. If $\{u_k\colon\ u_k = \sum\limits_I u_{kI} e_I\}$ is bounded in $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$, then there exists a subsequence of $\{u_{kI}\}$ (still denoted by $\{u_{kI}\}$) such that $u_{kI} \to u_I$ in $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$. Let $u = \sum\limits_I u_I e_I$. Then $u(x) \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ and $u_k \to u$ in $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$. □

Theorem 2.5. *If $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$, then*

$$\|u\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)} \leqslant C(n, \Omega)\|\partial u\|_{(L^{p(x)}(\Omega, \mathrm{C}\ell_n))^n}.$$

Proof. If $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$, then by Proposition 2.5 in [13] there exists a constant $C(\Omega) > 0$ such that $\|u_I\|_{L^{p(x)}(\Omega)} \leqslant C(\Omega)\|\partial u_I\|_{L^{p(x)}(\Omega)}$. Hence we obtain that $\|u\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)} \leqslant 2^n C(\Omega)\|\partial u\|_{(L^{p(x)}(\Omega, \mathrm{C}\ell_n))^n}$. □

Remark 2.2. We say that $f_n \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ converge modularly to $f \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ if $\lim\limits_{n\to\infty} \int_\Omega |f_n - f|^{p(x)}\,\mathrm{d}x = 0$. It is easy to see that the topology of $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ given by the norm coincides with the topology of modular convergence (see [23]).

Definition 2.3 (see [7], [18], [19])**.**

(i) Let $u \in C(\Omega, \mathrm{C}\ell_n)$. Teodorescu operator is defined by

$$\mathrm{T}u(x) = \int_\Omega G(x-y)u(y)\,\mathrm{d}y,$$

where $G(x)$ is the generalized Cauchy kernel mentioned above.

(ii) Let $u \in C^1(\Omega, \mathrm{C}\ell_n) \cap C(\overline{\Omega}, \mathrm{C}\ell_n)$. The boundary operator is defined by

$$Fu(x) = \int_{\partial\Omega} G(y-x)\alpha(y)u(y)\,\mathrm{d}y,$$

where $\alpha(y)$ denotes the outward normal unit vector at y.

(iii) Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The maximal operator is defined by

$$Mu(x) = \sup_{r>0} \frac{1}{\mathrm{meas}(B(x,r))} \int_{B(x,r)} |u(x)|\,\mathrm{d}x.$$

Lemma 2.1 (see [7]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $x \in \Omega$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then*

$$\int_\Omega \frac{1}{|x-y|^{n-1}} |u(y)|\,\mathrm{d}y \leqslant C(n)(\operatorname{diam}\Omega)Mu(x).$$

Lemma 2.2 (see [7]). *Let $p(x)$ satisfy* (1.3). *Then M is bounded in $L^{p(x)}(\mathbb{R}^n)$.*

Lemma 2.3 (see [19]). *Let $u \in C^1(\Omega, \mathrm{C}\ell_n)$. Then*

$$\partial_k \mathrm{T}u(x) = \frac{1}{\omega_n} \int_\Omega \frac{\partial}{\partial x_k} G(x-y)u(y)\,\mathrm{d}y + \frac{u(x)}{n}\bar{e}_k.$$

Lemma 2.4 (see [18]). *The operator $\mathrm{T}\colon L^p(\Omega, \mathrm{C}\ell_n) \to W^{1,p}(\Omega, \mathrm{C}\ell_n)$ $(1 < p < \infty)$ is continuous.*

Lemma 2.5 (see [7]). *Let Φ be a Calderón-Zygmund operator with Calderón-Zygmund kernel K on $\mathbb{R}^n \times \mathbb{R}^n$. Then Φ is bounded on $L^{p(x)}(\mathbb{R}^n)$.*

Theorem 2.6. *The operator $\partial_k \mathrm{T}\colon L^{p(x)}(\Omega, \mathrm{C}\ell_n) \to L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ is continuous.*

Proof. By Lemma 2.3 we have for $u \in C_0^\infty(\Omega, \mathrm{C}\ell_n)$

$$\partial_k \mathrm{T}u(x) = \frac{1}{\omega_n} \int_\Omega \frac{\partial}{\partial x_k} G(x-y)u(y)\,\mathrm{d}y + \frac{u(x)}{n}\bar{e}_k.$$

Let $K(x,y) = \omega_n^{-1}(\partial/\partial x_k)G(x-y)$. Since

$$\frac{\partial}{\partial x_k} G(x-y) = \frac{1}{|x-y|^n}\biggl(\sum_{j=1}^n \frac{(x_j-y_j)^2}{|x-y|^2}\bar{e}_k - n\sum_{i=1}^n \frac{(x_k-y_k)(x_i-y_i)}{|x-y|^2}\bar{e}_i\biggr),$$

we obtain

$$\Big|\frac{\partial}{\partial x_k}G(x-y)\Big| \leqslant \frac{n^2+1}{|x-y|^n} \quad (k=1,\dots,n).$$

Notice that

$$\int_{S_1}\bigg(\sum_{j=1}^{n}\frac{(x_j-y_j)^2}{|x-y|^2}\bar{e}_k - n\sum_{i=1}^{n}\frac{(x_k-y_k)(x_i-y_i)}{|x-y|^2}\bar{e}_i\bigg)\,\mathrm{d}S=0,$$

hence it is easy to verify that $K(x,y)$ satisfies the following conditions:

(a) $|K(x,y)|\leqslant C|x-y|^{-n}$;

(b) $K(t(x,y))=t^{-n}K(x,y),\ t>0$;

(c) $\int_{S_1}K(x,y)\,\mathrm{d}S=0$, where $S_1=\{y\in\Omega\colon\ |x-y|=1\}$.

Now we define $u(x)=0, x\in\mathbb{R}^n\setminus\Omega$. Then $K(x,y)$ satisfies the conditions of Calderón-Zygmund kernel on $\mathbb{R}^n\times\mathbb{R}^n$. By Theorem 2.1, we know the inequality can be extended to $L^{p(x)}(\Omega,\mathrm{C}\ell_n)$. Therefore, we obtain by Lemma 2.4 and Lemma 2.5

$$\bigg\|\frac{1}{\omega_n}\int_\Omega\frac{\partial}{\partial x_k}G(x-y)u(y)\,\mathrm{d}y\bigg\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}\leqslant C(n,p,\Omega)\|u\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}. \tag{2.3}$$

On the other hand, we have

$$\Big\|\frac{u(x)}{n}\bar{e}_k\Big\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}\leqslant\frac{1}{n}\|u\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}. \tag{2.4}$$

Combining (2.3) with (2.4), we obtain

$$\|\partial_k\mathrm{T}u\|_{(L^{p(x)}(\Omega,\mathrm{C}\ell_n))^n}\leqslant C(n,p,\Omega)\|u\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}.$$

□

Theorem 2.7. *The operator* $\mathrm{T}\colon\ L^{p(x)}(\Omega,\mathrm{C}\ell_n)\to W^{1,p(x)}(\Omega,\mathrm{C}\ell_n)$ *is continuous.*

Proof. First we prove that the operator $\mathrm{T}\colon\ L^{p(x)}(\Omega,\mathrm{C}\ell_n)\to L^{p(x)}(\Omega,\mathrm{C}\ell_n)$ is continuous. We define $u(x)=0, x\in\mathbb{R}^n\setminus\Omega$. Since

$$|G(x-y)|=\frac{1}{\omega_n}\frac{1}{|x-y|^{n-1}},$$

from (2.1) we have

$$\begin{aligned}|\mathrm{T}u(x)|&=\bigg|\int_\Omega G(x-y)u(y)\,\mathrm{d}y\bigg|\\&\leqslant C_5\int_\Omega|G(x-y)||u(y)|\,\mathrm{d}y=\frac{C_5}{\omega_n}\int_\Omega\frac{1}{|x-y|^{n-1}}|u(y)|\,\mathrm{d}y.\end{aligned}$$

Then we get by Lemma 2.1

$$|\mathrm{T}u(x)| \leqslant C(n,\Omega)M(|u|)(x), \quad \forall x \in \Omega.$$

In view of Lemma 2.2 we obtain

$$\|\mathrm{T}u\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} \leqslant C(n,p,\Omega)\|u\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}.$$

Secondly we prove that the operator $\mathrm{T}\colon L^{p(x)}(\Omega,\mathrm{C}\ell_n) \to W^{1,p(x)}(\Omega,\mathrm{C}\ell_n)$ is continuous.

By Theorem 2.6, we have

$$\begin{aligned}\|\mathrm{T}u\|_{W^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} &= \|\mathrm{T}u\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} + \sum_{k=1}^{n}\|\partial_k \mathrm{T}u\|_{(L^{p(x)}(\Omega,\mathrm{C}\ell_n))^n}\\ &\leqslant C(n,p,\Omega)\|u\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}.\end{aligned}$$

Then we get the desired conclusion. □

Lemma 2.6. *The operator $D\colon W^{1,p(x)}(\Omega,\mathrm{C}\ell_n) \to L^{p(x)}(\Omega,\mathrm{C}\ell_n)$ is continuous.*

Proof. If $u \in \sum\limits_I u_I e_I \in W^{1,p(x)}(\Omega,\mathrm{C}\ell_n)$, then

$$\partial u = \sum_I \partial u_I e_I = \sum_I (\partial_1 u_I,\ldots,\partial_n u_I)e_I, \quad Du = \sum_I \sum_{i=1}^{n} \partial_i u_I e_i e_I.$$

Since

$$\begin{aligned}\|Du\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} &= \sum_I \Bigl\| - \sum_{i\in I} \partial_i u_I e_{I\setminus\{i\}} + \sum_{i\in\{1,\ldots,n\}\setminus I} \partial_i u_I e_{I\cup\{i\}} \Bigr\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}\\ &\leqslant \sum_I \sum_{i=1}^{n} \|\partial_i u_I\|_{L^{p(x)}(\Omega)} = \|\partial u\|_{(L^{p(x)}(\Omega,\mathrm{C}\ell_n))^n},\end{aligned}$$

the conclusion follows from Remark 2.1. □

Lemma 2.7 (see [21])**.** *Let $u \in W^{1,p}(\Omega,\mathrm{C}\ell_n)$ $(1 < p < \infty)$. Then the Borel-Pompeiu formula*

$$Fu(x) + \mathrm{T}Du(x) = u(x)$$

holds for all $x \in \Omega$.

Now we define another norm on the space $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$:

$$\|\!|u|\!\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} = \|u\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} + \|Du\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}.$$

Remark 2.3. By Theorem 2.7, Lemma 2.6 and Lemma 2.7, we obtain that the Borel-Pompeiu formula still holds for $u \in W^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. Thus, we have for $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$

$$\begin{aligned}\|\partial u\|_{(L^{p(x)}(\Omega,\mathrm{C}\ell_n))^n} &= \|\partial \mathrm{T} Du\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} \leqslant C(n,p,\Omega)\|Du\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} \\ &\leqslant C(n,p,\Omega)\|\partial u\|_{(L^{p(x)}(\Omega,\mathrm{C}\ell_n))^n}.\end{aligned}$$

Hence $\|\!|u|\!\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}$ is equivalent to $\|u\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}$. Moreover, by Theorem 2.5 and Remark 2.1, we know that $\|\!|u|\!\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}$ and $\||Du|\|_{L^{p(x)}(\Omega)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$.

3. The main theorem

In [27], [28] C. A. Nolder introduced A-Dirac equations $DA(x, Du) = 0$ and investigated some properties of weak solutions to the scalar part of the above equations. Note that when u is a real-valued function, i.e. $u \in \mathrm{C}\ell_n^0(\Omega)$ and $A\colon \Omega \times \mathrm{C}\ell_n^1(\Omega) \to \mathrm{C}\ell_n^1(\Omega)$, the scalar part of an A-Dirac equation is $\operatorname{div} A(x, \nabla u) = 0$, i.e. an A-harmonic equation. These equations have been extensively studied with many applications, see [23].

In this section we will establish the existence of weak solutions to the scalar part of elliptic systems with variable growth.

Theorem 3.1. *Under conditions* (H1)–(H5), *there exists a weak solution to the scalar part of the Dirichlet problem* (1.1)–(1.2) *in* $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. *In other words, there exists at least one* $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ *satisfying*

$$\int_\Omega [\overline{A(x,u,Du)} D\varphi - B(x,u,Du)\varphi]_0 \,\mathrm{d}x = 0 \tag{3.1}$$

for each $\varphi \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$.

Let $V = W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. For $u \in V$, we define $T\colon V \to V^*$ in the following way: for each $\varphi \in V$

$$\langle Tu, \varphi\rangle = \int_\Omega [\overline{A(x,u(x),Du(x))} D\varphi - B(x,u(x),Du(x))\varphi]_0 \,\mathrm{d}x. \tag{3.2}$$

Now we need only to show that there exists $u_0 \in V$ such that $\langle Tu_0, \varphi\rangle = 0$ for any $\varphi \in V$.

Lemma 3.1. *T is strongly-weakly continuous on V.*

P r o o f. Suppose $u_k \to u$ strongly in V, then $\{u_k\}$ is uniformly bounded in V. Then, to see equiintegrability of the sequence $\{[\overline{A(x,u_k,Du_k)}D\varphi]_0\}$, we take a measurable subset $\Omega' \subset \Omega$. By (2.1) and (H2) we have for each $\varphi \in V$

$$
\begin{aligned}
(3.3)\qquad &\int_{\Omega'} |[\overline{A(x,u_k,Du_k)}D\varphi]_0|\,\mathrm{d}x \\
&\leqslant C_5 \int_{\Omega'} |\overline{A(x,u_k(x),Du_k(x))}|\,|D\varphi|\,\mathrm{d}x \\
&\leqslant C_5 \int_{\Omega'} (C_0|Du_k|^{p(x)-1} + C_1|u_k|^{p(x)-1} + G(x))\cdot|D\varphi|\,\mathrm{d}x \\
&\leqslant 2C_5(C_0\||Du_k|^{p(x)-1}\|_{L^{p'(x)}(\Omega')} + C_1\|\,|u_k|^{p(x)-1}\,\|_{L^{p'(x)}(\Omega')} \\
&\qquad + \|G\|_{L^{p'(x)}(\Omega')})\cdot\||D\varphi|\|_{L^{p(x)}(\Omega')}.
\end{aligned}
$$

By virtue of Remark 2.1, we obtain that the first term of (3.3) is bouded uniformly in k. The second term of (3.3) is arbitrarily small if the measure of Ω' is chosen small enough. A similar argument gives the equiintegrability of the sequence $\{[B(x,u_k,Du_k)\varphi]_0\}$. Hence by (H1) and the Vitali convergence theorem, we obtain

$$\lim_{k\to\infty}\langle Tu_k,\varphi\rangle = \langle T\lim_{k\to\infty}u_k,\varphi\rangle = \langle Tu,\varphi\rangle.$$

That is to say, T is strongly-weakly continuous. □

Lemma 3.2. *T is coercive on V, that is,*

$$\lim_{\|u\|_V\to\infty}\frac{\langle Tu,u\rangle}{\|u\|_V} = +\infty.$$

P r o o f. By (H3) and (H4), for any $\lambda \in (0,1)$ there exists a positive constant $C(\lambda)$ such that

$$
\begin{aligned}
\langle Tu,u\rangle &\geqslant \int_\Omega (C_2|Du|^{p(x)} + C_3|u|^{p(x)} - h(x) \\
&\qquad - \widetilde{C}_0C_5|Du|^{p(x)-1}|u| - \widetilde{C}_1C_5|u|^{p(x)} - C_5\widetilde{G}(x)|u|)\,\mathrm{d}x \\
&\geqslant \int_\Omega (C_2|Du|^{p(x)} + C_3|u|^{p(x)} - h(x) - \widetilde{C}_0C_5|Du|^{p(x)} - \widetilde{C}_0C_5|u|^{p(x)} \\
&\qquad - \widetilde{C}_1C_5|u|^{p(x)} - \lambda|u|^{p(x)} - C(\lambda,C_5)|\widetilde{G}(x)|^{p(x)})\,\mathrm{d}x \\
&= \int_\Omega ((C_2-\widetilde{C}_0C_5)|Du|^{p(x)} + (C_3-\widetilde{C}_0C_5-\widetilde{C}_1C_5-\lambda)|u|^{p(x)} \\
&\qquad - h(x) - C(\lambda,C_5)|\widetilde{G}(x)|^{p(x)})\,\mathrm{d}x.
\end{aligned}
$$

When $\widetilde{C}_0, \widetilde{C}_1$ are small enough such that $C_2 > \widetilde{C}_0 C_5$ and $C_3 > C_5(\widetilde{C}_0 + \widetilde{C}_1)$, then we take $\lambda < C_3 - C_5(\widetilde{C}_0 + \widetilde{C}_1)$. Hence we obtain

$$\frac{\langle Tu, u\rangle}{\|u\|_V} \geqslant \frac{C\int_\Omega (|Du|^{p(x)} + |u|^{p(x)})\,\mathrm{d}x - C}{\|u\|_V} \geqslant \frac{C\int_\Omega |Du|^{p(x)}\,\mathrm{d}x - C}{\|u\|_V}.$$

Since

$$\frac{\int_\Omega |Du|^{p(x)}\,\mathrm{d}x}{\||Du|\|_{L^{p(x)}(\Omega)}} = \int_\Omega \Big(\frac{|Du|}{2^{-1}\||Du|\|_{L^{p(x)}(\Omega)}}\Big)^{p(x)} \cdot \frac{(2^{-1}\||Du|\|_{L^{p(x)}(\Omega)})^{p(x)}}{\||Du|\|_{L^{p(x)}(\Omega)}}\,\mathrm{d}x,$$

we have when $\||Du|\|_{L^{p(x)}(\Omega)} \geqslant 1$

$$\frac{\int_\Omega |Du|^{p(x)}\,\mathrm{d}x}{\||Du|\|_{L^{p(x)}(\Omega)}} \geqslant 2^{-p_+}\||Du|\|^{p_- - 1}_{L^{p(x)}(\Omega)}.$$

In virtue of Remark 2.3 we have as $\|u\|_V \to \infty$

$$\frac{\langle Tu, u\rangle}{\|u\|_V} \to +\infty.$$

$\square$

Lemma 3.3 (see [26]). *If the mapping $F\colon \mathbb{R}^m \to \mathbb{R}^m$ is continuous and*

$$\lim_{|x|\to\infty} \frac{\langle F(x), x\rangle}{|x|} = +\infty$$

then the range of F is the whole of $\mathbb{R}^m$.

Lemma 3.4. *There exist a sequence $\{u_k\} \subset V$ and $u_0 \in V$ such that*

$$\langle Tu_k, u_k - u_0\rangle \to 0 \quad \text{as } k \to \infty.$$

P r o o f. By the separability of V, we can choose a basis $\{w_k\}$ of V such that the union of subspaces finitely generated from w_k are dense in V. Let $V_k = \operatorname{span}\{w_1, \ldots, w_k\}$. Since V_k is topologically isomorphic to $\mathbb{R}^k$, by Lemma 3.1, Lemma 3.2 and Lemma 3.3 there exists $u_k \in V_k$ such that for any $w \in V_k$

$$\langle Tu_k, w\rangle = 0 \tag{3.4}$$

By the coerciveness of T, u_k is bounded in V. Since V is reflective, we can extract a subsequence of $\{u_k\}$ (still denoted by $\{u_k\}$) such that

$$u_k \rightharpoonup u_0 \quad \text{weakly in } V \quad \text{as } k \to \infty.$$

By (H2) and (H3), T is a bounded operator. By the separability of V again and Corollary 3.30 in [4], we may suppose

$$Tu_k \rightharpoonup \xi \quad \text{weakly* in } V^* \quad \text{as } k \to \infty.$$

From (3.4), we have for any $w \in \operatorname{span}\{w_1, w_2, \dots\}$

$$\langle \xi, w\rangle = 0.$$

For fixed ξ, by the continuity of $\langle \xi, \cdot\rangle$, we get $\langle \xi, w\rangle = 0$ for all $w \in V$. Furthermore, we have

$$\langle Tu_k, u_k - u_0\rangle = \langle Tu_k, u_k\rangle - \langle Tu_k, u_0\rangle = -\langle Tu_k, u_0\rangle \to 0 \quad \text{as } k \to \infty.$$

This completes the proof of Lemma 3.4. $\square$

Let $z_k = u_k - u_0 = \sum\limits_I z_{kI} e_I$. Then

$$z_k \rightharpoonup 0 \quad \text{weakly in } V \quad \text{as } k \to \infty.$$

By Theorem 2.4, we obtain

$$z_k \to 0 \quad \text{strongly in } L^{p(x)}(\Omega, \mathrm{C}\ell_n). \tag{3.5}$$

Since

$$\begin{aligned}\langle Tu_k, u_k - u_0\rangle \\ &= \int_\Omega [\overline{A(x, u_0 + z_k, Du_0 + Dz_k)} Dz_k - B(x, u_0 + z_k, Du_0 + Dz_k) z_k]_0 \, \mathrm{d}x \to 0\end{aligned}$$

we have by virtue of (H3) and (3.5)

$$\int_\Omega [B(x, u_0 + z_k, Du_0 + Dz_k) z_k]_0 \, \mathrm{d}x \to 0.$$

Therefore, we obtain

$$\int_\Omega [\overline{A(x, u_0 + z_k, Du_0 + Dz_k)} Dz_k]_0 \, \mathrm{d}x \to 0 \quad \text{as } k \to \infty. \tag{3.6}$$

Now if we can prove that there exists a subsequence of $\{z_k\}$ which is strongly convergent in V, then from the strong-weak continuity of T we get $Tu_k \to Tu_0 = \xi$ weakly in V^* as $k \to \infty$ and u_0 will be a weak solution of (1.1) and (1.2). Next we need the following preliminary results.

Definition 3.1. A function $f\colon \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}$ is called a Carathéodory function if it satisfies: for every $(s,\xi) \in \mathbb{R}^l \times \mathbb{R}^m$ that $x \mapsto f(x,s,\xi)$ is measurable; and for almost every $x \in \mathbb{R}^n$, $(s,\xi) \mapsto f(x,s,\xi)$ is continuous.

Lemma 3.5 (see [9]). *$f\colon \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function if and only if for each compact set $K \subset \mathbb{R}^n$ and any $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset K$ satisfying $\operatorname{meas}(K \setminus K_\varepsilon) < \varepsilon$ such that f is continuous on $K_\varepsilon \times \mathbb{R}^l \times \mathbb{R}^m$.*

Lemma 3.6 (see [10]). *Let $E \subset \mathbb{R}^n$ be measurable and $\operatorname{meas} E < \infty$. Suppose that $\{E_k\}$ is a sequence of subsets of E such that for some $\varepsilon > 0$*

$$\operatorname{meas} E_k \geqslant \varepsilon \quad \textit{for each } k \in \mathbb{N}.$$

Then there exists a subsequence $\{E_{k_i}\}$ such that $\bigcap\limits_{i=1}^{\infty} E_{k_i} \neq \emptyset$.

Lemma 3.7 (see [3]). *Let $\{f_k\}$ be a sequence of bounded functions in $L^1(\mathbb{R}^n)$. For each $\varepsilon > 0$ there exist E_ε, δ, J (where E_ε is measurable and $\operatorname{meas} E_\varepsilon < \varepsilon$, $\delta > 0$, J is an infinite subset of $\mathbb{N}$) such that for each $k \in J$*

$$\int_E |f_k(x)| \, \mathrm{d}x < \varepsilon$$

where E and E_ε are disjoint and $\operatorname{meas} E < \delta$.

Definition 3.2. For $u \in C_0^1(\mathbb{R}^n)$, define

$$(M^* u)(x) = (Mu)(x) + \sum_{\alpha=1}^{n} (M \partial_\alpha u)(x).$$

Lemma 3.8 (see [25]). *If $u \in C_0^\infty(\mathbb{R}^n)$, then $M^* u \in C^0(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$,*

$$|u(x)| + \sum_{\alpha=1}^{n} (\partial_\alpha u)(x) \leqslant (M^* u)(x).$$

Furthermore, if $p > 1$, then

$$\|M^* u\|_{L^p(\mathbb{R}^n)} \leqslant C(n,p) \|u\|_{W_0^{1,p}(\mathbb{R}^n)};$$

and if $p = 1$, then

$$\operatorname{meas}\{x \in \mathbb{R}^n \colon\ (M^* u) \geqslant \lambda\} \leqslant \frac{C(n)}{\lambda} \|u\|_{W^{1,1}(\mathbb{R}^n)}$$

for all $\lambda > 0$.

Lemma 3.9 (see [9]). *Let $u \in C_0^\infty(\mathbb{R}^n)$ and $\lambda > 0$. Set*

$$H^\lambda = \{x \in \mathbb{R}^n \colon (M^*u)(x) < \lambda\}.$$

Then for $\forall x, y \in H^\lambda$, we have

$$|u(y) - u(x)| \leqslant C(n)\lambda|y - x|.$$

Lemma 3.10 (see [9]). *Let X be a metric space, E a subspace of X, and L a positive real number. Then any L-Lipschitz mapping from E into $\mathbb{R}$ can be extended to an L-Lipschitz mapping from X into $\mathbb{R}$.*

P r o o f of Theorem 3.1. We need only to prove that there exists a subsequence of $\{z_k\}$ which is strongly convergent in V.

For each measurable set $E \subset \Omega$, define

$$F(v, E) = \int_E [\overline{A(x, u_0 + v, Du_0 + Dv)}Dv]_0 \,\mathrm{d}x,$$

where $v \in V$. Similarly to the proof of Lemma 3.1, we can show that $F(\cdot, E)$ is continuous in V. Since $C_0^\infty(\Omega, \mathrm{C}\ell_n)$ is dense in V, there exists $\{f_k\} \subset C_0^\infty(\Omega, \mathrm{C}\ell_n)$ such that

$$\|f_k - z_k\|_V < \frac{1}{k}, \quad |F(f_k, \Omega) - F(z_k, \Omega)| < \frac{1}{k}.$$

So we can suppose that z_k is in $C_0^\infty(\Omega, \mathrm{C}\ell_n)$ and bounded in V.

Next we define

$$z_k(x) = 0, \quad \forall x \in \mathbb{R}^n \setminus \Omega.$$

In this way, we extend the domain of z_k to $\mathbb{R}^n$. Hence $\{z_k\}$ is bounded in $W_0^{1,p(x)}(\mathbb{R}^n, \mathrm{C}\ell_n)$ and $\operatorname{supp} z_k \subset \Omega$.

Let $\eta\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous increasing function satisfying $\eta(0) = 0$ and for each measurable set $E \subset \Omega$, let

$$\sup_k \int_E (|G(x)|^{p'(x)} + |h(x)| + (C_0 + C_1 + 1)(|u_0|^{p(x)} + |Du_0|^{p(x)} + |z_k|^{p(x)}))\,\mathrm{d}x \leqslant \eta(\operatorname{meas} E).$$

where C_0, C_1 are the constants in (H2).

Let $\{\varepsilon_j\}$ be a decreasing sequence with $\varepsilon_j > 0$ and let $\varepsilon_j \to 0$ as $j \to 0$. For ε_1 and each $\{(M^*z_{k_1 I})^{p(x)}\}$, by Lemma 3.7 we get a subsequence $\{z_{k_1}\}$, a set $E_{\varepsilon_1} \subset \Omega$ satisfying $\operatorname{meas} E_{\varepsilon_1} < \varepsilon_1$, and a real number $\delta_1 > 0$ such that

$$\int_U (M^*z_{k_1 I})^{p(x)}\,\mathrm{d}x < \varepsilon_1$$

for each k_1, I and $U \subset \Omega \setminus E_{\varepsilon_1}$ satisfying $\operatorname{meas} U < \delta_1$. By Lemma 3.8, we can choose $\lambda > 1$ so large that for all I and k_1,

$$\operatorname{meas}(\{x \in \mathbb{R}^n : (M^* z_{k_1 I})(x) \geqslant \lambda\}) \leqslant \min\{\varepsilon_1, \delta_1\}.$$

For each I and k_1, define

$$H^\lambda_{k_1 I} = \{x \in \mathbb{R}^n : (M^* z_{k_1 I})(x) < \lambda\}, \quad H^\lambda_{k_1} = \bigcap_I H^\lambda_{k_1 I}.$$

In view of Lemma 3.9, we have

$$\frac{|z_{k_1 I}(y) - z_{k_1 I}(x)|}{|y - x|} \leqslant C(n)\lambda.$$

By Lemma 3.10, there exists a Lipschitz function g_{k_1} which extends $z_{k_1 I}$ outside $H^\lambda_{k_1}$ and the Lipschitz constant of $g_{k_1 I}$ is not greater than $C(n)\lambda$. As $H^\lambda_{k_1}$ is an open set, we have $g_{k_1 I}(x) = z_{k_1 I}(x)$ and $\partial g_{k_1 I}(x) = \partial z_{k_1 I}(x)$ for all $x \in H^\lambda_{k_1}$, and $\|\partial g_{k_1 I}\|_{L^\infty(\mathbb{R}^n)} \leqslant C(n)\lambda$. By Lemma 3.8, we can further suppose that

$$\|g_{k_1 I}\|_{L^\infty(\mathbb{R}^n)} \leqslant \|z_{k_1 I}\|_{L^\infty(H^\lambda_{k_1})} \leqslant \lambda, \quad \|g_{k_1 I}\|_{W^{1,\infty}(\Omega)} \leqslant C(n, \lambda).$$

According to the uniform boundedness of $\{\|g_{k_1 I}\|_{W^{1,\infty}(\Omega)}\}$, there exists a subsequence of $\{g_{k_1 I}\}$ (still denoted by $\{g_{k_1 I}\}$) such that

$$g_{k_1 I} \rightharpoonup v_I \quad \text{weakly* in } W^{1,\infty}(\Omega), \quad \text{as } k_I \to \infty \text{ for all } I.$$

Setting $v = \sum_I v_I e_I$ and $g_{k_1} = \sum_I g_{k_1 I} e_I$, we have

$$\begin{aligned} F(z_{k_1}, \Omega) &= F(g_{k_1}, (\Omega \setminus E_{\varepsilon_1}) \cap H^\lambda_{k_1}) + F(z_{k_1}, E_{\varepsilon_1} \cup (\Omega \setminus H^\lambda_{k_1})) \\ &= F(g_{k_1}, \Omega \setminus E_{\varepsilon_1}) - F(g_{k_1}, (\Omega \setminus E_{\varepsilon_1}) \setminus H^\lambda_{k_1}) + F(z_{k_1}, E_{\varepsilon_1} \cup (\Omega \setminus H^\lambda_{k_1})). \end{aligned}$$

Next we estimate $F(z_{k_1}, \Omega)$ in four steps.

(i) The estimate of $F(g_{k_1}, (\Omega \setminus E_{\varepsilon_1}) \setminus H^\lambda_{k_1})$ and $F(z_{k_1}, E_{\varepsilon_1} \cup (\Omega \setminus H^\lambda_{k_1}))$. Since

$$\operatorname{meas}((\Omega \setminus E_{\varepsilon_1}) \setminus H^\lambda_{k_1}) \leqslant \sum_I \operatorname{meas}((\Omega \setminus E_{\varepsilon_1}) \setminus H^\lambda_{k_1 I}) \leqslant 2^n \min\{\varepsilon_1, \delta_1\},$$

from (H2), (H4) and (2.1), we have

$$
\begin{aligned}
(3.7)\qquad & |F(g_{k_1}, (\Omega \setminus E_{\varepsilon_1}) \setminus H^\lambda_{k_1})| \\
&\leqslant \int_{(\Omega\setminus E_{\varepsilon_1})\setminus H^\lambda_{k_1}} |\overline{A(x, u_0 + g_{k_1}, Du_0 + Dg_{k_1})} Dg_{k_1}| \,\mathrm{d}x \\
&\leqslant C_5 \int_{(\Omega\setminus E_{\varepsilon_1})\setminus H^\lambda_{k_1}} (C_0 |Du_0 + Dg_{k_1}|^{p(x)-1} |Dg_{k_1}| \\
&\quad + C_1 |u_0 + g_{k_1}|^{p(x)-1} |Dg_{k_1}| + G(x) |Dg_{k_1}|) \,\mathrm{d}x \\
&\leqslant C_5 \int_{(\Omega\setminus E_{\varepsilon_1})\setminus H^\lambda_{k_1}} (C_0 |Du_0 + Dg_{k_1}|^{p(x)} + C_0 |Dg_{k_1}|^{p(x)} \\
&\quad + C_1 |u_0 + g_{k_1}|^{p(x)} + C_1 |Dg_{k_1}|^{p(x)} + (G(x))^{p'(x)} + |Dg_{k_1}|^{p(x)}) \,\mathrm{d}x \\
&\leqslant C_5 \int_{(\Omega\setminus E_{\varepsilon_1})\setminus H^\lambda_{k_1}} (C_0 2^{p_+-1} |Du_0|^{p(x)} + 2^{p_+-1} C_0 |Dg_{k_1}|^{p(x)} \\
&\quad + C_0 |Dg_{k_1}|^{p(x)} + 2^{p_+-1} C_1 |u_0|^{p(x)} + 2^{p_+-1} C_1 |g_{k_1}|^{p(x)} \\
&\quad + C_1 |Dg_{k_1}|^{p(x)} + (G(x))^{p'(x)} + |Dg_{k_1}|^{p(x)}) \,\mathrm{d}x \\
&\leqslant 2^{p_+-1} C_5 \eta(\operatorname{meas}((\Omega \setminus E_{\varepsilon_1}) \setminus H^\lambda_{k_1})) \\
&\quad + 2^{p_+} C_5 (C_0 + C_1 + 1) \int_{(\Omega\setminus E_{\varepsilon_1})\setminus H^\lambda_{k_1}} (|g_{k_1}|^{p(x)} + |Dg_{k_1}|^{p(x)}) \,\mathrm{d}x \\
&\leqslant 2^{p_+-1} C_5 \eta(\operatorname{meas}((\Omega \setminus E_{\varepsilon_1}) \setminus H^\lambda_{k_1})) \\
&\quad + 2^{p_+} C_5 (C_0 + C_1 + 1) \int_{(\Omega\setminus E_{\varepsilon_1})\setminus H^\lambda_{k_1}} \Big(\Big(\sum_I |g_{k_1 I}| \Big)^{p(x)} \\
&\quad + \Big(\sum_I |\partial g_{k_1 I}| \Big)^{p(x)} \Big) \,\mathrm{d}x \\
&\leqslant 2^{p_+-1} C_5 \eta(2^n \varepsilon_1) + 2^{p_+} C(n, \Omega, C_0, C_1, C_5) \int_{(\Omega\setminus E_{\varepsilon_1})\setminus H^\lambda_{k_1}} \lambda^{p(x)} \,\mathrm{d}x \\
&\leqslant 2^{p_+-1} C_5 \eta(2^n \varepsilon_1) \\
&\quad + 2^{p_+} C(n, \Omega, C_0, C_1, C_5) \sum_I \int_{(\Omega\setminus E_{\varepsilon_1})\setminus H^\lambda_{k_1}} (M^* z_{k_1 I})^{p(x)} \,\mathrm{d}x \\
&\leqslant 2^{p_+-1} C_5 \eta(2^n \varepsilon_1) + 2^{n+p_+} C(n, \Omega, C_0, C_1, C_5) \varepsilon_1 := V_1(\varepsilon_1)
\end{aligned}
$$

and

$$
\begin{aligned}
& F(z_{k_1}, E_{\varepsilon_1} \cup (\Omega \setminus H^\lambda_{k_1})) \\
&\quad = \int_{E_{\varepsilon_1} \cup (\Omega\setminus H^\lambda_{k_1})} [\overline{A(x, u_0 + z_{k_1}, Du_0 + Dz_{k_1})} Dz_{k_1}]_0 \,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{E_{\varepsilon_1}\cup(\Omega\setminus H^{\lambda}_{k_1})} [\overline{A(x,u_0+z_{k_1},Du_0+Dz_{k_1})}(Du_0+Dz_{k_1})]_0 \,\mathrm{d}x \\
&\quad - \int_{E_{\varepsilon_1}\cup(\Omega\setminus H^{\lambda}_{k_1})} [\overline{A(x,u_0+z_{k_1},Du_0+Dz_{k_1})}Du_0]_0 \,\mathrm{d}x \\
&\geqslant \int_{E_{\varepsilon_1}\cup(\Omega\setminus H^{\lambda}_{k_1})} (C_2|Du_0+Dz_{k_1}|^{p(x)} + C_3|u_0+z_{k_1}|^{p(x)} - h(x)) \,\mathrm{d}x \\
&\quad - C_5 \int_{E_{\varepsilon_1}\cup(\Omega\setminus H^{\lambda}_{k_1})} (C_0|Du_0+Dz_{k_1}|^{p(x)-1}|Du_0| \\
&\quad + C_1|u_0+z_{k_1}|^{p(x)-1}|Du_0| + G(x)|Du_0|) \,\mathrm{d}x \\
&\geqslant (C_2 2^{1-p_+} - C_0C_5\mu 2^{p_+-1}) \int_{E_{\varepsilon_1}\cup(\Omega\setminus H^{\lambda}_{k_1})} |Dz_{k_1}|^{p(x)} \,\mathrm{d}x \\
&\quad - C\eta(\mathrm{meas}(E_{\varepsilon_1}\cup(\Omega\setminus H^{\lambda}_{k_1}))),
\end{aligned}
$$

where $\mu \in (0,1)$ is small enough. Furthermore, if we take $\mu < 2C_2/C_0C_5 4^{p_+}$, then

$$
F(z_{k_1}, E_{\varepsilon_1}\cup(\Omega\setminus H^{\lambda}_{k_1})) \geqslant C_2 2^{-p_+} \int_{E_{\varepsilon_1}\cup(\Omega\setminus H^{\lambda}_{k_1})} |Dz_{k_1}|^{p(x)} \,\mathrm{d}x - V_2(\varepsilon_1), \tag{3.8}
$$

where $V_2(\varepsilon_1) = C\eta(\mathrm{meas}(E_{\varepsilon_1}\cup(\Omega\setminus H^{\lambda}_{k_1})))$ and $V_1(\varepsilon)$, $V_2(\varepsilon)\to 0$ as $\varepsilon\to 0^+$.

Set $U^1_{\varepsilon_1,k_1} = E_{\varepsilon_1}\cup(\Omega\setminus H^{\lambda}_{k_1})$. From (3.7) and (3.8) we have

$$
\begin{aligned}
F(z_{k_1},\Omega) &\geqslant F(g_{k_1},\Omega\setminus E_{\varepsilon_1}) \\
&\quad + C_2 2^{-p_+} \int_{E_{\varepsilon_1}\cup(\Omega\setminus H^{\lambda}_{k_1})} |Dz_{k_1}|^{p(x)} \,\mathrm{d}x - V_1(\varepsilon_1) - V_2(\varepsilon_1).
\end{aligned} \tag{3.9}
$$

(ii) The estimate of $F(g_{k_1},\Omega\setminus E_{\varepsilon_1})$. Set $h_{k_1 I} = g_{k_1 I} - v_{k_1 I}$. Then

$$
h_{k_1 I} \rightharpoonup 0 \quad \text{weakly* in } W^{1,\infty}(\Omega) \quad \text{as } k_1\to\infty \text{ for each } I
$$

and

$$
\|h_{k_1 I}\|_{L^\infty(\mathbb{R}^n)} \leqslant 2\lambda, \quad \|Dh_{k_1 I}\|_{L^\infty(\mathbb{R}^n)} \leqslant 2C(n)\lambda.
$$

Set

$$
G = \bigcup_I G_I
$$

with $G_I = \{x\in\Omega\colon\ v_I \neq 0\}$. According to Acerbi and Fusco [3], we have

$$
\mathrm{meas}(G) \leqslant (2^n+1)\varepsilon_1.
$$

Set $h_{k_1} = \sum\limits_I h_{k_1 I} e_I$, then

$$\begin{aligned} F(g_{k_1}, \Omega \setminus E_{\varepsilon_1}) &= F(h_{k_1}, (\Omega \setminus E_{\varepsilon_1}) \setminus G) + F(g_{k_1}, (\Omega \setminus E_{\varepsilon_1}) \cap H_{k_1}^{\lambda} \cap G) \\ &\quad + F(g_{k_1}, (\Omega \setminus E_{\varepsilon_1}) \cap (G \setminus H_{k_1}^{\lambda})) \\ &= F(h_{k_1}, (\Omega \setminus E_{\varepsilon_1}) \setminus G) + F(z_{k_1}, (\Omega \setminus E_{\varepsilon_1}) \cap H_{k_1}^{\lambda} \cap G) \\ &\quad + F(g_{k_1}, (\Omega \setminus E_{\varepsilon_1}) \cap (G \setminus H_{k_1}^{\lambda})). \end{aligned}$$

Set

$U_{\varepsilon_1}^2 = (\Omega \setminus E_{\varepsilon_1}) \setminus G, \quad U_{\varepsilon_1,k_1}^3 = (\Omega \setminus E_{\varepsilon_1}) \cap H_{k_1}^{\lambda} \cap G,\ U_{\varepsilon_1,k_1}^4 = (\Omega \setminus E_{\varepsilon_1}) \cap (G \setminus H_{k_1}^{\lambda}).$

Similarly to the proof of (3.8), we get

$$F(z_{k_l}, U_{\varepsilon_1,k_1}^3) \geqslant C_2 2^{-p_+} \int_{U_{\varepsilon_1,k_1}^3} |Dz_{k_1}|^{p(x)}\, \mathrm{d}x - V_3(\varepsilon_1).$$

Since on U_{ε_1,k_1}^4 we have

$$\int_{U_{\varepsilon_1,k_1}^4} (|g_{k_1}|^{p(x)} + |Dg_{k_1}|^{p(x)})\, \mathrm{d}x \leqslant C(n,p)(2^n+1)\varepsilon_1,$$

hence similarly to the proof of (3.8) we obtain

$$|F(g_{k_1}, U_{\varepsilon_1,k_1}^4)| \leqslant C((2^n+1)\varepsilon_1 + \eta((2^n+1)\varepsilon_1)) := V_4(\varepsilon_1).$$

Furthermore, we have

$$F(g_{k_1}, \Omega \setminus E_{\varepsilon_1}) \geqslant F(h_{k_1}, U_{\varepsilon_1}^2) + C_2 2^{-p_+} \int_{U_{\varepsilon_1,k_1}^3} |Dz_{k_1}|^{p(x)}\, \mathrm{d}x - V_3(\varepsilon_1) - V_4(\varepsilon_1).$$

Denote $U_{\varepsilon_1,k_1}^5 = U_{\varepsilon_1,k_1}^1 \cup U_{\varepsilon_1,k_1}^3$. From (3.8) we have

$$F(z_{k_1}, \Omega) \geqslant F(h_{k_1}, U_{\varepsilon_1}^2) + C_2 2^{-p_+} \int_{U_{\varepsilon_1,k_1}^5} |Dz_{k_1}|^{p(x)}\, \mathrm{d}x - V_5(\varepsilon_1), \tag{3.10}$$

where $V_5(\varepsilon_1) = \sum\limits_{j=1}^{4} V_j(\varepsilon_1)$.

Choose an open set $\Omega' \subset \Omega$ which contains $U_{\varepsilon_1}^2$ such that

$$|F(h_{k_1}, \Omega') - F(h_{k_1}, U_{\varepsilon_1}^2)| \leqslant \varepsilon_1.$$

From (3.10) we have

$$F(z_{k_1},\Omega) \geqslant F(h_{k_1},\Omega') + C_2 2^{-p_+} \int_{U^5_{\varepsilon_1,k_1}} |Dz_{k_1}|^{p(x)}\,\mathrm{d}x - V_5(\varepsilon_1).$$

Approximate Ω' by a hypercube with edges parallel to the coordinate axes, i.e. construct

$$H_j \subset \Omega',$$
$$\operatorname{meas}(\Omega' \setminus H_j) \to 0,\ j \to \infty,$$
$$H_j = \bigcup_{m=1}^{s_j} D_{j,m},$$
$$\operatorname{meas}(D_{j,m}) = 2^{-nj}, \quad 1 \leqslant m \leqslant s_j.$$

Let $j > 0$ be large enough such that for all $k_1 > 0$ we have

$$|F(h_{k_1},\Omega') - F(h_{k_1},H_j)| \leqslant \varepsilon_1,\ \int_{\Omega'\setminus H_j} |Dh_{k_1}|^{p(x)}\,\mathrm{d}x < \varepsilon_1 \tag{3.11}$$

and

$$\operatorname{meas}(\Omega' \setminus H_j) < \min\{\varepsilon_1, \delta_1\}.$$

Then

$$F(z_{k_1},\Omega) \geqslant F(h_{k_1},H_j) + C_2 2^{-p_+} \int_{U^5_{\varepsilon_1,k_1}} |Dz_{k_1}|^{p(x)}\,\mathrm{d}x - V_5(\varepsilon_1) - 2\varepsilon_1. \tag{3.12}$$

(iii) The estimate of $F(h_{k_1},H_j)$. Let

$$M = 2^{n+1}C(n)\lambda \geqslant \||Dh_{k_1}|\|_{L^\infty(\Omega)}$$

and let $\alpha > 0$ be large enough such that for $E = \{x \in \Omega' \colon\ a(x) \leqslant \alpha\}$

$$\operatorname{meas}(\Omega' \setminus E) \leqslant \frac{\varepsilon_1}{M}, \quad \int_{\Omega'\setminus E} a(x)\,\mathrm{d}x \leqslant \varepsilon_1,$$

where $a(x) = 2^{p_+-1}C_5(C_0|Du_0(x)|^{p(x)} + C_1|u_0(x)|^{p(x)} + (G(x))^{p'(x)})$.

For every $x \in \Omega$, $s \in \mathrm{C}\ell_n$, $\xi \in \mathrm{C}\ell_n$, define

$$f(x,s,\xi) = [\overline{A(x,u_0+s,Du_0+\xi)}\xi]_0.$$

By Lemma 3.5 and (H1), there exists a compact subset $K \subset H_j$ such that $f(x,s,\xi)$ is continuous on $K \times \mathrm{C}\ell_n \times \mathrm{C}\ell_n$ and

$$\operatorname{meas}(H_j \setminus K) \leqslant \frac{\varepsilon_1}{\alpha + M}.$$

Divide each $D_{j,m}$ into 2^{nl} hypercubes $Q^l_{h,m,j}$ with edge length 2^{-jl}, $1 \leqslant h \leqslant 2^{nl}$. For all j, m, l, h, take $x^l_{h,m,j} \in Q^l_{h,m,j} \cap K \cap E$ (if this set is empty, take $x^l_{h,m,j} \in Q^l_{h,m,j}$) such that

$$a(x^l_{h,m,j}) \operatorname{meas}(Q^l_{h,m,j}) \leqslant \int_{Q^l_{h,m,j}} a(x)\,\mathrm{d}x.$$

Then we have by (H2) and (2.1)

$$\begin{aligned}
(3.13)\qquad F(h_{k_1}, H_j) &= F(h_{k_1}, H_j \cap K \cap E) + F(h_{k_1}, H_j \setminus E) + F(h_{k_1}, (H_j \cap E) \setminus K) \\
&\geqslant F(h_{k_1}, H_j \cap K \cap E) - \int_{H_j \setminus E} a(x)\,\mathrm{d}x - \int_{(H_j \cap E)\setminus K} a(x)\,\mathrm{d}x \\
&\quad - 2^{p_+} C_5 (1 + C_0 + C_1) \cdot \biggl(\int_{H_j \setminus E} (|Dh_{k_1}|^{p(x)} + |h_{k_1}|^{p(x)})\,\mathrm{d}x \\
&\quad + \int_{(H_j \cap E)\setminus K} (|Dh_{k_1}|^{p(x)} + |h_{k_1}|^{p(x)})\,\mathrm{d}x \biggr) \\
&= F(h_{k_1}, H_j \cap K \cap E) - V_6(\varepsilon_1) \\
&= a^j_{k_1} + b^{l,j}_{k_1} + c^{l,j}_{k_1} + d^{l,j}_{k_1} - V_6(\varepsilon_1)
\end{aligned}$$

where

$$\begin{aligned}
a^j_{k_1} &= \int_{H_j \cap K \cap E} (f(x, h_{k_1}(x), Dh_{k_1}(x)) - f(x, 0, Dh_{k_1}(x)))\,\mathrm{d}x, \\
b^{l,j}_{k_1} &= \sum_{h,m} \int_{Q^l_{h,m,j} \cap K \cap E} (f(x, 0, Dh_{k_1}(x)) - f(x^l_{h,m,j}, 0, Dh_{k_1}(x)))\,\mathrm{d}x, \\
c^{l,j}_{k_1} &= \sum_{h,m} \int_{Q^l_{h,m,j}} f(x^l_{h,m,j}, 0, Dh_{k_1}(x))\,\mathrm{d}x, \\
d^{l,j}_{k_1} &= -\sum_{h,m} \int_{Q^l_{h,m,j} \setminus (K \cap E)} f(x^l_{h,m,j}, 0, Dh_{k_1}(x))\,\mathrm{d}x.
\end{aligned}$$

Since $h_{k_1 I} \rightharpoonup 0$ weakly* in $W^{1,\infty}(\Omega)$, we get that $\|h_{k_1 I}\|_{L^\infty(\Omega)} \to 0$ as $k_1 \to \infty$ for each I. Thus

$$R^{k_1,l}_{h,m,j} := \||h_{k_1}|\|_{L^\infty(Q^l_{h,m,j})} \to 0 \quad \text{as } k_1 \to \infty \text{ for fixed } l.$$

Since f is uniformly continuous on bounded subsets of $K \times \mathrm{C}\ell_n(\Omega) \times \mathrm{C}\ell_n(\Omega)$, we have

$$\lim_{k_1 \to \infty} a^j_{k_1} = 0.$$

Because of $x^l_{h,m,j} \in Q^l_{h,m,j}$, we obtain for any $x \in Q^l_{h,m,j}$

$$|x - x^l_{h,m,j}| \leqslant \sqrt{n} 2^{-lj} \to 0 \quad \text{as } l \to \infty.$$

Using the pointwise convergence of $u_0(x^l_{h,m,j})$ and $Du_0(x^l_{h,m,j})$, we have

$$\lim_{l\to\infty} b^{l,j}_{k_1} = 0$$

uniformly with respect to k_1 for fixed j, and

$$\begin{aligned}
|d^{l,j}_{k_1}| &\leqslant \sum_{h,m} \int_{Q^l_{h,m,j}\setminus(K\cap E)} |f(x^l_{h,m,j}, 0, Dh_{k_1}(x))| \,\mathrm{d}x \\
&\leqslant \sum_{h,m} \int_{Q^l_{h,m,j}\setminus(K\cap E)} |\overline{A(x^l_{h,m,j}, 0, Du_0(x^l_{h,m,j}) + Dh_{k_1}(x))} Dh_{k_1}(x)| \,\mathrm{d}x \\
&\leqslant C_5 \sum_{h,m} \int_{Q^l_{h,m,j}\setminus(K\cap E)} (C_0 |Du_0(x^l_{h,m,j}) + Dh_{k_1}(x)|^{p(x)-1} |Dh_{k_1}(x)| \\
&\qquad + C_1 |u_0(x^l_{h,m,j})|^{p(x)-1} |Dh_{k_1}(x)| + G(x^l_{h,m,j}) |Dh_{k_1}(x)|) \,\mathrm{d}x \\
&\leqslant \sum_{h,m} \int_{Q^l_{h,m,j}\setminus(K\cap E)} (a(x^l_{h,m,j}) + 2^{p_+} C_5 (1 + C_0 + C_1) M) \,\mathrm{d}x \\
&= \sum_{h,m} \int_{(Q^l_{h,m,j}\cap E)\setminus K} (a(x^l_{h,m,j}) + 2^{p_+} C_5 (1 + C_0 + C_1) M) \,\mathrm{d}x \\
&\qquad + \sum_{h,m} \int_{Q^l_{h,m,j}\setminus E} (a(x^l_{h,m,j}) + 2^{p_+} C_5 (1 + C_0 + C_1) M) \,\mathrm{d}x \\
&\leqslant C(\alpha + M) \operatorname{meas}((H_j \cap E) \setminus K) + \int_{H_j\setminus E} (a(x) + 2^{p_+} C_5 (1 + C_0 + C_1) M) \,\mathrm{d}x \\
&\leqslant C\varepsilon_1.
\end{aligned}$$

From (3.6), we have

$$F(z_{k_1}, \Omega) = \int_\Omega \Big[\overline{A(x, u_0 + z_{k_1}, Du_0 + Dz_{k_1})} Dz_{k_1}\Big]_0 \,\mathrm{d}x \to 0 \quad \text{as } k_1 \to \infty.$$

Now we suppose that l is large enough so that $|b^{l,j}_{k_1}| \leqslant \varepsilon_1$ for each k_1 and there exists $\bar{k}_1$ such that $F(z_{k_1}, \Omega) < \varepsilon_1$ for $k_1 > \bar{k}_1$. Therefore we have

$$\begin{aligned}
\varepsilon_1 &> F(z_{k_1}, \Omega) \\
&\geqslant c^{l,j}_{k_1} + 2^{-p_+} C_2 \int_{U^5_{\varepsilon_1,k_1}} |Dz_{k_1}|^{p(x)} \,\mathrm{d}x - C\varepsilon_1 - V_5(\varepsilon_1) - V_6(\varepsilon_1) - 3\varepsilon_1.
\end{aligned} \tag{3.14}$$

(iv) The estimate of $c^{l,j}_{k_1}$. Define a hypercube $E^{k_1,l}_{h,m,j}$ contained in $Q^l_{h,m,j}$ with edge length $2^{-jl} - 2R^{k_1,l}_{h,m,j}$ such that

$$\operatorname{dist}(\partial Q^l_{h,m,j}, E^{k_1,l}_{h,m,j}) = R^{k_1,l}_{h,m,j}.$$

Next define

$$\psi_{k_1 I}(x) = \begin{cases} 0, & x \in \partial Q^l_{h,m,j}, \\ h_{k_1 I}(x), & x \in E^{k_1,l}_{h,m,j}. \end{cases}$$

Then $\psi_{k_1 I}$ is a Lipschitz mapping on the set where it is defined and its Lipschitz constant is not greater than $2C(n)\lambda$. By Lemma 3.10, $\psi_{k_1 I}$ can be extended to the whole $Q^l_{h,m,j}$, where it is also a Lipschiz mapping with the same Lipschitz constant. We still denote the extension by $\psi_{k_1 I}$ and suppose that it is defined on the whole H_j. Then by [5]

$$\partial\psi_{k_1 I} - \partial h_{k_1 I} \to 0, \quad \text{a.e. on } H_j.$$

So there exists a $\bar{\bar{k}}_1 > \bar{k}_1$ such that for all $\bar{k}_1 > \bar{\bar{k}}_1$ we have

$$\int_{H_j} |D\psi_{k_1} - Dh_{k_1}|^{p(x)}\,\mathrm{d}x \leqslant \frac{\varepsilon_1}{2},$$

and

$$\left|\sum_{h,m}\int_{Q^l_{h,m,j}} f(x^l_{h,m,j}, 0, Dh_{k_1}(x)) - f(x^l_{h,m,j}, 0, D\psi_{k_1}(x))\,\mathrm{d}x\right| \leqslant \frac{\varepsilon_1}{2}.$$

We obtain by (H5)

$$\begin{aligned} c^{l,j}_{k_1} &= \sum_{h,m}\int_{Q^l_{h,m,j}} f(x^l_{h,m,j}, 0, Dh_{k_1}(x))\,\mathrm{d}x \\ &\geqslant \sum_{h,m}\int_{Q^l_{h,m,j}} f(x^l_{h,m,j}, 0, D\psi_{k_1}(x))\,\mathrm{d}x - \frac{\varepsilon_1}{2} \\ &= \sum_{h,m}\int_{Q^l_{h,m,j}} [\overline{A(x^l_{h,m,j}, u_0(x^l_{h,m,j}), Du_0(x^l_{h,m,j}) + D\psi_{k_1}(x))}D\psi_{k_1}(x)]_0\,\mathrm{d}x - \frac{\varepsilon_1}{2} \\ &\geqslant \sum_{h,m}\int_{Q^l_{h,m,j}} C_4|D\psi_{k_1}(x)|^{p(x)}\,\mathrm{d}x - \frac{\varepsilon_1}{2} \\ &\geqslant \frac{C_4}{2^{p_+-1}}\int_{H_j} |Dh_{k_1}(x)|^{p(x)}\,\mathrm{d}x - \frac{(C_4+1)\varepsilon_1}{2}. \end{aligned}$$

Thus by (3.14) we obtain, as $k_1 > \bar{\bar{k}}_1$,

$$\varepsilon_1 \geqslant 2^{-p^+}C_2\int_{U^5_{\varepsilon_1,k_1}} |Dz_{k_1}|^{p(x)}\,\mathrm{d}x + 2^{1-p^+}C_4\int_{H_j} |Dh_{k_1}|^{p(x)}\,\mathrm{d}x - V_7(\varepsilon_1),$$

where $V_7(\varepsilon_1) = V_5(\varepsilon_1) + V_6(\varepsilon_1) + (3+C)\varepsilon_1 + \frac{1}{2}(1+C_4)\varepsilon_1$.

Set

$$k(\varepsilon_1) = \frac{V_7(\varepsilon_1) + \varepsilon_1}{\min\{2^{-p^+}C_2, 2^{1-p^+}C_4\}}.$$

Then we have as $k_1 > \bar{\bar{k}}_1$

$$\int_{U^5_{\varepsilon_1,k_1}} |Dz_{k_1}|^{p(x)}\,\mathrm{d}x + \int_{H_j} |Dh_{k_1}|^{p(x)}\,\mathrm{d}x \leqslant k(\varepsilon_1). \tag{3.15}$$

Hence we get from (3.11) and (3.15) that

$$\int_{U^5_{\varepsilon_1,k_1}} |Dz_{k_1}|^{p(x)}\,\mathrm{d}x \leqslant k(\varepsilon_1),$$

and

$$\int_{\Omega'} |Dh_{k_1}|^{p(x)}\,\mathrm{d}x = \int_{H_j} |Dh_{k_1}|^{p(x)}\,\mathrm{d}x + \int_{\Omega'\setminus H_j} |Dh_{k_1}|^{p(x)}\,\mathrm{d}x \leqslant k(\varepsilon_1) + \varepsilon_1.$$

According to the definition of Ω', we have

$$\int_{U^2_{\varepsilon_1}} |Dg_{k_1}|^{p(x)}\,\mathrm{d}x = \int_{U^2_{\varepsilon_1}} |Dh_{k_1}|^{p(x)}\,\mathrm{d}x \leqslant k(\varepsilon_1) + \varepsilon_1.$$

Since $Dg_{k_1}(x) = Dz_{k_1}(x)$ for each $x \in H^\lambda_{k_1}$, we obtain

$$\int_{U^2_{\varepsilon_1}\cap H^\lambda_{k_1}} |Dz_{k_1}|^{p(x)}\,\mathrm{d}x \leqslant k(\varepsilon_1) + \varepsilon_1.$$

By the definition of $U^2_{\varepsilon_1}$ and $U^5_{\varepsilon_1,k_1}$, it is immediate that

$$\Omega = (U^2_{\varepsilon_1} \cap H^\lambda_{k_1}) \cup U^5_{\varepsilon_1,k_1},$$

which implies that

$$\int_{\Omega} |Dz_{k_1}|^{p(x)}\,\mathrm{d}x \leqslant 2k(\varepsilon_1) + \varepsilon_1 := O(\varepsilon_1),$$

where $O(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. For $\varepsilon_2 > 0$ and the sequence $\{z_{k_1}\}$, repeating the above argument we can extract a subsequence of $\{z_{k_1}\}$, denote it by $\{z_{k_2}\}$, such that

$$\int_{\Omega} |Dz_{k_2}|^{p(x)}\,\mathrm{d}x \leqslant O(\varepsilon_2)$$

whenever $k_2 > \bar{\bar{k}}_2$ for some $\bar{\bar{k}}_2$. If $\{z_{k_j}\}$ has been obtained, repeating the above process we can extract a subsequence of $\{z_{k_j}\}$, denote it by $\{z_{k_{j+1}}\}$, such that

$$\int_{\Omega} |Dz_{k_{j+1}}|^{p(x)}\,\mathrm{d}x \leqslant O(\varepsilon_{j+1})$$

whenever $k_{j+1} > \bar{\bar{k}}_{j+1}$ for some $\bar{\bar{k}}_{j+1}$. Finally, by a diagonal argument we get a subsequence $\{z_{k_i}\}$ which satisfies

$$\int_\Omega |Dz_{k_i}|^{p(x)}\,\mathrm{d}x \to 0 \quad \text{as } i \to \infty.$$

By Remark 2.2, we get

$$\||Dz_{k_i}|\|_{L^{p(x)}(\Omega)} \to 0 \quad \text{as } i \to \infty.$$

Therefore, by Remark 2.3 we have

$$z_{k_i} \to 0 \quad \text{strongly in } V.$$

Now we have completed the proof of Theorem 3.1. □

Remark 3.1. In the case that u is a real-valued function, the scalar part of elliptic systems (1.1) implies that

$$\int_\Omega \left[\sum_{i=1}^n A_i(x,u,\partial u)\frac{\partial\varphi}{\partial x_i} - B(x,u,\partial u)\varphi\right]\mathrm{d}x = 0$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$, where $A = (A_1, \dots, A_n)$. So in this case (3.1) can be identified with the equation

$$-\operatorname{div}(A(x,u,\partial u)) = B(x,u,\partial u).$$

Hence by Theorem 3.1, we obtain the existence of a weak solution in $W_0^{1,p(x)}(\Omega)$ for the above equation under the corresponding assumptions.

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