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MEASURING CONSISTENCY AND INCONSISTENCY OF PAIR COMPARISON SYSTEMS

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In this paper we deal with mathematical modeling of real processes that are based on preference relations in the sense that, for every pair of distinct alternatives, the processes are linked to a value of preference degree of one alternative over the other one. The use of preference relations is usual in decision making, psychology, economics, knowledge acquisition techniques for knowledge-based systems, social choice and many other social sciences. For designing useful mathematical models of such processes, it is very important to adequately represent properties of preference relations. We are mainly interested in the properties of such representations which are usually called reciprocity, consistency and transitivity. In decision making processes, the lack of reciprocity, consistency or transitivity may result in wrong conclusions. That is why it is so important to study the conditions under which these properties are satisfied. However, the perfect consistency or transitivity is difficult to obtain in practice, particularly when evaluating preferences on a set with a large number of alternatives. Under different preference representation structures, the multiplicative and additive preference representations are incorporated in the decision problem by means of a transformation function between multiplicative and additive representations. Some theoretical results on relationships between multiplicative and additive representations of preferences on finite sets are presented and some possibilities of measuring their consistency or transitivity are proposed and discussed. Illustrative numerical examples are provided.

Keywords: multi-criteria optimization, pair-wise comparison matrix, AHP

Classification: 90B50, 90C29, 91B08

1. INTRODUCTION

Decision making in situations with multiple variants is a prominent area of research in decision theory, particularly, in the multi-criteria decision making when we wish to rank the variants or criteria, or to select the best one. This topic has been widely studied; see, for example, [7, 16, 24, 27, 29]. In this paper, we are concerned with methods in which a decision-maker (DM) is required to compare alternatives in pairs; that is, for each pair, the DM is required to provide information which of the two alternatives he or she prefers, or about the degree of this preference. The resulting pair comparison system (a pairwise comparison matrix in the case of a finite number of alternatives) is a powerful inference tool that can be also used as a knowledge acquisition technique for knowledge-based systems. It can also be useful for assessing the relative importance of several objects, when this cannot be done by direct rating. In fact, this perspective has been recently used for measuring the importance ranking of web sites [4]. It can be used also in coalitional preferences and coalitional domination concepts of coalitional games [21, 22].

As it is known, most of real decision making processes are based on preference relations in the sense that, for every pair of distinct alternatives, the processes are linked to a value of preference degree of one alternative over the other one. The use of preference relations is usual in decision making [8, 16, 19, 27, 29]. Therefore, to establish properties to be satisfied by such preference relations is very important for designing good mathematical models. Three of these properties we investigate in this paper is the so called reciprocity, consistency and transitivity property. The lack of these properties in a decision making process may result in wrong conclusions. Therefore, it is important to study conditions under which they are satisfied, see [3, 15, 17, 23, 26, 28, 29]. However, in practice, the perfect consistency or perfect transitivity is difficult to obtain, particularly, when it is necessary to evaluate preferences on a set with a large or even infinite number of alternatives. Then it is important to know, whether our preferences are sufficiently coherent, in other words, we ask whether our preferences are consistent or transitive. Here, we deal with two types of consistency: multiplicative one, [26], and additive one, [15, 18]. Our goal here is to derive some simple tools enabling us to measure the grade of consistency and transitivity of pair comparison systems, or, giving us some information about inconsistency of our preferences, i. e., how much the consistency or transitivity of our preferences is damaged. If a calculated inconsistency grade is sufficiently low, then the corresponding relation deduced from the pair comparison system can be applied to the DM problem. Otherwise, the system should be reconsidered and newly adjusted.

The paper is structured as follows. In Section 2 we deal with representation of preferences that involves measurement performed by assigning numbers to represent properties of possibly nonnumerical systems by properties of numerical systems. In fact, this means assigning numbers to alternatives so that some properties of relation "preferred to" are preserved in the corresponding numerical system. Two basic problems of these measurements are investigated: First, finding conditions under which such assignment is possible, and second, determining the type of uniqueness of the resulting measure. Here, we present both some results from the literature concerning the set of alternatives X with arbitrary cardinality and results for a more specific situation where X is a denumerable or finite set. In Section 3 general pair comparison systems are dealt with. We consider a function p, $(x, y) \mapsto p_{xy}$, that maps $X \times X$ into the unit interval $[0, 1]$ of real numbers. Such functions arise quite naturally in a number of decision making situations. We define several consistency conditions which are closely related to the standard notion of transitivity. In Section 4 we define two types of pairwise comparison matrices as well as the corresponding concepts of reciprocity, consistency and transitivity for the situation of finite set of alternatives X where the information about preference relations is represented by a square matrix. In Section 5 we investigate mutual relationships between consistency and transitivity and derive some necessary and sufficient conditions for their existence. Section 6 deals with the problem of measuring inconsistency. For this purpose we define three inconsistency indices. Section 7 provides several illustrative examples. Finally, in Section 8 some conclusions are presented.

2. REPRESENTATION OF PREFERENCES

As stated in the Introduction, we are concerned with situations in which a (cardinal) weighted ranking and the corresponding (ordinal) ordering of a nonempty set of alternatives should be deduced from information provided through pairwise comparisons; that is, from information provided by a decision-maker about his or her preferences (or their intensity) by means of a real-valued function on the Cartesian product of the set of alternatives with itself. Numerous models of various complexity for dealing with this problem have been proposed and analyzed in various branches of measurement theory, utility theory, and theory of decision making.

Various abstract models of preference structures differ in assumptions about the sets of feasible alternatives, individual preferences, types of preference representations, fields of application, and other features. Nevertheless, all reasonably applicable models suppose that the preference relations satisfy some consistency conditions; for example, the conditions of asymmetry, transitivity or completeness. Application of such models often involves measurement which is performed by assigning numbers to represent properties of possibly nonnumerical systems by properties of numerical systems. In the case of preference, this means assigning numbers to alternatives so that some properties of relation "preferred to" are preserved in the corresponding numerical system. Two basic problems of these measurements are that of finding conditions under which such assignment is possible, and that of determining the type of uniqueness of the resulting measure.

We begin with a brief reminder of some of the basic representations of preference structures on a fixed nonempty set X; for details and here undefined terms, see [13, 14, 20] and [25].

2.1. Ordinal measurement

It is well-known that if a binary relation $\succ$ on a finite or denumerable set X is asymmetric and negatively transitive, then there is a real-valued function f on X such that

$$x \succ y \text{ if and only if } f(x) > f(y) \text{ for all } x, y \in X. \tag{1}$$

Moreover, if (1) holds, then $[x \succ y \Leftrightarrow g(x) > g(y)$ for all $x, y \in X]$ for a real valued function g on X if and only if $[g(x) > g(y) \Leftrightarrow f(x) > f(y)$ for all $x, y \in X]$. In other words, the function f in (1) is unique up to an increasing transformation of real numbers. The triple consisting of the (empirical) system $(X, \succ)$, (numerical) system $(Re, >)$, and function f satisfying (1), (which maps X into the set Re of real numbers) is called an *ordinal scale*[1].

For sets of arbitrary cardinality, we have the following necessary and sufficient conditions of the existence of an ordinal scale: If $\succ$ is a complete asymmetric transitive binary relation on X, then there is a real-valued function f on X satisfying (1) if and only if $(X, \succ)$ has a countable order-dense subset.

[1]When the systems $(X, \succ)$, and $(Re, >)$ are clear from the context, we often refer to f alone as the scale. We make use of this impreciseness also for other types of scales.

2.2. Difference measurement

Let D be a quaternary relation on X, and let E and W be quaternary relations on X defined by[2]

$$\begin{aligned} xyEuv &\Leftrightarrow [\text{not}(xyDuv) \text{ and not}(uvDxy)], \\ xyWuv &\Leftrightarrow [(xyDuv) \text{ or } (xyEuv)]. \end{aligned}$$

In the case of preference, the statement $xyDuv$ may mean that the DM prefers x to y more than he or she prefers u to v. In practice, the relation D may be obtained by comparison of pairs of alternatives, or by asking the DM to make numerical estimates $d(x,y) \geq 0$ of absolute differences, and then defining D by

$$xyDuv \text{ if and only if } \delta(x,y) > \delta(u,v) \tag{2}$$

where

$$\delta(x,y) = \begin{cases} 0 & \text{if } x = y \text{ or } x \text{ and } y \text{ are judged equally important,} \\ d(x,y) & \text{if } x \text{ is judged more important than } y, \\ -d(x,y) & \text{if } y \text{ is judged more important than } x. \end{cases}$$

To guarantee that system (X,D) can be represented by a numerical system, one has to impose some conditions on D. For this purpose we associate with D a binary relation $\succ_D$ on $X \times X$ defined by

$$(x,y) \succ_D (u,v) \Leftrightarrow xyDuv. \tag{3}$$

We shall say that (X,D) is an *algebraic difference structure* if D satisfies the following five conditions.

C1 The relation $\succ_D$ is asymmetric and negatively transitive.

C2 If $xyDuv$, then $vuDyx$ for all x,y,u,v from X.

C3 If $xyWx'y'$ and $yzWy'z'$, then $xzWx'z'$ for all $x,y,z,x',y'z'$ from X.

C4 If $xyWuv$ holds and $uvWzz$ holds, then there are a,b in X such that $xaEuv$ and $byEuv$.

To state the fifth condition we need the notion of the strictly bounded standard sequence. A sequence $(x_1, x_2, \ldots)$ of elements from X is called *standard* if $x_{i+1}x_iEx_2x_1$ holds for all x_i, x_{i+1} in the sequence and $x_2x_1Ex_1x_1$ does not hold. A standard sequence is called *strictly bounded* if there exist u,v in X such that, for all x_i in the sequence, we have $uvDx_ix_1$ and x_ix_1Dvu.

C5 Every strictly bounded standard sequence is finite.

[2]To support intuition, we write $xyRuv$ instead of $(x,y,u,v) \in R$ whenever R is a quaternary relation.

The following result on representation of algebraic difference structures is proved in [20]: If a system (X, D) satisfies conditions C1 – C5, then there is a real-valued function f on X so that, for all x, y, u, v from X,

$$xyDuv \text{ if and only if } f(x) - f(y) > f(u) - f(v). \tag{4}$$

Moreover, if f is a function satisfying (4) for an algebraic difference structure (X, D) and if Δ is a quaternary relation on Re defined by

$$ab\Delta cd \Leftrightarrow a - b > c - d, \tag{5}$$

then the triple $\langle (X, D), (Re, \Delta), f \rangle$ is an *interval scale*; that is, f is unique up to a positive affine transformation of real numbers.

For finite algebraic difference structures, we have the following necessary and sufficient conditions for the existence of a function f satisfying (4).

If X is a nonempty finite set, then the following three conditions are necessary and sufficient for there to be a real valued function f on X satisfying (4):

- $xyDuv$ or $uvDxy$ holds for all x, y, u, v from X.
- If $xyDuv$, then $vuDyx$ for all x, y, u, v from X.
- Let n be a positive integer and π, σ be permutation of $\{0, 1, \ldots, n-1\}$. For all sequences $(x_0, x_1, \ldots, x_{n-1}, y_0, y_1, \ldots, y_{n-1})$, if $x_i y_i W x_{\pi(i)} y_{\sigma(i)}$ holds for all $0 < i < n$, then $x_{\pi(0)} y_{\sigma(0)} W x_0 y_0$.

It is worth noting that the properties C1 – C5 also guarantee the existence of a function g that maps X into the set of positive real numbers so that, for all $x, y, u, v \in X$,

$$xyDuv \text{ if and only if } \frac{g(x)}{g(y)} > \frac{g(u)}{g(v)}, \tag{6}$$

and that such function g is unique up to a transformation of real numbers of the form $\xi \mapsto \alpha\xi^{\beta}$ with positive α, β. Such scales are called *log-interval.*

In addition to the ordinal, interval, and log-interval scales, we shall need also some other scale types. Namely, the *ratio scale*, *difference scale* and *absolute scale*. The ratio scale is unique up to a positive linear transformation of real numbers. The difference scale is unique up to a transformation $\xi \mapsto \xi + \beta$. The absolute scale is unique up to the transformation $\xi \mapsto \xi$; that is, the absolute scale is absolute in the sense that the only permissible transformation is the identity transformation.

Remark 2.1. Notice that the difference scales correspond to ratio scales by transformation of the latter by logarithmic transformation because if a scale f is unique up to multiplication by a positive number α, then $\log f$ is unique up to addition of β where $\beta = \log \alpha$. Similarly, log-interval scales correspond to exponential transformations of interval scales. Also notice that if f and g are ratio scales with values in the set of positive numbers, then the derived scale $\frac{f}{g}$ is an absolute scale.

3. PAIR COMPARISON SYSTEMS

Let p be a function $(x, y) \mapsto p_{xy}$ that maps $X \times X$ into the unit interval $[0, 1]$ of real numbers. Such functions may arise quite naturally in a number of situations. For example, p_{xy} may be the proportion of individuals from a given group who prefer x to y, or it may be the frequency with which x is preferred to y in repeated experiments, or it may be a membership function of a fuzzy subset of $X \times X$. Following the terminology of [25] (which we are doing throughout this section), we call (X, p) a *pair comparison system*.

When we are trying to recover a preference relation on X from values of p by requiring that x is preferred to y whenever $p_{xy} > p_{yx}$, then the resulting relation can be the empty relation or it may have no consistency property required from preference relations like, for example, asymmetry or transitivity. Therefore it is necessary to require from p to satisfy some properties which would guarantee some kind of consistency.

An obvious desirable property is the requirement that $p_{xy} + p_{yx} = 1$ for all pairs (x, y) of distinct elements from X. The pair comparison system (X, p) with this property is called a *forced choice pair comparison system*. It would be natural to leave p_{xx} undefined for all x but, for convenience, it is usually assumed that $p_{xx} = 0.5$ for all x.

Let (X, p) be a forced choice pair comparison system. We say that (X, p) satisfies

- the *weak utility model* if there is a real-valued function f on X satisfying

$$p_{xy} > p_{yx} \text{ if and only if } f(x) > f(y), \tag{7}$$

- the *strong utility model* if there is a real-valued function f on X satisfying

$$p_{xy} > p_{uv} \text{ if and only if } f(x) - f(y) > f(u) - f(v), \tag{8}$$

- the *strict utility model* if there is a real-valued function f on X satisfying

$$p_{xy} = \frac{f(x)}{f(x) + f(y)}. \tag{9}$$

It can be shown that:

– A forced choice pair comparison system (X, p) with finite or denumerable set X satisfies the weak utility model if and only if the binary relation R defined by $xRy \Leftrightarrow p_{xy} \geq p_{yx}$ is transitive and complete.

– If (X, p) is a forced choice pair comparison system and a function f satisfies (7), then f defines an ordinal scale.

– If the quaternary relation D defined by $xyDuv \Leftrightarrow p_{xy} > p_{uv}$ is an algebraic difference structure, then the forced choice pair comparison system (X, p) satisfies the strong utility model.

– If (X, p) is an algebraic difference structure and a function f satisfies (8), then f defines an interval scale.

– A forced choice pair comparison system (X, p) with $p_{xy} \in (0,1)$ satisfies the strict utility model if and only if

$$p_{xy}p_{yz}p_{zx} = p_{xz}p_{zy}p_{yx} \quad \text{for all } x, y, z \in X. \tag{10}$$

– If (X, p) with $p_{xy} \in (0,1)$ is a forced choice comparison system satisfying (10) and if f is a positive function satisfying (9) for (X, p), then f defines a ratio scale.

In the next section we shall need several consistency conditions which are closely related to the standard notion of transitivity.

A forced choice pair comparison system (X, p) is said to satisfy

- *weak transitivity* if, for all x, y, z in X,

$$(p_{xy} \geq 0.5 \text{ and } p_{yz} \geq 0.5) \;\text{ implies }\; p_{xz} \geq 0.5; \tag{11}$$

- *moderate transitivity* if, for all x, y, z in X,

$$(p_{xy} \geq 0.5 \text{ and } p_{yz} \geq 0.5) \;\text{ implies }\; p_{xz} \geq \min(p_{xy}, p_{yz}); \tag{12}$$

- *strong transitivity* if, for all x, y, z in X,

$$(p_{xy} \geq 0.5 \text{ and } p_{yz} \geq 0.5) \;\text{ implies }\; p_{xz} \geq \max(p_{xy}, p_{yz}). \tag{13}$$

In [14], these transitivity concepts are called weak (moderate, strong) stochastic transitivity. As we do not consider here the stochastic context of the relations, the word “stochastic” is omitted.

4. PAIRWISE COMPARISON MATRICES

From now on, we shall be concerned with pair comparison systems on a finite set $X = \{x_1, x_2, \ldots, x_n\}$ of n mutually distinct alternatives. In this case, it is natural to represent a pair comparison systems (X, p) discussed in the previous section by an $n \times n$ matrix $P = \{p_{ij}\}$ with $0 \leq p_{ij} \leq 1$, and apply the results of the measurement theory to problems we are dealing with. For example, it can be shown (see [14]) that, in the finite case,

strict utility model $\Rightarrow$ strong transitivity $\Rightarrow$ moderate transitivity $\Rightarrow$ weak transitivity $\Leftrightarrow$ weak utility model[3].

Now we have a number of conditions for deriving weighted ranking of X in ordinal, interval, or ratio sense from matrices representing forced choice pair comparison systems. However we take more general view and consider also matrices whose elements are numbers that are not necessarily in the interval $[0,1]$. For example, if the elements are to represent the preference ratios, then it is natural to allow matrices with elements from $(0, \infty)$ or from some finite subsets of $(0, \infty)$, whereas if the elements are to represent the difference of preference, then it is appropriate to allow elements from $(-\infty, \infty)$. Nevertheless, we begin with matrices whose elements are in the interval $[0,1]$.

[3]The strong utility model implies strong transitivity even without restriction to finite sets.

4.1. Additively reciprocal matrices

An $n \times n$ matrix whose entries are from the unit interval $[0,1]$ can be viewed as the membership function of a fuzzy subsets of $X \times X$, and it may be interpreted as a fuzzy preference relation on X^4; see, for example, [11, 15, 17, 23, 27]. However, we shall look at such matrices differently. We shall assume that the DM has a preference relation on X and such a matrix provides information about the DM preference relation through the pairwise comparison process.

Let $B = \{b_{ij}\}$ be an $n \times n$ matrix with $0 \leq b_{ij} \leq 1$ for all i, j. If the values of entries of B are interpreted according to the rule that $b_{ij} = 0.5$ indicates indifference between x_i and x_j, $b_{ij} = 1$ indicates that x_i is absolutely preferred to x_j, $b_{ij} = 0$ indicates that x_j is absolutely preferred to x_i, and $b_{ij} > 0.5$ indicates that x_i is preferred to x_j, then we immediately face the problem of possible inconsistencies. To give a drastic example, let us consider the case in which $b_{ij} = b_{ji} = 1$ for some i different from j. Then we face an extreme inconsistency: x_i is absolutely preferred to x_j and at the same time x_j is absolutely preferred to x_i. Consequently, to make the interpretation meaningful, a number of conditions (like those presented in the previous section for not necessarily finite sets of alternatives) have been proposed and analyzed in the literature; see, for example, [14, 15, 18, 25, 29, 30].

To avoid obvious inconsistency that could destroy even asymmetry of preference relation, we confine ourselves to the matrices with $0 \leq b_{ij} \leq 1$ that satisfy the condition

$$b_{ij} + b_{ji} = 1 \text{ for all } i, j. \tag{14}$$

We shall call such matrices *additively reciprocal (a-reciprocal)*. Notice that if $B = \{b_{ij}\}$ is an a-reciprocal matrix, then (X, p) with $p(x_i, x_j) = b_{ij}$ is a forced choice pair comparison system. Consequently, we shall use the following terminology for a-reciprocal matrices.

Weak transitivity

$$(b_{ij} \geq 0.5 \text{ and } b_{jk} \geq 0.5) \quad \text{implies} \quad b_{ik} \geq 0.5. \tag{15}$$

The interpretation of this condition is the following: If x_i is preferred or indifferent to x_j and x_j is preferred or indifferent to x_k, then x_i should be preferred or indifferent to x_k. This kind of transitivity is the usual transitivity condition that a logical and consistent person should use if he or she does not want to express inconsistent opinions. Therefore, it is the minimum requirement that a consistent fuzzy preference relation should satisfy. Moreover, because the weak transitivity implies the weak utility model on finite sets, we know that it guarantees the existence of weights that provide ranking in the ordinal sense.

Strong transitivity

$$(b_{ij} \geq 0.5 \text{ and } b_{jk} \geq 0.5) \quad \text{implies} \quad b_{ik} \geq \max(b_{ij}, b_{jk}). \tag{16}$$

When an alternative x_i is preferred or indifferent to x_j with a value b_{ij} and x_j is preferred or indifferent to x_k with a value b_{jk}, then x_i should be preferred to

[4] also called a valued relation on X

x_k with at least an intensity of preference b_{ik} being equal to the maximum of the above values. It is clear that this concept is stronger than the concept of weak transitivity. This concept has been considered by Tanino [29] as a compulsory condition to be verified by a "consistent" fuzzy preference relation.

Moderate transitivity

$$(b_{ij} \geq 0.5 \text{ and } b_{jk} \geq 0.5) \text{ implies } \quad b_{ik} \geq \min(b_{ij}, b_{jk}). \tag{17}$$

Multiplicative transitivity (*m-transitivity*)

$$b_{ij}b_{jk}b_{ki} = b_{ik}b_{kj}b_{ji} \text{ for all } i, j, k. \tag{18}$$

Notice that this correspond to (10). If $b_{ij} > 0$ for all i and j, then (18) can be rewritten as

$$\frac{b_{ij}}{b_{ji}}.\frac{b_{jk}}{b_{kj}} = \frac{b_{ik}}{b_{ki}} \text{ for all } i, j, k, \tag{19}$$

or, equivalently,

$$\frac{b_{ij}}{b_{ji}}.\frac{b_{jk}}{b_{kj}}.\frac{b_{ki}}{b_{ik}} = 1 \text{ for all } i, j, k. \tag{20}$$

Here, the ratio is interpreted as the preference intensity for x_i to that of x_j; that is, "x_i is times as good as x_j". For instance, when comparing x_i and x_j, and x_i is assigned 60 % of the property and x_j is assigned 40 %, and at the same time, comparing x_j and x_k, x_j is assigned 70 % of the property and x_k is assigned 30 %, then comparing x_i and x_k, x_i is assigned $\frac{700}{9}$ % of the property (that is, 77, 8 %) and x_k is assigned $\frac{200}{9}$ % of that property (that is, 22, 2 %).

It is easy to prove that if $B = \{b_{ij}\}$ is an m-transitive matrix, then it is strongly transitive. Evidently, the converse is not true.

Additive transitivity (*a-transitivity*)

$$(b_{ij} - 0.5) + (b_{jk} - 0.5) = (b_{ik} - 0.5) \text{ for all } i, j, k. \tag{21}$$

Equivalently, (21) can be rewritten as

$$b_{ij} + b_{jk} + b_{ki} = 1.5 \text{ for all } i, j, k. \tag{22}$$

The interpretation of the additive transitivity is rather difficult [18]. It is easy to prove that if $B = \{b_{ij}\}$ is additively transitive, then it is strongly transitive. Therefore, the additive transitivity is a stronger concept than strong transitivity, Evidently, there is no inclusion between multiplicatively and additively transitive relations.

4.2. Multiplicatively reciprocal matrices

We begin with recalling the classical Saaty's Analytic Hierarchy Process. The essential component of Saaty's method is the extraction of a vector of weights (also called priorities) for a nonempty finite set of alternatives from pairwise comparisons stated in an absolute scale.

In detail, let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of mutually distinct alternatives, and let us assume that the intensities of DM's preferences are given by an $n \times n$ matrix $A = \{a_{ij}\}$ with positive elements in such a way that, for all i and j, the entry a_{ij} indicates the ratio of preference intensity for alternative x_i to that of x_j. In other words, a_{ij} indicates that "x_i is a_{ij} times as good as x_j ". Saaty [26] suggests to represent the preference intensities on the absolute scale $\{1/9, 1/8, \ldots, 1/2, 1, 2, \ldots, 8, 9\}$ where $a_{ij} = 1$ indicates equal intensity of preference, $a_{ij} = 9$ indicates extreme intensity of preference for x_i over x_j, and $a_{ij} \in \{2, 3, \ldots, 8\}$ indicates intermediate evaluations, and the elements of A satisfy the reciprocity condition $a_{ij}a_{ji} = 1$ for all i, j. If, for example, x_i is 3 times as good as x_j, then the goodness of x_j is $1/3$ with respect to the goodness of x_i.

In general, for arbitrary positive matrices $A = \{a_{ij}\}$, we introduce the notions of multiplicative reciprocity and multiplicative consistency as follows. A positive $n \times n$ matrix $A = \{a_{ij}\}$ is called

- *multiplicatively reciprocal* (*m-reciprocal*), if

$$a_{ij}a_{ji} = 1 \text{ for all } i, j, \tag{23}$$

- *multiplicatively consistent* (or, *m-consistent*) [15, 27], if

$$a_{ij} = a_{ik}a_{kj} \text{ for all } i, j, k. \tag{24}$$

Notice that $a_{ii} = 1$ for all i, and that

– every m-consistent matrix is also m-reciprocal (however, not vice-versa);

– the equality (24) can be rewritten equivalently as

$$a_{ik}.a_{kj}.a_{ji} = 1 \text{ for all } i, j, k; \tag{25}$$

– if A with $0 < a_{ij} < 1$ is m-consistent then A is multiplicatively transitive;

– if A with $0 < a_{ij} < 1$ is is m-reciprocal, then A is multiplicatively transitive (that is, a-consistent) if and only if A is m-consistent.

In what follows, we shall investigate mutual relationships between multiplicatively and additively reciprocal nonnegative $n \times n$ matrices with respect to some transitivity properties.

5. RELATIONS BETWEEN A-RECIPROCAL AND M-RECIPROCAL MATRICES

In this section we shall investigate some relationships between a-reciprocal and m-reciprocal pairwise comparison matrices. We start with extending recent results of E. Herrera-Viedma et al. [18], see also [6] – [10].

For this purpose, given $\sigma > 1$, we define the following transformation functions φ and φ^{-1}

$$\varphi(t) = \frac{1}{2}\Big(1 + \frac{\ln t}{\ln \sigma}\Big) \text{ for } t \in [\frac{1}{\sigma}, \sigma], \tag{26}$$

$$\varphi^{-1}(t) = \sigma^{2t-1} \text{ for } t \in [0, 1]. \tag{27}$$

We prove that a-transitive matrices and m-consistent matrices are mutually related in the following way [18].

Proposition 5.1. Let $\sigma > 1$ and let $A = \{a_{ij}\}$ be an $n \times n$ matrix with $\frac{1}{\sigma} \le a_{ij} \le \sigma$ for all i and j. Then $A = \{a_{ij}\}$ is an m-consistent if and only if $B = \{\varphi(a_{ij})\}$ is a-transitive.

Proof. Let $A = \{a_{ij}\}$ be an $n \times n$ matrix with $\frac{1}{\sigma} \le a_{ij} \le \sigma$ for all i and j.

Suppose that A is m-consistent and set $b_{ij} = \varphi(a_{ij})$ for all i and j. Then by (19)

$$b_{ij} + b_{jk} + b_{ki} = \frac{1}{2}\Big(3 + \frac{\ln a_{ij}a_{jk}a_{ki}}{\ln \sigma}\Big) \text{ for all } i, j, k. \tag{28}$$

Hence, by (25) we have $a_{ij}a_{jk}a_{ki} = 1$ for all i, j, k and then

$$b_{ij} + b_{jk} + b_{ki} = 1.5 \text{ for all } i, j, k. \tag{29}$$

Now, suppose that $B = \{\varphi(a_{ij})\}$ is a-transitive. Then

$$\frac{1}{2}\Big(1 + \frac{\ln a_{ij}}{\ln \sigma}\Big) + \frac{1}{2}\Big(1 + \frac{\ln a_{jk}}{\ln \sigma}\Big) + \frac{1}{2}\Big(1 + \frac{\ln a_{ki}}{\ln \sigma}\Big) = \frac{3}{2} \text{ for all } i, j, k,$$

which is equivalent to

$$\frac{\ln a_{ij}a_{jk}a_{ki}}{\ln \sigma} = 0 \text{ for all } i, j, k.$$

Therefore, $a_{ij}a_{jk}a_{ki} = 1$ and by (25), $A = \{a_{ij}\}$ is m-consistent. □

The following "inverse result" will be useful in the next section for measuring the grade of intransitivity.

Proposition 5.2. Let $\sigma > 1$ and let $B = \{b_{ij}\}$ be an a-reciprocal $n \times n$ matrix with $0 \le b_{ij} \le 1$ for all i and j. Then $B = \{b_{ij}\}$ is an a-transitive matrix if and only if $A = \{\varphi^{-1}(b_{ij})\}$ is m-consistent.

Proof. Let $B = \{b_{ij}\}$ be additively transitive. Then, by setting $a_{ij} = \varphi^{-1}(b_{ij})$, we obtain

$$a_{ij}a_{jk}a_{ki} = \sigma^{2b_{ij}-1}\sigma^{2b_{jk}-1}\sigma^{2b_{ki}-1} = \sigma^{2(b_{ij}+b_{jk}+b_{ki})-3} = \sigma^{2(\frac{3}{2})-3} = 1 \text{ for all } i, j, k.$$

Therefore, $A = \{a_{ij}\}$ is m-consistent.

On the other hand, if $A = \{\varphi^{-1}(b_{ij})\}$ is m-consistent, then by (25)

$$a_{ij}a_{jk}a_{ki} = \sigma^{2b_{ij}-1}\sigma^{2b_{jk}-1}\sigma^{2b_{ki}-1} = \sigma^{2(b_{ij}+b_{jk}+b_{ki})-3} = 1 \text{ for all } i, j, k.$$

Hence, $2(b_{ij} + b_{jk} + b_{ki}) - 3 = 0$ for all i, j, k, and thus $B = \{b_{ij}\}$ is a-transitive. □

Propositions 5.1 and 5.2 give a characterization of additive-transitive matrix by some transformed m-consistent matrix dependent on the given scale $[{}^1/_\sigma; \sigma]$, where $\sigma > 1$. Usually, for example. in [9, 10, 18], and [27], $\sigma = 9$.

In the following proposition we characterize an additive-reciprocal matrix by some transformed multiplicatively reciprocal matrix using transformations independent of the scale.

Proposition 5.3. (i) An $n \times n$ matrix $B = \{b_{ij}\}$ with $0 < b_{ij} < 1$ for all i and j is a-reciprocal if and only if $A = \{a_{ij}\} = \{\frac{b_{ij}}{1-b_{ij}}\}$ is m-reciprocal.

(ii) A positive $n \times n$ matrix $A = \{a_{ij}\}$ is m-reciprocal if and only if $B = \{b_{ij}\} = \{\frac{a_{ij}}{1+a_{ij}}\}$ is a-reciprocal.

P r o o f. (i) Let B be a-reciprocal. Then

$$a_{ji} = \frac{b_{ji}}{1-b_{ji}} = \frac{1-b_{ij}}{1-(1-b_{ij})} = \frac{1-b_{ij}}{b_{ij}} = \frac{1}{a_{ij}},$$

hence, $A = \{a_{ij}\}$ is m-reciprocal.

Now, assume that $A = \{a_{ij}\} = \{\frac{b_{ij}}{1-b_{ij}}\}$ is m-reciprocal, then

$$\frac{b_{ij}}{1-b_{ij}} = \frac{1-b_{ji}}{b_{ji}},$$

hence, $b_{ji} = 1 - b_{ij}$ and B is a-reciprocal.

(ii) Let $A = \{a_{ij}\}$ be m-reciprocal. Then

$$b_{ji} = \frac{a_{ji}}{1+a_{ji}} = \frac{\frac{1}{a_{ij}}}{1+\frac{1}{a_{ij}}} = \frac{1}{1+a_{ij}} = 1 - \frac{a_{ij}}{1+a_{ij}} = 1 - b_{ij},$$

hence, $B = \{b_{ij}\}$ is a-reciprocal.

Now, assume that $B = \{b_{ij}\}$ is a-reciprocal, then

$$\frac{a_{ij}}{1+a_{ij}} = 1 - \frac{a_{ji}}{1+a_{ji}},$$

hence, $a_{ij}.a_{ji} = 1$ and A is m-reciprocal. □

In the following proposition we characterize a multiplicatively transitive matrix by some transformed multiplicatively consistent matrix using different transformation not dependent on a scale. In the next section it will be useful for measuring the grade of intransitivity.

Proposition 5.4. (i) An a-reciprocal matrix $B = \{b_{ij}\}$ with $0 < b_{ij} < 1$ for all i and j is m-transitive if and only if $A = \{a_{ij}\} = \{\frac{b_{ij}}{1-b_{ij}}\}$ is m-consistent.

(ii) A positive m-reciprocal $n \times n$ matrix $A = \{a_{ij}\}$ is m-consistent if and only if $B = \{b_{ij}\} = \{\frac{a_{ij}}{1+a_{ij}}\}$ is m-transitive.

Proof. (i) By a-reciprocity of B, (25) and (20) are clearly equivalent.

(ii) We have $0 < \frac{a_{ij}}{1+a_{ij}} < 1$ for all i, j. Moreover, $B = \{b_{ij}\} = \{\frac{a_{ij}}{1+a_{ij}}\}$ is a-reciprocal, as

$$b_{ji} = \frac{a_{ji}}{1+a_{ji}} = \frac{\frac{1}{a_{ij}}}{1+\frac{1}{a_{ij}}} = \frac{1}{1+a_{ij}} = 1 - \frac{a_{ij}}{1+a_{ij}} = 1 - b_{ij}.$$

By (i), $B = \{b_{ij}\} = \{\frac{a_{ij}}{1+a_{ij}}\}$ is m-transitive iff $\{\frac{b_{ij}}{1-b_{ij}}\}$ is m-consistent. However, $\frac{b_{ij}}{1-b_{ij}} = \frac{\frac{a_{ij}}{1+a_{ij}}}{1-\frac{a_{ij}}{1+a_{ij}}} = a_{ij}$, hence, A is m-consistent. □

The following result gives a characterization of m-consistent matrix by a vector of weights, that is, by a positive vector with sum of elements equal to one.

Proposition 5.5. A positive $n \times n$ matrix $A = \{a_{ij}\}$ is m-consistent if and only if there exists a vector $w = (w_1, w_2, \ldots, w_n)$ with $w_i > 0$ for all $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} w_i = 1$ such that

$$a_{ij} = \frac{w_i}{w_j} \text{ for all } i, j = 1, 2, \ldots, n. \tag{30}$$

Proof. (i) $A = \{a_{ij}\}$ be m-consistent. For $i = 1, 2, \ldots, n$, set

$$v_i = (a_{i1}a_{i2} \ldots a_{in})^{\frac{1}{n}}. \tag{31}$$

Moreover, set

$$S = \sum_{i=1}^{n} v_i$$

and, finally, define

$$w_i = \frac{v_i}{S} \text{ for } i = 1, 2, \ldots, n. \tag{32}$$

Then, for $i, j = 1, 2, \ldots, n$, by reciprocity (23) and consistency (24) we obtain successively

$$\frac{w_i}{w_j} = \left(\frac{a_{i1}a_{i2} \ldots a_{in}}{a_{j1}a_{j2} \ldots a_{jn}}\right)^{\frac{1}{n}} = ((a_{i1}a_{1j})(a_{i2}a_{2j}) \ldots (a_{in}a_{nj}))^{\frac{1}{n}} = (a_{ij} \cdot a_{ij} \ldots a_{ij})^{\frac{1}{n}} = a_{ij}.$$

Moreover, $\sum_{i=1}^{n} w_i = 1$, consequently, (30) is true.

(ii) If (30) holds, then evidently (24) is satisfied, hence, $A = \{a_{ij}\}$ is m-consistent. □

In the following proposition we derive a similar characterization of m-transitive matrices.

Proposition 5.6. Let $B = \{b_{ij}\}$ be an a-reciprocal $n \times n$ matrix with $0 < b_{ij} < 1$ for all i and j. Then $B = \{b_{ij}\}$ is m-transitive if and only if there exists a vector $v = (v_1, v_2, \ldots, v_n)$ with $v_i > 0$ for all $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} v_i = 1$ such that

$$b_{ij} = \frac{v_i}{v_i + v_j} \text{ for all } i, j = 1, 2, \ldots, n. \tag{33}$$

P r o o f. By Proposition 5.4 (i), we know that $B = \{b_{ij}\}$ is m-transitive if and only if $A = \{a_{ij}\} = \{\frac{b_{ij}}{1-b_{ij}}\}$ is m-consistent. By Proposition 5.5, this result is equivalent to the existence of a vector $v = (v_1, v_2, \ldots, v_n)$ with $v_i > 0$ for all $i = 1, 2, \ldots, n$, such that $\frac{b_{ij}}{1-b_{ij}} = \frac{v_i}{v_j}$ for all $i, j = 1, 2, \ldots, n$, or, equivalently, $b_{ij} = \frac{v_i}{v_i+v_j}$ for all $i, j = 1, 2, \ldots, n$, i. e. (33) is true[5]. □

When considering Propositions 5.3, 5.4 and 5.6, it is clear that the concept of multiplicative transitivity plays a similar role for a-reciprocal matrices as the concept of m-consistency does for m-reciprocal matrices. That is why it is reasonable to introduce the following definition.

Definition 5.7. Any $n \times n$ nonnegative a-reciprocal matrix $B = \{b_{ij}\}$ which is m-transitive is called *additively consistent* (*a-consistent*).

Proposition 5.6 can be then reformulated as follows.

Proposition 5.8. Let $B = \{b_{ij}\}$ be an a-reciprocal $n \times n$ matrix with $0 < b_{ij} < 1$ for all i and j. Then $B = \{b_{ij}\}$ is a-consistent if and only if there exists a vector $v = (v_1, v_2, \ldots, v_n)$ with $v_i > 0$ for all $i = 1, 2, \ldots, n$, and such that (33) is true.

Propositions 5.1 to 5.4 can be also similarly reformulated according to Definition 5.7, however, we leave it to the reader.

In practice, perfect consistency or transitivity is difficult to obtain, particularly when measuring preferences on a set with a large number of alternatives. In the following section we deal with the problem of measuring consistency/transitivity, or, inconsistency/intransitivity of such pairwise comparison matrices.

6. INCONSISTENCY/INTRANSITIVITY OF PAIRWISE COMPARISON MATRICES

If, for some positive $n \times n$ matrix $A = \{a_{ij}\}$ and for some $i, j, k = 1, 2, \ldots, n$, the multiplicative consistency condition (24) does not hold, then A is said to be *multiplicatively inconsistent* (or, *m-inconsistent*). Furthermore if, for some $n \times n$ matrix $B = \{b_{ij}\}$ with $0 \le b_{ij} \le 1$ for all i and j, and for some $i, j, k = 1, 2, \ldots, n$, (18) does not hold, then B is said to be *additively inconsistent* (or, *a-inconsistent*). Finally if, for some $n \times n$ matrix $B = \{b_{ij}\}$ with $0 \le b_{ij} \le 1$ for all i and j, and for some $i, j, k = 1, 2, \ldots, n$, (22) does not hold, then B is said to be *additively intransitive* (or, *a-intransitive*). In order to

[5]Compare with (9) in the definition of the strict utility mode.

measure the grade of inconsistency/intransitivity of a given matrix several measurement methods have been proposed in the literature.

We have already mentioned that multiplicative m-reciprocal matrices are considered in Analytic hierarchy process [26]. In order to measure the grade of inconsistency, T. Saaty proposed the consistency ratio (CR) [27]. Moreover, for prioritization procedure based on the geometric mean, the geometric consistency ratio was proposed in [5] and [1], with interpretation analogous to that of CR. In [3], Koczkodaj proposed a new consistency index based on 3×3 sub-matrices and derived its relation to CR. Singular value decomposition method by Gass and Rapcsak [16] is another method for measuring consistency of positive m-reciprocal matrix. Recently, Stein and Mizzi [28] proposed the harmonic consistency index. Ramík and Korviny [23] proposed an inconsistency index for measuring of pairwise comparison matrix with fuzzy elements.

As far as additively reciprocal matrices are concerned, appropriate methods for measuring inconsistency, or intransitivity, are not known to the authors. Some methods for measuring a-inconsistency/a-intransitivity will be dealt with in this section.

Instead of positive matrices we first consider generalized preference matrices with nonnegative elements; that is, some of their elements are permitted to be zeros. A typical example is so called won/lost matrix $A = \{a_{ij}\}$ with the elements a_{ij} defined for all i and j by

$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ beats } j \\ 0 & \text{if } j \text{ beats } i \\ 0.5 & \text{if } i \text{ and } j \text{ are equal.} \end{cases} \tag{34}$$

Measuring inconsistency of this matrix can be based on the Perron–Frobenius theory which is known in several versions [2, 26, 12]. For a comfort of the reader, let us recall it shortly. One of the assumptions of the Perron–Frobenius theory is that the matrix does not have an off-diagonal zero-block. More precisely, a square matrix A is *cogredient* to a matrix B if there exists a permutation matrix P with elements 0 or 1 such that $PAP^T = B$, where P^T is the transpose - and hence also the inverse - of P. That is, two matrices are cogredient if one can be obtained from the other by applying the same permutation to its rows and columns. The matrix A is *reducible* if it is cogredient to a partitioned matrix

$$B = \begin{pmatrix} C & D \\ 0 & F \end{pmatrix} \tag{35}$$

where the diagonal blocks C and F are square. If a matrix is not reducible, it is said to be *irreducible*. Obviously, every positive matrix is irreducible. For instance, a won-lost matrix (35) is reducible if and only if the players can be partitioned into two groups such that players from the group one are always beaten by players from the group two.

The Perron–Frobenius theorem, see e. g. [12], describes some of the remarkable properties enjoyed by the eigenvalues and eigenvectors of irreducible nonnegative matrices.

Theorem 6.1. (Perron–Frobenius) Let A be an irreducible nonnegative square matrix. Then the spectral radius, $\rho(A)$, is a real eigenvalue, which has a positive (real) eigenvector. This eigenvalue called the principal eigenvalue of A is simple (it is not a multiple root of the characteristic equation), and its eigenvector is unique up to a multiplicative constant.

The m-consistency of a nonnegative m-reciprocal $n \times n$ matrix A is measured by the *m-consistency index* $I_{mc}(A)$ by

$$I_{mc}(A) = \frac{\rho(A) - n}{n - 1} \tag{36}$$

where $\rho(A)$ is the spectral radius of A (particularly, *principal eigenvalue* of A).

The "relative importance" of the alternatives in X is determined by the vector of weights $w = (w_1, w_2, \dots, w_n)$, with $w_i > 0$, for all $i = 1, 2, \dots, n$, such that $\sum_{i=1}^{n} w_i = 1$, which is called the (*normalized*) *principal eigenvector* of A. It holds

$$Aw = \rho(A)w \tag{37}$$

Since the weight w_i is interpreted as the relative importance of alternative x_i, the alternatives $x_1, x_2, \dots, x_n$ in X are ranked by their relative importance, or, they are ordered by their magnitude. The following important result has been derived in [27].

Theorem 6.2. Let $A = \{a_{ij}\}$ be an $n \times n$ positive m-reciprocal matrix. Then $I_{mc}(A) \geq 0$ and A is m-consistent if and only if $I_{mc}(A) = 0$.

To provide a (in)consistency measure independently of the dimension of the matrix, n, T. Saaty in [27] proposed the m-consistency ratio CR_{mc}. T. Saaty himself called it in [27] consistency ratio. Here, in order to distinguish it from the other consistency measures, we shall call it *m-consistency ratio.* This consistency ratio is obtained by taking the ratio I_{mc} to its mean value R_{mc} over a large number of positive m-reciprocal matrices of dimension n, whose entries are randomly and uniformly chosen, i. e.

$$CR_{mc} = \frac{I_{mc}}{R_{mc}}. \tag{38}$$

For this consistency measure T. Saaty proposed an estimation of 10% threshold for the CR_{mc}. A pairwise comparison matrix should be accepted in decision making process if its m-consistency ratio does not surpass this threshold. In other words, in practical decision making situations, inconsistency is "acceptable" if $CR_{mc} < 0.1$, see [27].

The m-consistency index I_{mc} has been defined by (36) only for m-reciprocal matrices. Now, we shall investigate inconsistency/intransitivity also for a-reciprocal matrices. For this purpose we use relations between m-consistent and a-transitive/a-consistent matrices derived in Proposition 5.4 and Proposition 5.2.

Let us denote

$$\phi(t) = \frac{t}{1+t} \text{ for } t > 0, \tag{39}$$

$$\phi^{-1}(t) = \frac{t}{1-t} \text{ for } 0 < t < 1. \tag{40}$$

Let $B = \{b_{ij}\}$ be an a-reciprocal $n \times n$ matrix with $0 < b_{ij} < 1$ for all i and j. Now, we define the *a-consistency index* $I_{ac}(B)$ of the positive a-reciprocal matrix $B = \{b_{ij}\}$ as follows

$$I_{ac}(B) = I_{mc}(A), \text{where } A = \{\phi^{-1}(b_{ij})\}. \tag{41}$$

From (40), (41) we obtain the following result which is parallel to Theorem 6.2.

Theorem 6.3. Let $B = \{b_{ij}\}$ be an a-reciprocal $n \times n$ matrix with $0 < b_{ij} < 1$ for all i and j. Then $I_{ac}(A) \geq 0$ and B is a-consistent if and only if $I_{ac}(B) = 0$.

Proof. 1. Let B be a-consistent, i. e. m-transitive. By Proposition 5.4 (i), $A = \{\phi^{-1}(b_{ij})\}$ is m-consistent. By Theorem 6.2, $I_{mc}(A) = 0$, hence, by (41), $I_{ac}(B) = I_{mc}(A) = 0$.

2. Let $I_{ac}(B) = 0$. Then by (41), $I_{mc}(A) = 0$, where $A = \{\phi^{-1}(b_{ij})\}$. By Theorem 6.2, A is m-consistent. Now, by Proposition 5.4 (ii), $\{\phi(\phi^{-1}(b_{ij}))\} = \{b_{ij}\} = B$ is m-transitive, i. e. a-consistent. □

Now, we shall deal with measuring a-intransitivity of a-reciprocal matrices. Recall transformation functions φ and φ^{-1} defined by (26), (27), where $\sigma > 1$.

$$\varphi(t) = \frac{1}{2}\left(1 + \frac{\ln t}{\ln \sigma}\right) \text{ for } t \in [1/\sigma; \sigma],$$

$$\varphi^{-1}(t) = \sigma^{2t-1} \text{ for } t \in [0; 1].$$

Let $B = \{b_{ij}\}$ be an a-reciprocal $n \times n$ matrix with $0 < b_{ij} < 1$ for all i and j. We define the a-transitivity index $I_{at}(B)$ of the positive a-reciprocal matrix $B = \{b_{ij}\}$ as follows:

$$I_{at}(B) = I_{mc}(A), \text{where } A = \{\varphi^{-1}(b_{ij})\}. \tag{42}$$

From (27), (42) we obtain the following result which is parallel to Theorem 6.2 and Theorem 6.3.

Theorem 6.4. Let $B = \{b_{ij}\}$ be an a-reciprocal $n \times n$ matrix with $0 < b_{ij} < 1$ for all i and j. Then $I_{at}(A) \geq 0$ and B is a-transitive if and only if $I_{at}(B) = 0$.

Proof. 1. Let B be a-transitive. By Proposition 5.2, $A = \{\varphi^{-1}(b_{ij})\}$ is m-consistent. By Theorem 6.2, $I_{mc}(A) = 0$, hence, by (42), $I_{at}(B) = I_{mc}(A) = 0$.

2. Let $I_{at}(B) = 0$. Then by (42), $I_{mc}(A) = 0$, where $A = \{\varphi^{-1}(b_{ij})\}$. By Theorem 6.2, A is m-consistent. Now, by Proposition 5.3, $\{\varphi(\varphi^{-1}(b_{ij}))\} = \{b_{ij}\} = B$ is a-transitive. □

Let $A = \{a_{ij}\}$ be an a-reciprocal $n \times n$ matrix. In (38), the m-consistency ratio of A denoted by $CR_{mc}(A)$ is obtained by taking the ratio $I_{mc}(A)$ to its mean value $R_{mc}(n)$ over a large number of randomly and uniformly generated positive m-reciprocal matrices of dimension n, that is,

$$CR_{mc}(A) = \frac{I_{mc}(A)}{R_{mc}(n)}. \tag{43}$$

The following table gives the dimension n of the matrix in the first row and corresponding mean value $R_{mc}(n)$ over a large number of randomly generated positive m-reciprocal matrices of dimension n = 3,4,...,10.

n	3	4	5	6	7	8	9	10
$R_{mc}(n)$	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49

Similarly, we define a-consistency ratio and a-transitivity ratio. Let $B = \{b_{ij}\}$ be an a-reciprocal $n \times n$ matrix with $0 < b_{ij} < 1$ for all i and j. We define the a-consistency ratio CR_{ac} of B as follows:

$$CR_{ac}(B) = \frac{I_{ac}(B)}{R_{mc}(n)}. \tag{44}$$

Moreover, we define a-transitivity ratio CR_{at} of B as

$$CR_{at}(B) = \frac{I_{at}(B)}{R_{mc}(n)}. \tag{45}$$

In practical decision making situations, a-inconsistency of a positive a-reciprocal pairwise comparison matrix B is "acceptable" if $CR_{ac}(B) < 0.1$. Also, a-intransitivity of a positive a-reciprocal pairwise comparison matrix B is "acceptable" if $CR_{at}(B) < 0.1$.

7. ILLUSTRATIVE EXAMPLES

Example 7.1. Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of 4 alternatives (products) and let A be a pairwise comparison matrix obtained as the result of comparisons of all pairs x_i , x_j evaluated on the scale $[^1/_{10}; 10]$ by a DM according to the criterion "design",

$$A = \begin{pmatrix} 1 & \frac{5}{3} & \frac{10}{3} & 10 \\ \frac{3}{5} & 1 & 2 & 6 \\ \frac{3}{10} & \frac{1}{2} & 1 & 3 \\ \frac{1}{10} & \frac{1}{6} & \frac{1}{3} & 1 \end{pmatrix}. \tag{46}$$

Here, as it can be easily verified by (24), $A = \{a_{ij}\}$ is m-consistent and m-reciprocal. Then, by Theorem 6.2, $I_{mc}(A) = 0$, hence by (36), $CR_{mc}(A) = 0$.

Also, $w = (0.5; 0.3; 0.15; 0.05)$ is the corresponding vector of weights given by Proposition 5.5 with $A = \{w_i/w_j\}$. Consequently, according to the criterion "design" , the best alternative is x_1, (with the corresponding weight 0.5), the second best is x_2(with 0.3), then x_3(with 0.15), and the worst alternative is x_4(with 0.05).

Now, let $B = \{b_{ij}\} = \{\frac{a_{ij}}{1+a_{ij}}\} = \{\phi(a_{ij})\}$, that is,

$$B = \begin{pmatrix} 0.5 & 0.63 & 0.77 & 0.91 \\ 0.38 & 0.5 & 0.67 & 0.86 \\ 0.23 & 0.33 & 0.5 & 0.75 \\ 0.09 & 0.14 & 0.25 & 0.5 \end{pmatrix}. \tag{47}$$

By Proposition 5.3 (i), B is a-reciprocal and by Proposition 5.4 (ii), B is m-transitive (that is, a-consistent). This fact can be also easily verified directly by (18). Moreover, the above stated vector w satisfies Proposition 5.6 where B is a matrix evaluated by the corresponding additive pairwise comparison relation. By (41), the additive consistency index $I_{ac}(B) = I_{mc}(A) = 0$, where $A = \{\phi^{-1}(b_{ij})\}$, hence by (44), $CR_{ac}(A) = 0$. Similarly, by (42) and (45), for additive transitivity, we obtain $I_{at}(B) = CR_{at}(A) = 0$, where $A = \{\varphi^{-1}(b_{ij})\}$.

Example 7.2. Let us consider the same decision problem as in Example 7.1, with the different DM, who evaluated all pairs x_i, x_j of the pairwise comparison matrix C as follows:

$$C = \begin{pmatrix} 1 & 2 & 3 & 9 \\ 0.5 & 1 & 2 & 6 \\ 0.\overline{3} & 0.5 & 1 & 3 \\ 0.\overline{1} & 0.1\overline{6} & 0.\overline{3} & 1 \end{pmatrix}. \tag{48}$$

Here, as it can be easily verified by (24), $C = \{c_{ij}\}$ is m-inconsistent as $a_{12}a_{23} = 2 \times 2 = 4 \neq 3 = a_{13}$.

The vector of weights $w = (w_1, w_2, w_3, w_4)$, with $w_i > 0$, for all $i = 1, 2, 3, 4$, and satisfying

$$Cw = \rho(C)w \tag{49}$$

is calculated as follows: $w = (0.503; 0.290; 0.155; 0.052)$ with $\rho(C) = \lambda_{max} = 4.01$ and m-consistency ratio $CR_{mc} = 0.0038 < 0.1$. Hence, in this DM situation, m-inconsistency of matrix C is acceptable; in other words, it is slightly m-inconsistent.

Example 7.3. Let $Z = \{z_1, z_2, z_3, z_4\}$ be again a set of 4 alternatives, D be a pairwise comparison matrix: the result of comparisons of all pairs z_i, z_j evaluated on the scale $[0, 1]$ by a DM according to the criterion "comfort" ,

$$D = \begin{pmatrix} 0.5 & 0.4 & 0.6 & 0.7 \\ 0.6 & 0.5 & 0.6 & 0.9 \\ 0.4 & 0.4 & 0.5 & 0.5 \\ 0.3 & 0.1 & 0.5 & 0.5 \end{pmatrix}. \tag{50}$$

Here, $D = \{d_{ij}\}$ is a-reciprocal and it is a-inconsistent, as it may be directly verified by (18); for example, $d_{12}.d_{23}.d_{31} \neq d_{32}.d_{21}.d_{13}$. At the same time, D is a-intransitive because $d_{12} + d_{23} + d_{31} = 1.9 \neq 1.5$.

We can calculate the following matrices

$$E = \{\phi^{-1}(d_{ij})\} = \begin{pmatrix} 1 & 0.67 & 1.5 & 2.33 \\ 1.5 & 1 & 1.5 & 9 \\ 0.67 & 0.67 & 1 & 1 \\ 0.43 & 0.11 & 1 & 1 \end{pmatrix},$$

$$F = \{\varphi^{-1}(d_{ij})\} = \begin{pmatrix} 1 & 0.64 & 1.55 & 2.41 \\ 1.55 & 1 & 1.55 & 5.80 \\ 0.64 & 0.64 & 1 & 1 \\ 0.42 & 0.17 & 1 & 1 \end{pmatrix},$$

where $\sigma = 9$.

We also calculate $\rho(E) = 4.29$, $\rho(F) = 4.15$, and by (36), (44) and (45) we obtain

$CR_{ac}(D) = 0.11 > 0.1$, with the vector of weights $w_E = (0.249; 0.473; 0.177; 0.102)$, and $CR_{at}(D) = 0.0055 < 0.1$, with the vector of weights $w_F = (0.267; 0.435; 0.181; 0.117)$.

As it is evident, a-consistency ratio $CR_{ac}(D) > 0.1$, it is too high that matrix D should be considered "a-consistent". On the other hand, a-transitivity ratio $CR_{at}(D) < 0.1$, it is sufficiently low so that matrix D is considered "a-transitive".

According to the criterion "comfort", the best alternative is z_2 (with the corresponding weight 0.473, resp. 0.435), the second best is z_1 (0.249, resp. 0.267), then z_3 (0.177, resp. 0.181), and the worst alternative is z_4 (with 0.102, resp. 0.117).

8. CONCLUSION

In this paper we have investigated properties of two types of pair comparison systems and their mutuals relations. In particular, we have been interested in several consistency properties of these systems and the ways of measurement of inconsistency. New features and results presented in this paper can be summarized as follows:

- We dealt with the representation of preferences that involves measurement performed by assigning numbers to represent properties of possibly nonnumerical systems by properties of numerical systems. Two basic problems of these measurements were investigated: First, finding conditions under which such assignment is possible, and second, determining the type of uniqueness of the resulting measure. Here, we presented both some results from the literature concerning the set of alternatives X with arbitrary cardinality and results for a more specific situation where X was a denumerable or finite set.

- General pair comparison systems were dealt with. We considered functions that map $X \times X$ into the unit interval $[0, 1]$ of real numbers. We also defined several consistency conditions which were closely related to the standard notion of transitivity.

- Defining function φ by formula (26), we extended the concept of the transformation function introduced by Herrera et al. [18], allowing the Saaty's scale $[{}^1/_9, 9]$ to become $[{}^1/_\sigma, \sigma]$ with $\sigma > 1$. Moreover, by (27) we defined a new transformation functions φ and φ^{-1}. Some properties of equivalence between multiplicative consistency and additive transitivity have been derived in Propositions 5.1 and 5.2.

- The concept of additive consistency (a-consistency) as a counterpart to m-consistency proved to be equivalent to the multiplicative transitivity known from the literature.

- In Proposition 5.3 and Proposition 5.4 we characterized an a-consistent matrix by some transformed m-consistent matrix using a transformation independent of the scale.

- Proposition 5.5 gives a characterization of m-consistent matrix by a vector of weights, i. e. a positive vector with sum of elements equal to one.

- In Proposition 5.6 we gave a characterization of a-consistent matrix by a vector of weights.

- We summarized some results of Perron–Frobenius theory concerning the spectral radius of irreducible matrices applicable both to m-reciprocal and a-reciprocal positive square matrices, i. e. pairwise comparison matrices. In the literature, an inconsistency measure, i. e. inconsistency index, is known only for m-reciprocal matrix. Here we defined the inconsistency index also for a-reciprocal matrices.

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REFERENCES

[1] J. Aguarón and J. M. Moreno-Jimenéz: The geometric consistency index: Approximated thresholds. European J. Oper. Res. *147* (2003), 137–145.

[2] J. P. Boyd: Numerical methods for Bayesian ratings from paired comparisons. J. Quantitative Anthropology *3* (1991), 117–133.

[3] S. Bozóki and T. Rapcsák: On Saaty's and Koczkodaj's inconsistencies of pairwise comparison matrices. WP 2007-1, June 2007, Computer and Automation Research Institute, Hungarian Academy of Sciences, http://www.oplab.sztaki.hu/WP_2007_1_Bozoki_Rapcsak.pdf

[4] S. Brin and L. Page: The anatomy of a large-scale hypertextual web search engine. Comput. Networks and ISDN Systems *30* (1998), 107–117.

[5] G. Crawford and C. Williams: A note on the analysis of subjective judgment matrices. J. Math. Psychol. *29* (1985), 387–405.

[6] F. Chiclana, F. Herrera, and E. Herrera-Viedma: Integrating three representation models in fuzzy multipur-pose decision making nased on fuzzy preference relations. Fuzzy Sets and Systems *97* (1998), 33–48.

[7] F. Chiclana, F. Herrera, and E. Herrera-Viedma: Integrating multiplicative preference relations in a multipur-pose decision making model based on fuzzy preference relations. Fuzzy Sets and Systems *112* (2001), 277–291.

[8] F. Chiclana, F. Herrera, E. Herrera-Viedma, and S. Alonso: Some induced ordered weighted averaging opera-tors and their use for solving group decision-making problems based on fuzzy preference relations. European J. Oper. Res. *182* (2007), 383–399.

[9] F. Chiclana, E. Herrera-Viedma, and S. Alonso: A note on two methods for estimating missing pairwise preference values. IEEE Trans. Systems, Man and Cybernetics – Part B: Cybernetics *39* (2009), 6, 1628–1633.

[10] F. Chiclana, E. Herrera-Viedma, S. Alonso, and F. Herrera: Cardinal consistency of reciprocal preference relations: A characterization of multiplicative transitivity. IEEE Trans. Fuzzy Systems *17* (2009), 1, 14–23.

[11] E. Dopazo and J. Gonzales-Pachón: Consistency-driven approximation of a pairwise comparison matrix. Kybernetika *39* (2003), 5, 561–568.

[12] M. Fiedler, J. Nedoma, J. Ramík, J. Rohn, and K. Zimmermann: Linear Optimization Problems with Inexact Data. Springer, Berlin – Heidelberg – New York – Hong Kong – London – Milan- Tokyo 2006.

[13] P. C. Fishburn: Utility Theory for Decision Making. Wiley, New York 1970.

[14] P. C. Fishburn: Binary choice probabilities: On the varieties of stochastic transitivity. J. Math. Psychol. *10* (1973), 329–352.

[15] J. Fodor and M. Roubens: Fuzzy Preference Modelling and Multicriteria Decision Support. Kluwer, Dordrecht 1994.

[16] S. I. Gass and T. Rapcsák: Singular value decomposition in AHP. European J. Oper. Res. *154* (2004), 573–584.

[17] B. Golany: A multicriteria evaluation methods from obtaining weights from ratio scale matrices. European J. Oper. Res. *69* (1993), 210–220.

[18] E. Herrera-Viedma, F. Herrera, F. Chiclana, and M. Luque: Some issues on consistency of fuzzy preference relations. European J. Oper. Res. *154* (2004), 98–109.

[19] E. Herrera-Viedma, F. Chiclana, F. Herrera, and S. Alonso: Group decision-making model with incomplete fuzzy preference relations based on additive consistency. IEEE Trans. Systems, Man and Cybernetics, Part B – Cybernetics *37* (2007), 1, 176–189.

[20] D. H. Krantz, R. D. Luce, P. Suppes, and A. Tversky: Foundations of Measurement. Vol. I. Academic Press, New York 1971.

[21] M. Mareš: Coalitional fuzzy preferences. Kybernetika *38* (2002), 3, 339–352.

[22] M. Mareš: Fuzzy coalitional structures. Fuzzy Sets and Systems *114* (2000), 3, 23-33.

[23] J. Ramík and P. Korviny: Inconsistency of pairwise comparison matrix with fuzzy elements based on geo-metric mean. Fuzzy Sets and Systems *161* (2010), 1604–1613.

[24] J. Ramík and M. Vlach: Generalized Concavity in Optimization and Decision Making. Kluwer Publ. Comp., Boston – Dordrecht – London, 2001.

[25] F. S. Roberts: Measurement theory: with application to decisionmaking, utility and the social sciences. In: Encyklopedia of Mathematics and its Applications, Vol. 7, Addison-Wesley, Reading 1979.

[26] T. L. Saaty: Fundamentals of Decision Making and Priority Theory with the AHP. RWS Publications, Pittsburgh 1994.

[27] T. L. Saaty: Multicriteria Decision Making – The Analytical Hierarchy Process. Vol. I. RWS Publications, Pittsburgh 1991.

[28] W. E. Stein and P. J. Mizzi: The harmonic consistency index for the analytic hierarchy process. European J. Oper. Res. *117* (2007), 488–497.

[29] T. Tanino: Fuzzy preference orderings in group decision making. Fuzzy Sets and Systems *12* (1984), 117–131.

[30] T. Tanino: Fuzzy preference relations in group decision making. In: Non-Conventional Preference Relations in Decision Making (J. Kacprzyk and M. Roubens, eds.), Springer-Verlag, Berlin 1988, pp. 54–71.

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