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ADAPTIVE FINITE-TIME SYNCHRONIZATION OF CROSS-STRICT FEEDBACK HYPERCHAOTIC SYSTEMS WITH PARAMETER UNCERTAINTIES

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This paper is concerned with the finite-time synchronization problem for a class of cross-strict feedback underactuated hyperchaotic systems. Using finite-time control and backstepping control approaches, a new robust adaptive synchronization scheme is proposed to make the synchronization errors of the systems with parameter uncertainties zero in a finite time. Appropriate adaptive laws are derived to deal with the unknown parameters of the systems. The proposed method can be applied to a variety of chaotic systems which can be described by the so-called cross-strict feedback systems. Numerical simulations are given to demonstrate the efficiency of the proposed control scheme.

Keywords: finite-time synchronization, cross-strict feedback hyperchaotic system, backstepping, adaptive control

Classification: 34H10, 34C28, 34D06, 34K35

1. INTRODUCTION

Since Pecora et al. introduced a method to synchronize two identical chaotic systems with different initial conditions in 1990 [19], chaos synchronization has attracted a great deal of attention from various fields during the last two decades [5, 8, 29, 40]. Boccaletti et al. gave a general review of major ideas involved in the synchronization field of chaotic systems, and discussed in detail several types of synchronization features [2]. Different synchronization control methods, such as linear and nonlinear feedback control [24,33],adaptive control [10,30,39], backstepping [3,22,23,28,37,38], neural network [11], observer-based control scheme [4], sliding mode control methods [6, 9, 21, 25, 31, 32, 34], and so on have been successfully applied to the chaos synchronization.

Most methods guarantee the asymptotic stability of the synchronization error dynamics, which means that the trajectories of the slave system can reach the trajectories of the master system as $t \to \infty$. From a practical point of view, however, it is more valuable that the synchronization objective is realized in a finite time [1]. In recent years, some researchers have applied finite-time control techniques, such as nonsingular terminal sliding mode control method [25], CLF-based method [26, 36], sliding mode control [1] and the finite-time stability theory based methods [27], to realize synchronization. However, n control inputs are required to achieve the asymptotic stability in their works.

Backstepping is one of the most important and popular approaches in controller design of nonlinear control systems [12]. As a result, many backstepping-based methods have been presented for the synchronization of chaotic systems. Wang et al. developed a new control law to achieve chaos control for a vibro-impact system on the basis of the backstepping technique [18]. Li et al. used active control to achieve synchronization between two different chaotic systems [13]. However, the ordinary backstepping technique can only be used to control strict feedback systems, which severely restricts its application in synchronization control of chaotic systems [14]. A cross active backstepping control method was proposed by Wang et al. for a class of cross-strict feedback systems without unknown parameters using n control inputs, which extended the application areas of backstepping technique [28]. Li et al. gave robust backstepping control methods for cross-strict feedback systems with less control inputs [14,15], but it couldn't guarantee that all the error states converge to zero.

Compared with the existing results in the literature, the following advantages make our approach attractive. First of all, combining backstepping, adaptive control and finite-time control theory, a new synchronization method is presented for a wide class of underactuated nonlinear systems. As far as we know, the existing methods can't solve this design problem. Secondly, it needs only two controllers to realize synchronization between 4-dimensional chaotic systems in a finite time. To our knowledge, there are no results in this direction. And then, it achieves finite-time convergence in one stage rather than two stages as in Ref. [1]. Finally, it guarantees that all the error states are driven to zero in a finite time even for the systems with unknown parameters, and avoids the possible singularity of the controllers in Ref. [1].

In this paper, a backstepping-based finite-time synchronization scheme is proposed for a class of hyperchaotic systems. The hyperchaotic systems considered here are the hyperchaotic Rössler systems [17]. The rest of the paper is organized as follows: In Section 2, we introduce the hyperchaotic Rössler system and preliminary lemmas. In Section 3, the proposed finite-time controller is designed to synchronize two identical cross-strict feedback hyperchaotic systems. We give the simulation results and the conclusions in Sections 4 and 5 respectively.

2. SYSTEM DESCRIPTION

The hyperchaotic Rössler system is emanative and initial values should be in the domain of attraction [36]. It can be described by

$$\begin{aligned}\dot{x} &= -y - z,\\ \dot{y} &= x + ay + w,\\ \dot{z} &= b + xz,\\ \dot{w} &= cz + dw\end{aligned} \tag{1}$$

where a, b, c and d are parameters, and Eq. (1) (master system) shows hyperchaotic state if the parameters are appropriately chosen. For example, the parameters are often selected with $a = 0.25$, $b = 3$, $c = -0.5$ and $d = 0.05$ to induce hyperchaotic state in

Eq. (1). The slave system is given with

$$\begin{aligned} \dot{x}_1 &= -y_1 - z_1, \\ \dot{y}_1 &= x_1 + a y_1 + w_1, \\ \dot{z}_1 &= b + x_1 z_1 + u_3, \\ \dot{w}_1 &= c z_1 + d w_1 + u_4 \end{aligned} \tag{2}$$

where u_3, u_4 are control laws to be designed.

Subtracting Eq. (1) from Eq. (2) yields the error dynamical system as follows

$$\begin{aligned} \dot{e}_{10} &= -e_{20} - e_{30}, \\ \dot{e}_{20} &= e_{10} + a e_{20} + e_{40}, \\ \dot{e}_{30} &= z e_{10} + x_1 e_{30} + u_3, \\ \dot{e}_{40} &= c e_{30} + d e_{40} + u_4 \end{aligned} \tag{3}$$

where $e_{10} = x_1 - x$, $e_{20} = y_1 - y$, $e_{30} = z_1 - z$ and $e_{40} = w_1 - w$.

Assumption 2.1. The unknown parameters a, c and d are norm bounded, i. e.

$$|a| \le d_a, \ \ |c| \le d_c, \ \ |d| \le d_d \tag{4}$$

where d_a, d_c and d_d are known positive constants.

Remark 2.2. Assumption 2.1 is easy to be satisfied. Generally speaking, we can choose bigger numbers d_a, d_c and d_d to satisfy the inequality (4).

Definition 2.3. (Aghababa et al. [1]) Consider the master and slave chaotic systems described by Eqs. (1) and (2), respectively. If there exists a constant $T = T(e(0)) > 0$, such that

$$\lim_{t \to T} \|e(t)\| = 0 \tag{5}$$

and$\|e(t)\| \equiv 0$, if $t \ge T$, then the chaos synchronization between the systems (1) and (2) is achieved in a finite time.

Lemma 2.4. (Aghababa et al. [1]) Consider the system

$$\dot{x} = f(x), \ f(0) = 0, \ x \in R^n, \tag{6}$$

where $f : D \to R^n$ is continuous on an open neighborhood $D \in R^n$. Suppose there exists a continuous differential positive-definite function $V(x) : D \to R$, and real numbers $p > 0$, $0 < \eta < 1$, such that

$$\dot{V}(x) + pV^{\eta}(x) \le 0, \ \forall x \in D. \tag{7}$$

Then, the origin of system (6) is a locally finite-time stable equilibrium, and the settling time depending on the initial state $x(0) = x_0$, satisfies

$$T(x_0) \le \frac{V^{1-\eta}(x_0)}{p(1-\eta)}. \tag{8}$$

In addition, if $D = R^n$ and $V(x)$ is radially unbounded (i. e. $V(x) \to +\infty$ as $\|x\| \to +\infty$), then the origin is a globally finite-time stable equilibrium of system (6).

Lemma 2.5. (Aghababa et al. [1]) Suppose $a_1, a_2, \cdots, a_n$ and $0 < q < 2$ are all real numbers, then the following inequality holds

$$|a_1|^q + |a_2|^q + \cdots + |a_n|^q \geq \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)^{q/2}. \tag{9}$$

Obviously, the error system is not a strict feedback system and the backstepping technique cannot be used. But after careful examination, we notice that this system is made up of two coupled subsystems as follows

$$\begin{cases} \dot{e}_{10} = -e_{20} - e_{30}, \\ \dot{e}_{30} = ze_{10} + x_1 e_{30} + u_3, \end{cases} \quad \begin{cases} \dot{e}_{20} = e_{10} + ae_{20} + e_{40}, \\ \dot{e}_{40} = ce_{30} + de_{40} + u_4 \end{cases} \tag{10}$$

and each subsystem is of strict feedback form. This kind of system is called cross-strict feedback system [28]. Since each subsystem of this hyperchaotic system is of strict feedback form, backstepping technique can be applied to each subsystem. The following design procedure is based on this idea.

3. SYNCHRONIZATION OF TWO IDENTICAL CROSS-STRICT FEEDBACK HYPERCHAOTIC SYSTEMS

Consider two identical cross-strict feedback hyperchaotic systems (1) and (2) from different initial states. The aim of controller design is to determine appropriate u_3 and u_4 such that

$$\lim_{t \to T} e_{i0} = 0, \ i = 1, \ 2, \ 3, \ 4 \tag{11}$$

holds.

In the following, we will design the active controller u_3 and u_4 on the basis of adaptive backstepping technique.

Remark 3.1. Instead of using 4 control inputs as in Ref. [28], only 2 control inputs are used to achieve synchronization of cross-strict feedback hyperchaotic systems, which is more difficult than the case using 4 control inputs. In addition, the proposed method guarantees that all the error states converge to zero in a finite time.

Now we are ready to give the design steps.

Step 1. Let $e_1 = e_{10}$. Its derivative is

$$\dot{e}_1 = -e_{20} - e_{30}. \tag{12}$$

Let $e_3 = e_{30} - \alpha_1$, where α_1 is the virtual control law. Define Lyapunov function

$$V_1 = \frac{1}{2} e_1^2. \tag{13}$$

Taking the time derivative of Eq. (13) gives

$$\begin{aligned} \dot{V}_1 &= e_1 \dot{e}_1 \\ &= e_1 \left(-e_{20} - e_{30}\right) \\ &= e_1 \left(-\alpha_1 - e_3 - e_{20}\right). \end{aligned} \tag{14}$$

Choose the virtual control law as

$$\alpha_1 = -e_{20} + c_1 sgn\left(e_1\right)\left|e_1\right|^{\alpha} + k_1 e_1 \tag{15}$$

where $k_1 > 0$ and $c_1 > 0$ are constants, and $0 < \alpha < 1$ is a constant.

Substituting Eq. (15) into Eq. (14) yields

$$\dot{V}_1 = -k_1 e_1^2 - e_1 e_3 - c_1 \left|e_1\right|^{1+\alpha}. \tag{16}$$

Step 2. Taking the derivative of e_3 gives

$$\dot{e}_3 = z e_{10} + x_1 e_{30} + u_3 - \dot{\alpha}_1. \tag{17}$$

Choosing the Lyapunov function as $V_3 = V_1 + \frac{1}{2}e_3^2$, and taking its derivative, we have

$$\dot{V}_3 = -k_1 e_1^2 - e_1 e_3 - c_1 \left|e_1\right|^{1+\alpha} + e_3\left(z e_{10} + x_1 e_{30} + u_3 - \dot{\alpha}_1\right). \tag{18}$$

Let

$$u_3 = -z e_{10} + e_1 - x_1 e_{30} + \hat{\dot{\alpha}}_1 - k_3 e_3 - c_3 sgn\left(e_3\right)\left|e_3\right|^{\alpha} - \rho_3 sgn\left(e_3\right) \tag{19}$$

where $k_3 > 0$, $\rho_3 > 0$ and $c_3 > 0$ are constants, and $\hat{\dot{\alpha}}_1$ is the output of a tracking differentiator [7] with α_1 as its input.

According to Ref. [7], we can make the following Assumption.

Assumption 3.2. $\max\left(\left|\hat{\dot{\alpha}}_1 - \dot{\alpha}_1\right|\right) \le \delta_{\alpha_1}$, where $\delta_{\alpha_1} > 0$.

Choosing $\rho_3 > \delta_{\alpha_1}$ and substituting Eq. (19) into Eq. (18), we have

$$\begin{aligned} \dot{V}_3 &= -k_1 e_1^2 - e_1 e_3 - c_1 \left|e_1\right|^{1+\alpha} \\ &\quad + e_3\left[e_1 - k_3 e_3 - c_3 sgn\left(e_3\right)\left|e_3\right|^{\alpha} + \hat{\dot{\alpha}}_1 - \dot{\alpha}_1 - \rho_3 sgn\left(e_3\right)\right] \\ &\le -k_1 e_1^2 - c_1\left|e_1\right|^{1+\alpha} - k_3 e_3^2 - c_3\left|e_3\right|^{1+\alpha}. \end{aligned} \tag{20}$$

According to Lemma 2.5, the inequality (20) can be rewritten as

$$\begin{aligned} \dot{V}_3 &\le -c_1\left|e_1\right|^{1+\alpha} - c_3\left|e_3\right|^{1+\alpha} \\ &\le -c_{13}\left(\left|e_1\right|^{1+\alpha} + \left|e_3\right|^{1+\alpha}\right) \\ &\le -c_{13}\left(\left|e_1\right|^2 + \left|e_3\right|^2\right)^{\frac{1+\alpha}{2}} \\ &\le -c_{13} V_3^{\frac{1+\alpha}{2}} \end{aligned} \tag{21}$$

where $c_{13} = \min\left\{c_1, c_3\right\}$.

According to Lemma 2.4, $e_1 \to 0$, $e_3 \to 0$ as $t \to T_1$, where $T_1 = \frac{V_{30}^{(1-\alpha)/2}}{c_{13}(1-\alpha)/2}$, and V_{30} denotes the initial value of V_3.

Step 3. Let $e_2 = e_{20}$. Its derivative is

$$\dot{e}_2 = e_1 + ae_2 + e_{40}. \tag{22}$$

Let $e_4 = e_{40} - \alpha_2$, where α_2 is the virtual control law to be designed. Define Lyapunov function

$$V_2 = \frac{1}{2}e_2^2 + \frac{1}{2}\tilde{a}^2 \tag{23}$$

where $\tilde{a} = \hat{a} - a$ and $\hat{a}$ is the estimation of a. Taking the time derivative of Eq. (23) gives

$$\begin{aligned} \dot{V}_2 &= e_2\dot{e}_2 + \tilde{a}\dot{\hat{a}} \\ &= e_2\left(e_1 + ae_2 + e_{40}\right) + \tilde{a}\dot{\hat{a}} \\ &= e_2\left(e_1 + ae_2 + e_4 + \alpha_2\right) + \tilde{a}\dot{\hat{a}}. \end{aligned} \tag{24}$$

Choose the virtual control law as

$$\alpha_2 = \begin{cases} -e_1 - \hat{a}e_2 - c_2 sgn\left(e_2\right)\left|e_2\right|^{\alpha} - k_2 e_2 - \mu_2\left(\left|\hat{a}\right| + d_a\right)^{1+\alpha}/e_2, & \text{if } \ e_2 \neq 0, \\ -e_1, & \text{if } \ e_2 = 0, \end{cases} \tag{25}$$

where $k_2 > 0$ and $c_2 > 0$ are constants.

If $e_2 \neq 0$, substituting Eq. (25) into Eq. (24) gives

$$\dot{V}_2 = -k_2 e_2^2 + e_2 e_4 - c_2\left|e_2\right|^{1+\alpha} - \tilde{a}e_2^2 + \tilde{a}\dot{\hat{a}} - \mu_2\left(\left|\hat{a}\right| + d_a\right)^{1+\alpha}. \tag{26}$$

Choosing the updating law as

$$\dot{\hat{a}} = e_2^2. \tag{27}$$

Substituting Eq. (27) into Eq. (26) yields

$$\dot{V}_2 = -k_2 e_2^2 + e_2 e_4 - c_2\left|e_2\right|^{1+\alpha} - \mu_2\left(\left|\hat{a}\right| + d_a\right)^{1+\alpha}. \tag{28}$$

If $e_2 = 0$, it is obvious that $\dot{V}_2 = 0$.

Step 4. Taking the derivative of e_4 gives

$$\dot{e}_4 = ce_{30} + de_{40} + u_4 - \dot{\alpha}_2. \tag{29}$$

Choosing the Lyapunov function as $V_4 = V_2 + \frac{1}{2}e_4^2 + \frac{1}{2}\tilde{c}^2 + \frac{1}{2}\tilde{d}^2$, and taking its derivative, we have

$$\dot{V}_4 = \dot{V}_2 + e_4\left(ce_{30} + de_{40} + u_4 - \dot{\alpha}_2\right) + \tilde{c}\dot{\hat{c}} + \tilde{d}\dot{\hat{d}} \tag{30}$$

where $\tilde{c} = \hat{c} - c$, $\tilde{d} = \hat{d} - d$, $\hat{c}$ is the estimation of c, and $\hat{d}$ is the estimation of d.

Let

$$u_4 = \begin{cases} -e_2 - \hat{c}e_{30} - \hat{d}e_{40} + \dot{\hat{\alpha}}_2 - k_4 e_4 - c_4 sgn\left(e_4\right)\left|e_4\right|^{\alpha} - \rho_4 sgn\left(e_4\right) \\ \qquad - \mu_2\left[\left(\left|\hat{c}\right| + d_c\right)^{1+\alpha} + \left(\left|\hat{d}\right| + d_d\right)^{1+\alpha}\right]\Big/ e_4, & \text{if } e_4 \neq 0, \\ -e_2 - \hat{c}e_{30} - \hat{d}e_{40} + \dot{\hat{\alpha}}_2, & \text{if } e_4 = 0, \end{cases} \tag{31}$$

where $\hat{\dot{\alpha}}_2$ is the output of a tracking differentiator [7] with α_2 as its input, $k_4 > 0$ and $c_4 > 0$ are constants, and $\rho_4 > 0$.

Similarly, we make the following Assumption.

Assumption 3.3. $\max\left(|\hat{\dot{\alpha}}_2 - \dot{\alpha}_2|\right) \le \delta_{\alpha_2}$, where $\delta_{\alpha_2} > 0$.

Remark 3.4. Assumptions 3.2 and 3.3 are not strict conditions. It is not necessary to know the exact value of δ_{α_2}. In fact, the Assumptions 3.2 and 3.3 are the results of Ref. [7].

Choosing $\rho_4 > \delta_{\alpha_2}$ and substituting Eq. (31) into Eq. (30), we have

$$\begin{aligned} \dot{V}_4 &= -k_2e_2^2 + e_2e_4 - c_2\left|e_2\right|^{1+\alpha} - \mu_2\left(|\hat{a}| + d_a\right)^{1+\alpha} \\ &\quad + \tilde{c}\dot{\hat{c}} + \tilde{d}\dot{\hat{d}} + e_4\left(-e_2 - \tilde{c}e_{30} - \tilde{d}e_{40} + \hat{\dot{\alpha}}_2 - k_4e_4 - c_4 sgn\left(e_4\right)\left|e_4\right|^{\alpha} - \dot{\alpha}_2\right) \\ &\quad -\mu_2\left[\left(|\hat{c}| + d_c\right)^{1+\alpha} + \left(\left|\hat{d}\right| + d_d\right)^{1+\alpha}\right] - e_4\rho_4 sgn\left(e_4\right) \\ &\le -k_2e_2^2 - k_4e_4^2 - \mu_2\left[\left(|\hat{a}| + d_a\right)^{1+\alpha} + \left(|\hat{c}| + d_c\right)^{1+\alpha} + \left(\left|\hat{d}\right| + d_d\right)^{1+\alpha}\right] \\ &\quad -c_2\left|e_2\right|^{1+\alpha} - c_4\left|e_4\right|^{1+\alpha} - \tilde{c}e_{30}e_4 - \tilde{d}e_{40}e_4 + \tilde{c}\dot{\hat{c}} + \tilde{d}\dot{\hat{d}}. \end{aligned} \tag{32}$$

Choosing the updating laws as

$$\dot{\hat{c}} = e_{30}e_4, \tag{33}$$

$$\dot{\hat{d}} = e_{40}e_4. \tag{34}$$

Substituting Eqs. (33) and (34) into Eq. (32) gives

$$\begin{aligned} \dot{V}_4 &\le -k_2e_2^2 - k_4e_4^2 - c_2\left|e_2\right|^{1+\alpha} - c_4\left|e_4\right|^{1+\alpha} \\ &\quad -\mu_2\left[\left(|\hat{a}| + d_a\right)^{\alpha} + \left(|\hat{c}| + d_c\right)^{\alpha} + \left(\left|\hat{d}\right| + d_d\right)^{\alpha}\right]. \end{aligned} \tag{35}$$

Since $|\hat{a} - a| \le |\hat{a}| + |a| \le |\hat{a}| + d_a$, $|\hat{c} - c| \le |\hat{c}| + |c| \le |\hat{c}| + d_c$ and $|\hat{d} - d| \le |\hat{d}| + |d| \le |\hat{d}| + d_d$, we can conclude that $-\left(|\hat{a}| + d_a\right)^{1+\alpha} \le -|\hat{a} - a|^{1+\alpha}$, $-\left(|\hat{c}| + d_c\right)^{1+\alpha} \le -\||\hat{c} - c|^{1+\alpha}$ and $-\left(|\hat{d}| + d_d\right)^{1+\alpha} \le -|\hat{d} - d|^{1+\alpha}$. Therefore, the inequality (35) can be rewritten as

$$\begin{aligned} \dot{V}_4 &\le -c_2\left|e_2\right|^{1+\alpha} - c_4\left|e_4\right|^{1+\alpha} - \mu_2\left[\left(|\hat{a}| + d_a\right)^{1+\alpha} + \left(|\hat{c}| + d_c\right)^{1+\alpha} + \left(\left|\hat{d}\right| + d_d\right)^{1+\alpha}\right] \\ &\le -c_2\left|e_2\right|^{1+\alpha} - c_4\left|e_4\right|^{1+\alpha} - \mu_2\left|\hat{a} - a\right|^{1+\alpha} - \mu_2\left|\hat{c} - c\right|^{1+\alpha} - \mu_2\left|\hat{d} - d\right|^{1+\alpha} \\ &\le -c_{24\mu}\left(\left|e_2\right|^{1+\alpha} + \left|e_4\right|^{1+\alpha} + \left|\hat{a} - a\right|^{1+\alpha} + \left|\hat{c} - c\right|^{1+\alpha} + \left|\hat{d} - d\right|^{1+\alpha}\right) \\ &\le -c_{24\mu}\left(\left|e_2\right|^{2} + \left|e_4\right|^{2} + \left|\hat{a} - a\right|^{2} + \left|\hat{c} - c\right|^{2} + \left|\hat{d} - d\right|^{2}\right)^{\frac{1+\alpha}{2}} \\ &\le -c_{24\mu}V_4^{\frac{1+\alpha}{2}} \end{aligned} \tag{36}$$

where $c_{24\mu} = \min\{c_2, c_4, \mu_2\}$.

According to Lemma 2.4, $e_2 \to 0$, $e_4 \to 0$ as $t \to T_2$, where $T_2 = \frac{V_{40}^{(1-\alpha)/2}}{c_{24\mu}(1-\alpha)/2}$, and V_{40} denotes the initial value of V_4.

From the discussion above, we have the following result.

Theorem 3.5. For the systems (1) and (2), under Assumptions 2.1 − 3.3, if the virtual control laws are chosen as Eqs. (15) and (25), updating laws are chosen as Eqs. (27), (33) and (34), and the control laws are chosen as Eq. (19) and (31), then $e_i (i = 1, 2, 3, 4)$ will converge to zero in a finite time T,where $T = \max\{T_1, T_2\}$.

The Theorem 3.5 does not tell us whether the systems (1) and (2) are synchronized or not, but it is easy to obtain the following Corollary, which is a straightforward consequence of the Theorem 3.5.

Corollary 3.6. Under the conditions of Theorem 3.5, $e_{i0} (i = 1, 2, 3, 4)$ will converge to zero in a finite time, namely, systems (1) and (2) are synchronized in a finite time.

Proof. Because of $e_1 = e_{10}$ and $e_1 \to 0$ as $t \to T_1$, it is natural that $e_{10} \to 0$ holds as $t \to T_1$.

Since $e_2 = e_{20}$ and $e_2 \to 0$ as $t \to T_2$, it is natural that $e_{20} \to 0$ holds as $t \to T_2$.

According to Theorem 1, it can be seen from Eq. (15) that $\alpha_1 \to 0$ holds as $t \to T$, which implies that $e_{30} \to 0$ as $t \to T$.

Because of $e_i \to 0$ as $t \to T$, $i = 1, 2, 3, 4$, it can be seen from Eq. (25) that $\alpha_2 \to 0$ holds as $t \to T$, which implies that $e_{40} \to 0$ as $t \to T$.

This completes the proof. □

Remark 3.7. Since the control signals (15), (19), (25) and (31) contain the discontinuous "sgn" functions, as a hard switcher, it may cause undesirable chattering. In order to avoid the chattering, the "sgn" function is replaced by a continuous function "tanh" to remove discontinuity.

Remark 3.8. Replacing the "sgn" function by the "tanh" function in (15), (19), (25) and (31) has no effect on the robustness and stability.

Remark 3.9. In addition, a saturation function proposed by Lu et al. [16] can be used to smooth out the "sgn" function in a thin boundary layer of the neighborhood of stability state,which can also improve the performance of the system.

Before proving Remark 3.8, an auxiliary lemma is provided.

Lemma 3.10. (Pourmahmood et al. [20]) For every given scalar a and positive scalar b the following inequality holds

$$a \ \tanh(ab) = |a \ \tanh(ab)| = |a| \, |\tanh(ab)| \,. \tag{37}$$

Proof of Remark 3.8. Consider the Lyapunov function introduced in the Theorem 3.5. Replacing the "sgn" function by the "tanh" function in (15) and (19), from inequality (20) one obtains

$$\begin{aligned} \dot{V}_3 &= -k_1 e_1^2 - e_1 e_3 - c_1 \left|\tanh(\varepsilon e_1)\right| \left|e_1\right|^{1+\alpha} \\ &\quad + e_3 \left[e_1 - k_3 e_3 - c_3 \tanh(\varepsilon e_3) \left|e_3\right|^{\alpha} + \hat{\dot{\alpha}}_1 - \dot{\alpha}_1 - \rho_3 \tanh(\varepsilon e_3) \right] \end{aligned} \tag{38}$$

From Lemma 3.10, one has

$$\begin{aligned} \dot{V}_3 &\le -k_1 e_1^2 - c_1 \left|\tanh(\varepsilon e_1)\right| \left|e_1\right|^{1+\alpha} - k_3 e_3^2 - c_3 \left|\tanh(\varepsilon e_3)\right| \left|e_3\right|^{1+\alpha} \\ &= -k_1 e_1^2 - c_1' \left|e_1\right|^{1+\alpha} - k_3 e_3^2 - c_3' \left|e_3\right|^{1+\alpha} \end{aligned} \tag{39}$$

where $c_1' = c_1 \left|\tanh(\varepsilon e_1)\right|$ and $c_3' = c_3 \left|\tanh(\varepsilon e_3)\right|$. The inequality (39) is of the same form as the inequality (20), so replacing the "sgn" function by the "tanh" function in (15) and (19) has no effect on the result.

Similarly, one can prove that replacing the "sgn" function by the "tanh" function in (25) and (31) has no effect on the robustness and stability. □

4. NUMERICAL SIMULATION

In this section, we present numerical results to verify the proposed synchronization approach.

The initial states in drive system (1) are $x(0) = -20$, $y(0) = 0$, $z(0) = 0$ and $w(0) = 15$. The initial states in driven system (2) are $x_1(0) = -15$, $y_1(0) = 1$, $z_1(0) = 1$ and $w_1(0) = 16$. The initial parameter estimation values of the systems (1) and (2) are $\hat{a}(0) = 0.1$, $\hat{c}(0) = -0.7$, $\hat{d}(0) = 0.1$, $d_a = 0.5$, $d_c = 1.0$, and $d_d = 0.3$.

According to Remark 3.7, the virtual control laws and control laws (15), (19), (25) and (31) are modified as follows

$$\alpha_1 = -e_{20} + c_1 \tanh(\varepsilon e_1) \left|e_1\right|^{\alpha} + k_1 e_1, \tag{40}$$

$$\alpha_2 = \begin{cases} -e_1 - \hat{a} e_2 - c_2 \tanh(\varepsilon e_2) \left|e_2\right|^{\alpha} - k_2 e_2 - \mu_2 \left(\left|\hat{a}\right| + d_a\right)^{1+\alpha} \Big/ e_2, & \text{if } e_2 \neq 0, \\ -e_1, & \text{if } e_2 = 0, \end{cases} \tag{41}$$

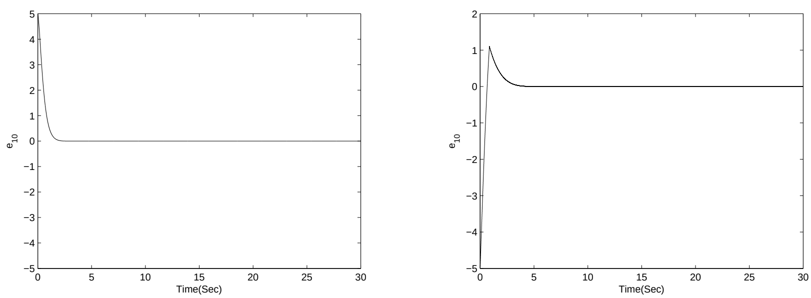
$$u_3 = -z e_{10} + e_1 - x_1 e_{30} + \hat{\dot{\alpha}}_1 - k_3 e_3 - c_3 \tanh(\varepsilon e_3) \left|e_3\right|^{\alpha} - \rho_3 \tanh(\varepsilon e_3), \tag{42}$$

$$u_4 = \begin{cases} -e_2 - \hat{c} e_{30} - \hat{d} e_{40} + \hat{\dot{\alpha}}_2 - k_4 e_4 - c_4 \tanh(\varepsilon e_4) \left|e_4\right|^{\alpha} - \rho_4 \tanh(\varepsilon e_4) \\ \qquad - \mu_2 \left[\left(\left|\hat{c}\right| + d_c\right)^{1+\alpha} + \left(\left|\hat{d}\right| + d_d\right)^{1+\alpha} \right] \Big/ e_4, & \text{if } e_4 \neq 0, \\ -e_2 - \hat{c} e_{30} - \hat{d} e_{40} + \hat{\dot{\alpha}}_2, & \text{if } e_4 = 0, \end{cases} \tag{43}$$

where $k_1 = 4.5$, $k_2 = 1.8$, $k_3 = 2.0$, $k_4 = 3.8$, $c_1 = 0.5$, $c_2 = 0.5$ $c_3 = 0.5$, $c_4 = 0.1$, $\mu_2 = 0.01$, $\rho_3 = 0.1$, $\rho_4 = 0.1$, and $\varepsilon = 100$.

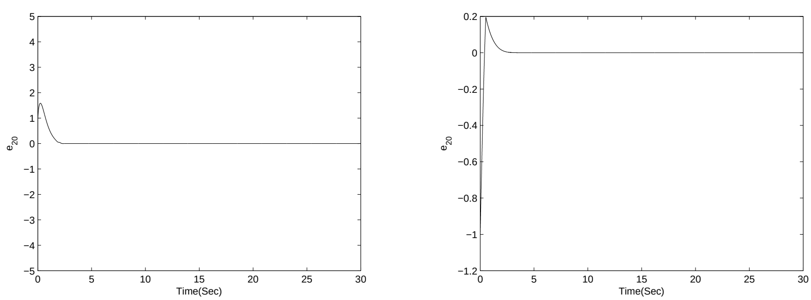
In what follows, we will give numerical simulations to verify the proposed result and compare it with the result of Ref. [1]. The synchronization errors of the proposed method between two hyperchaotic Rössler systems are illustrated in Figures 1a–4a, where the control inputs are activated at $t = 0$s. We can see that the synchronization errors converge to zero in a finite time, which implies that the chaos synchronization between the two hyperchaotic Rössler systems is realized. The time responses of adaptive parameters $\hat{a}$, $\hat{c}$ and $\hat{d}$ are depicted in Figure 5a. It is obvious that the parameters almost approach their true values. The synchronization errors of the method in Ref. [1] between two hyperchaotic Rössler systems are illustrated in Figures 1b−4b, where the control inputs are activated at $t = 0$s. We can see that the synchronization errors converge to zero in a finite time, which implies that the chaos synchronization between the two hyperchaotic Rössler systems is realized. The time responses of adaptive parameters $\hat{a}$, $\hat{c}$ and $\hat{d}$ are depicted in Figure 5b. It is noted that the parameters approach some constants.

From Figures 1−5, it can be seen that the proposed method achieves better performance although it only uses two control inputs instead of using 4 control inputs as in Ref. [1].



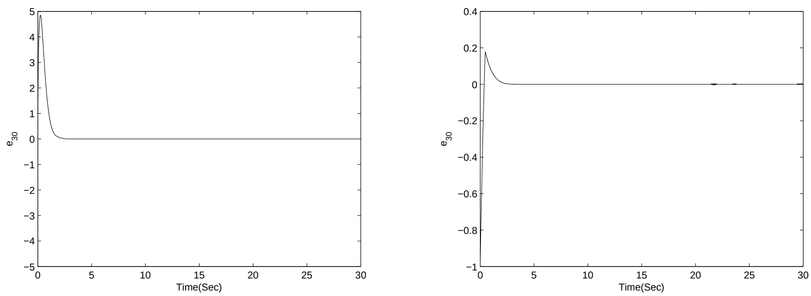
(a) the result of this paper (b) the result of Ref. [1]

Fig. 1. Synchronization error e_{10}.



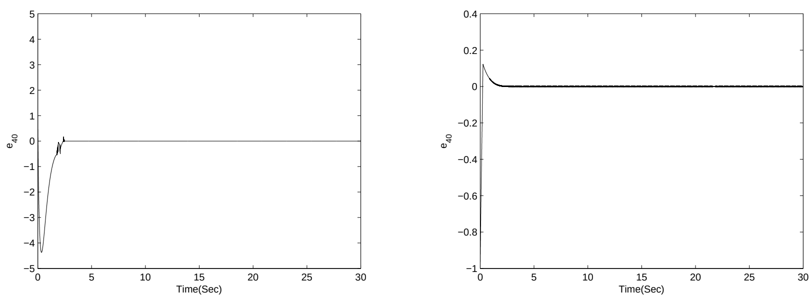
(a) the result of this paper (b) the result of Ref. [1]

Fig. 2. Synchronization error e_{20}.



(a) the result of this paper (b) the result of Ref. [1]

Fig. 3. Synchronization error e_{30}.

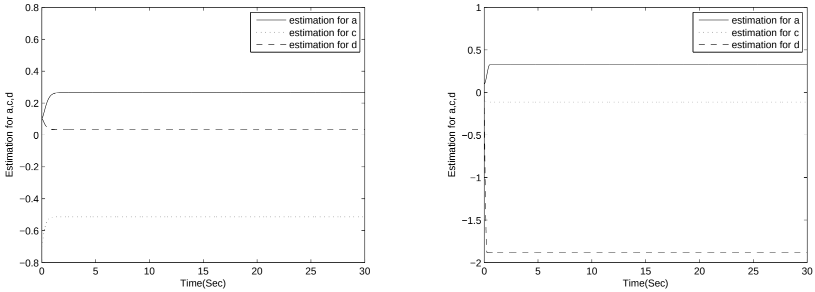


(a) the result of this paper (b) the result of Ref. [1]

Fig. 4. Synchronization error e_{40}.

5. CONCLUSIONS

In this paper, an adaptive backstepping design method has been proposed to synchronize the hyperchaotic Rössler systems with parameter uncertainties. The proposed method can be applied to a variety of chaotic systems which can be described by the so-called cross-strict feedback systems. It is very important to note that the proposed control technique can realize chaos synchronization by using only 2 control inputs and guarantee that all the error states are driven to zero in a finite time. Numerical simulations are given to show the proposed synchronization approach works well for synchronizing hyperchaotic Rössler systems in a finite time, even when the parameters of both the master and slave systems are unknown. But how to design a controller in absence of upbounds and how to make the parameters converge to their true values in a finite time are not solved in this paper, which would be one part of our future works. In addition, Yu et al. propsed a systematic methodology to generate various grid attractors for the



(a) the result of this paper

(b) the result of Ref. [1]

Fig. 5. Time response of the adaptive parameters.

multiwing hyperchaotic system, which has much more complex topological structures [35]. Extending our method to multiwing hyperchaotic systems will be our future work too.

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