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REMARKS ON STAR COUNTABLE DISCRETE CLOSED SPACES

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Abstract. In this paper, we prove the following statements:

(1) There exists a Tychonoff star countable discrete closed, pseudocompact space having a regular-closed subspace which is not star countable.

(2) Every separable space can be embedded into an absolutely star countable discrete closed space as a closed subspace.

(3) Assuming $2^{\aleph_0} = 2^{\aleph_1}$, there exists a normal absolutely star countable discrete closed space having a regular-closed subspace which is not star countable.

Keywords: pseudocompact, normal, Tychonoff, star countable, absolutely star countable, star countable discrete closed, absolutely star countable discrete closed space

MSC 2010: 54D20

1. Introduction

By a space, we mean a topological space. In this section, we give definitions of terms which are used in this paper. Let X be a space and $\mathscr{U}$ a collection of subsets of X. For $A \subseteq X$, let $\mathrm{St}(A, \mathscr{U}) = \bigcup\{U \in \mathscr{U} \colon U \cap A \neq \emptyset\}$. As usual, we write $\mathrm{St}(x, \mathscr{U})$ instead of $\mathrm{St}(\{x\}, \mathscr{U})$.

Definition 1.1 ([1], [2], [3], [15])**.** Let P be a topological property. A space X is said to be *star P* if whenever $\mathscr{U}$ is an open cover of X, there exists a subspace $A \subseteq X$ with property P such that $X = \mathrm{St}(A, \mathscr{U})$. The set A will be called a *star kernel* of the cover $\mathscr{U}$.

Definition 1.2. Let P be a topological property. A space X is said to be *absolutely star P* if whenever $\mathscr{U}$ is an open cover of X and D is a dense subset of X,

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there exists a subspace $A \subseteq X$ with property P such that A is a subset of D and $X = \mathrm{St}(A, \mathscr{U})$.

The term star P was coined in [15] but certain star properties, specifically those corresponding to "$P =$ finite", "$P =$ countable" and "$P =$ countable discrete closed" were first studied by van Douwen et al. in [14] and Yasui et al. in [17], and later by many other authors. The term absolutely star P for "P = finite", "P = countable" and "P = countable discrete closed" were first studied by Matveev in [6], Bonanzinga [4] and Song [10], [11] respectively. A survey of star covering properties with a comprehensive bibliography can be found in [7], [14]. The author believes the terminology from [1], [2], [3], [15] and the terminology used in the paper to be simple and logical. Nonetheless we must mention that the authors of previous works have used many different terms to define properties of this sort. For example, in [7] and earlier [14], a star finite space is called starcompact and strongly 1-starcompact, a star countable space is called star Lindelöf and strongly 1-star Lindelöf; in [8], [9], [17], a star countable discrete closed space is called discretely star-Lindelöf and a space with a countable web; in [6], [7], an absolutely star finite space is called absolutely countably compact; in [4], [7], an absolutely star countable space is called absolutely star Lindelöf, and in [10], [11], an absolutely star countable discrete closed space is called absolutely discretely star-Lindelöf.

From the definitions, it is clear that every star finite space is star countable; every star countable discrete closed space is star countable; every absolutely star countable space is star countable and every absolutely star countable discrete closed space is both absolutely star countable and star countable discrete closed.

In this paper we shall be concerned with property P related to the countable discrete closed property, specifically, "star countable discrete closed" and "absolutely star countable discrete closed". In the paper, spaces are assumed only to be T_1.

Throughout the paper, the cardinality of a set A is denoted by $|A|$. For a cardinal κ, let κ^+ denote the smallest cardinal greater than κ and $cf(\kappa)$ the cofinality of κ. Let $\mathfrak{c}$ denote the cardinality of the continuum, ω_1 the first uncountable cardinal and ω the first infinite cardinal. For a pair of ordinals α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma\colon\ \alpha < \gamma < \beta\}$, $(\alpha, \beta] = \{\gamma\colon\ \alpha < \gamma \leqslant \beta\}$ and $[\alpha, \beta] = \{\gamma\colon\ \alpha \leqslant \gamma \leqslant \beta\}$. Other terms and symbols that we do not define will be used as in [5].

2. Some results on star countable discrete closed spaces

The author [9], [10] showed that there exists a Tychonoff absolutely star countable discrete closed (star countable discrete closed) space having a regular-closed subspace which is not star countable (hence, not star countable discrete closed). However, his space is neither pseudocompact nor normal. First we give a stronger example to show that a regular-closed subspace of a Tychonoff star countable discrete closed, pseudocompact space need not be star countable. The example uses Matveev's space. We now sketch the construction of Matveev's space M_κ defined in [8]. Let κ be an infinite cardinal and let $D = \{0, 1\}$ be a discrete space. For every $\alpha < \kappa$, let z_α be the point of D^κ defined by $z_\alpha(\alpha) = 1$ and $z_\alpha(\beta) = 0$ for $\beta \neq \alpha$. Put $Z = \{z_\alpha \colon \alpha < \kappa\}$. For a given ordinal κ, Matveev's space M_κ is the subspace

$$M_\kappa = (D^\kappa \times \omega) \cup (Z \times \{\omega\})$$

of the product space $D^\kappa \times (\omega + 1)$. Then M_κ is Tychonoff. Matveev [8] showed that M_κ is star countable discrete closed. For a Tychonoff space X, let βX denote the Čech-Stone compactification of X.

Example 2.1. There exists a Tychonoff star countable discrete closed, pseudocompact space X having a regular-closed subspace which is not star countable (hence not star countable discrete closed).

Proof. Let $D = \{d_\alpha \colon \alpha < \mathfrak{c}\}$ be a discrete space of cardinality $\mathfrak{c}$ and let

$$S_1 = (\beta D \times (\mathfrak{c} + 1)) \setminus ((\beta D \setminus D) \times \{\mathfrak{c}\})$$

be a subspace of the product space $\beta D \times (\mathfrak{c}+1)$. Then S_1 is Tychonoff pseudocompact. In fact, it has a countably compact, dense subspace $\beta D \times \mathfrak{c}$. To show that S_1 is not star countable discrete closed, we only show that S_1 is not star countable, since every star countable discrete closed space is star countable. For each $\alpha < \mathfrak{c}$, let

$$U_\alpha = \{d_\alpha\} \times [0, \mathfrak{c}].$$

Let us consider the open cover

$$\mathscr{U} = \{U_\alpha \colon \alpha < \mathfrak{c}\} \cup \{\beta D \times \mathfrak{c}\}$$

of S_1 and let F be any countable subset of S_1. Let $\alpha' = \sup\{\alpha \colon \langle d_\alpha, \mathfrak{c}\rangle \in F\}$. Then $\alpha' < \mathfrak{c}$, since F is countable. If we pick $\beta > \alpha'$, then $\langle d_\beta, \mathfrak{c}\rangle \notin \mathrm{St}(F, \mathscr{U})$, since U_β is the only element of $\mathscr{U}$ containing $\langle d_\beta, \mathfrak{c}\rangle$ and $U_\beta \cap F = \emptyset$, which shows that S_1 is not star countable.

Let

$$S_2 = (\beta M_\mathfrak{c} \times (\omega_1 + 1)) \setminus ((\beta M_\mathfrak{c} \setminus M_\mathfrak{c}) \times \{\omega_1\})$$

be a subspace of the product space $\beta M_\mathfrak{c} \times (\omega_1 + 1)$. Then S_2 is Tychonoff pseudocompact. In fact, it has a countably compact, dense subspace $\beta M_\mathfrak{c} \times \omega_1$. We show that S_2 is star countable discrete closed. To this end, let $\mathscr{U}$ be an open cover of S_2. Since $\beta M_{\mathfrak{c}} \times \omega_1$ is countably compact, there exists a finite subset $E \subseteq \beta M_{\mathfrak{c}} \times \omega_1$ such that

$$\beta M_{\mathfrak{c}} \times \omega_1 \subseteq \mathrm{St}(E, \mathscr{U}),$$

since every countably compact space is star finite (see [13]). On the other hand, $M_{\mathfrak{c}} \times \{\omega_1\}$ is star countable discrete closed, since it is homeomorphic to $M_{\mathfrak{c}}$. Thus there exists a countable subset $E' \subseteq M_{\mathfrak{c}} \times \{\omega_1\}$ such that E' is discrete closed in $M_{\mathfrak{c}} \times \{\omega_1\}$ and

$$M_{\mathfrak{c}} \times \{\omega_1\} \subseteq \mathrm{St}(E', \mathscr{U}).$$

If we put $F = E \cup E'$, then F is a countable subset of S_2 such that $S_2 = \mathrm{St}(F, \mathscr{U})$. Since $M_{\mathfrak{c}} \times \{\omega_1\}$ is closed in S_2, so E' is discrete closed in S_2, hence F is discrete closed in S_2, which shows that S_2 is star countable discrete closed.

We assume $S_1 \cap S_2 = \emptyset$. Let $\pi\colon\ D \times \{\mathfrak{c}\} \to (Z \times \{\omega\}) \times \{\omega_1\}$ be a bijection and let X be the quotient image of the disjoint sum $S_1 \oplus S_2$ by identifying $\langle d_\alpha, \mathfrak{c}\rangle$ of S_1 with $\pi(\langle d_\alpha, \mathfrak{c}\rangle)$ of S_2 for each $\langle d_\alpha, \mathfrak{c}\rangle$ from $D \times \{\mathfrak{c}\}$. Let $\varphi\colon\ S_1 \oplus S_2 \to X$ be the quotient map. Then X is pseudocompact, since S_1 and S_2 are pseudocompact. It is clear that $\varphi(S_1)$ is a regular-close subspace of X which is not star countable (hence not star countable discrete closed).

We shall show that X is star countable discrete closed. To this end, let $\mathscr{U}$ be an open cover of X. Since $\varphi(S_2)$ is homeomorphic to S_2, so $\varphi(S_2)$ is star countable discrete closed, hence there exists a countable discrete closed subset F_1 of $\varphi(S_2)$ such that

$$\varphi(S_2) \subseteq \mathrm{St}(F_1, \mathscr{U}).$$

On the other hand, since $\varphi(\beta D \times \mathfrak{c})$ is homeomorphic to $\beta D \times \mathfrak{c}$, so $\varphi(\beta D \times \mathfrak{c})$ is countably compact, hence there exists a finite subset F_2 of $\varphi(\beta D \times \mathfrak{c})$ such that

$$\varphi(\beta D \times \mathfrak{c}) \subseteq \mathrm{St}(F_2, \mathscr{U}).$$

If we put $F = F_1 \cup F_2$, then F is countable and $X = \mathrm{St}(F, \mathscr{U})$. Since $\varphi(S_2)$ is closed in X, F is discrete closed in X, thus X is star countable discrete closed. $\square$

Remark 2.2. Example 2.1 shows that regular-closed subspaces of Tychonoff star countable discrete closed (star countable), pseudocompact spaces need not be star countable discrete closed (star countable, respectively). The author does not know if a regular-closed subspace of a Tychonoff absolutely star countable discrete closed, pseudocompact space is absolutely star countable discrete closed.

Remark 2.3. For normal spaces, there is no normal star countable discrete closed, pseudocompact space X having a regular-closed subspace which is not star countable discrete closed, since it is well-known that a normal pseudocompact space is countably compact [5] and countable compactness is preserved by a closed subspace.

Example 2.4. There exists a Tychonoff countably compact (hence star countable discrete closed) space X which is not absolutely star countable discrete closed.

Proof. Let $X = \omega_1 \times (\omega_1 + 1)$ be the product space of ω_1 and $\omega_1 + 1$. Then X is Tychonoff countably compact. Hence X is star countable discrete closed, since every countably compact space is star finite and every star finite space is star countable discrete closed.

We will show that X is not absolutely star countable discrete closed. For $\alpha < \omega_1$, let

$$U_\alpha = [0, \alpha) \times (\alpha, \omega_1] \quad \text{and} \quad D = \omega_1 \times \omega_1.$$

Let us consider the open cover

$$\mathscr{U} = \{U_\alpha \colon \alpha < \omega_1\} \cup \{D\}$$

of X and a dense subset D of X. It remains to show that $\mathrm{St}(A, \mathscr{U}) \neq X$ for any countable, closed discrete (in X) subset A of D. To show this, let A be a countable, closed discrete (in X) subset of D. Then A is finite, since X is countably compact. Then $\pi(A)$ is a finite subset of ω_1, where $\pi \colon \omega_1 \times (\omega_1 + 1) \to \omega_1 + 1$ is the projection. Hence there exists an $\alpha' < \omega_1$ such that $A \cap (\omega_1 \times (\alpha', \omega_1]) = \emptyset$. Pick $\beta > \alpha'$. If $\langle \beta, \omega_1 \rangle \in U_\alpha$, then $\alpha > \beta$ and $U_\alpha \cap A = \emptyset$ by the construction of the open cover $\mathscr{U}$. Hence $\langle \beta, \omega_1 \rangle \notin \mathrm{St}(A, \mathscr{U})$, which shows that X is not absolutely star countable discrete closed. $\square$

Remark 2.5. The author does not know if there exists a normal star countable discrete closed space X which is not absolutely star countable discrete closed.

Vaughan [16] proved that every countably compact GO-space is absolutely star finite. Thus, every cardinal with uncountable cofinality is absolutely star finite.

Remark 2.6. The author [9] gave an example showing that the product of a star countable discrete closed space and a compact space need not be star countable (hence not star countable discrete closed). Since ω_1 is absolutely star finite, Example 2.4 shows that the product of an absolutely star finite (hence absolutely star countable discrete closed) space and a compact space need not be absolutely star countable discrete closed. However, the author does not know if the product of a star countable discrete closed (absolutely star countable discrete closed) space and a compact metric space is star countable discrete closed (absolutely star countable discrete closed, respectively).

Next, we give a machine which produces absolutely star countable discrete closed spaces. For a separable space X and its countable dense subset D, we define

$$S(X,D) = X \cup (\kappa^+ \times D), \quad \text{where } \kappa \text{ is regular such that } cf(\kappa) \geqslant |X|$$

and topologize $S(X,D)$ as follows: A basic neighborhood of $x \in X$ in $S(X,D)$ is the set of the form

$$G_{U,\alpha}(x) = U \cup ((\alpha, \kappa^+) \times (U \cap D)),$$

for a neighborhood U of x in X and for $\alpha < \kappa^+$, and a basic neighborhood of $\langle \alpha, x\rangle \in \kappa^+ \times D$ in $S(X,D)$ is the set of the form

$$G_V(\langle \alpha, x\rangle) = V \times \{x\}$$

for a neighborhood V of α in κ^+. When it is not necessary to specify D, we simply write $S(X)$ instead of $S(X,D)$.

Theorem 2.7. *Let X be a separable space with a countable dense set D. Then the space $S(X,D)$ is absolutely star countable discrete closed (star countable discrete closed). Moreover,*

(1) *if X is a Tychonoff space, so is $S(X,D)$;*
(2) *if X is a normal space, so is $S(X,D)$.*

Proof. Put $S = S(X,D)$. We will show that S is absolutely star countable discrete closed. To this end, let $\mathscr{U}$ be an open cover of S. Let S' be the set of all isolated points of κ^+ and let $D' = S' \times D$. Then D' is dense in S and every dense subspace of S includes D'. Thus it is sufficient to show that there exists a countable subset $F \subseteq D'$ such that F is discrete closed in S and $\mathrm{St}(F,\mathscr{U}) = S$. For each $d \in D$, since $\kappa^+ \times \{d\}$ is absolutely star finite, there exists a finite subset $D_d \subseteq S' \times \{d\}$ such that

$$\kappa^+ \times \{d\} \subseteq \mathrm{St}(D_d, \mathscr{U}).$$

Let

$$E_1 = \bigcup\{D_d \colon d \in D\}.$$

Then E is countable, discrete closed in S and

$$\kappa^+ \times D \subseteq \mathrm{St}(E_1, \mathscr{U}).$$

On the other hand, for each $x \in X$ there exists a neighborhood U of x in X and $\alpha(x) < \kappa^+$ such that $G_{U,\alpha(x)}(x)$ is included in some member of $\mathscr{U}$. Since $|X| = \kappa$, we can find $\alpha \in S'$ such that $\alpha > \alpha(x)$ for each $x \in X$. Then the set $E_2 = \{\alpha\} \times D$ is countable, discrete closed in S and

$$X \subseteq \mathrm{St}(E_2, \mathscr{U}).$$

If we put $F = E_1 \cup E_2$, then F is countable discrete closed in S such that $S = \mathrm{St}(F, \mathscr{U})$, which shows that S is absolutely star countable discrete closed. The proof of statement (1) is left to the reader since it is not difficult.

Finally, to prove the statement (2), assume that X is normal. Let A_0 and A_1 be disjoint closed subsets of S. Since X is normal and $\kappa^+ > |X|$, we can find disjoint open subsets U_0 and U_1 of X and $\alpha < \kappa^+$ such that $A_i \cap X \subseteq U_i$ and

$$(U_i \cup ((\alpha, \kappa^+) \times (U_i \cap D))) \cap A_{1-i} = \emptyset$$

for each $i = 0, 1$. Let $X_0 = \kappa^+ \times D$ and let

$$B_i = ((\alpha, \kappa^+) \times (U_i \cap D)) \cup (A_i \cap X_0) \quad \text{for } i = 0, 1.$$

Then B_0 and B_1 are disjoint closed in X_0. Since X_0 is normal, there exist disjoint open subsets V_0 and V_1 in X_0 such that

$$B_0 \subseteq V_0 \quad \text{and} \quad B_1 \subseteq V_1.$$

Let

$$G_0 = U_0 \cup V_0 \quad \text{and} \quad G_1 = U_1 \cup V_1.$$

Then G_0 and G_1 are disjoint open subsets in S such that $A_0 \subseteq G_0$ and $A_1 \subseteq G_1$, which shows that S is normal. □

We have the following corollaries of Theorem 2.7.

Corollary 2.8. *Every separable space can be embedded in an absolutely star countable discrete closed (hence star countable discrete closed) space as a closed subspace.*

Corollary 2.9. *Every Tychonoff space X with $w(X) \leqslant \mathfrak{c}$ can be embedded in a Tychonoff absolutely star countable discrete closed (hence star countable discrete closed) space as a closed subspace.*

Proof. Let X be a Tychonoff space with $w(X) \leqslant \mathfrak{c}$. Then it is known that X can be embedded in a separable Tychonoff space Y as a closed subspace. Indeed, embed X into $[0,1]^{\mathfrak{c}}$ and take a countable dense subset D of $[0,1]^{\mathfrak{c}}$. Then the space Y is obtained from the subspace $X \cup D$ by making each point of $D \setminus X$ isolated. Next consider the space $S(Y)$ defined above. Then $S(Y)$ is absolutely star countable discrete closed by Theorem 2.7 and X is closed in $S(Y)$. □

For a normal space, we have the following consistent example.

Example 2.10. Assuming $2^{\aleph_0} = 2^{\aleph_1}$, there exists an absolutely star countable discrete closed space having a regular-closed subspace which is not star countable.

Proof. Let $Y = L \cup \omega$ be a separable normal, uncountable T_1 space where L is closed and discrete and each element of ω is isolated. See Example E [13] for the construction of such a space. Let

$$S_1 = L \cup (\omega_1 \times \omega)$$

and topologize S_1 as follows: A basic neighborhood of $l \in L$ in S_1 is the set of the form

$$G_{U,\alpha}(l) = (U \cap L) \cup ((\alpha, \omega_1) \times (U \cap \omega))$$

for a neighborhood U of l in X and $\alpha < \omega_1$, and a basic neighborhood of $\langle \alpha, n\rangle \in \omega_1 \times \omega$ in S_1 is the set of the form

$$G_V(\langle \alpha, n\rangle) = V \times \{n\},$$

where V is a neighborhood of α in ω_1. Then S_1 is normal, but it is not star countable (see [12]).

Let

$$S_2 = S(Y, \omega) = Y \cup (\kappa^+ \times \omega).$$

Then S_2 is normal absolutely star countable discrete closed by Theorem 2.7.

We assume $S_1 \cap S_2 = \emptyset$. Let X be the quotient image of the disjoint sum $S_1 \oplus S_2$ by identifying each l in the copy of L in S_1 with the corresponding point l in the copy of L in S_2. Let $\varphi\colon S_1 \oplus S_2 \to X$ be the quotient map. Then X is normal, since S_1 and S_2 are normal. It is clear that $\varphi(S_1)$ is a regular-close subspace of X which is not star countable (hence not absolutely star countable).

We show that X is absolutely star countable discrete closed. To this end, let $\mathscr{U}$ be an open cover of X. Let S' be the set of all isolated points of ω_1 and let $D' = S' \times \omega$. Let S'' be the set of all isolated points of κ^+ and let $D'' = S'' \times \omega$. If we put

$$D = \varphi(D' \cup D''),$$

then D is dense in X and every dense subspace of X includes D. Thus it is sufficient to show that there exists a countable subset $F \subseteq D$ such that F is discrete closed in X and $\mathrm{St}(F, \mathscr{U}) = X$. For each $n \in \omega$, since $\varphi(\omega_1 \times \{n\})$ is absolutely star finite, there exists a finite subset $E_n \subseteq \varphi(S' \times \{n\})$ such that

$$\varphi(\omega_1 \times \{n\}) \subseteq \mathrm{St}(E_n, \mathscr{U}).$$

Let

$$F_1 = \bigcup\{E_n\colon\ n \in \omega\}.$$

Then

$$\varphi(\omega_1 \times \omega) \subseteq \mathrm{St}(F_1, \mathscr{U}).$$

On the other hand, since $\varphi(S_2)$ is homeomorphic to S_2, so $\varphi(S_2)$ is absolutely star countable discrete closed, hence there exists a countable subset F_2 of $\varphi(D'')$ such that F_2 is discrete closed in $\varphi(S_2)$ and

$$\varphi(S_2) \subseteq \mathrm{St}(F_2, \mathscr{U}).$$

If we put $F = F_1 \cup F_2$, then F is countable and $X = \mathrm{St}(F, \mathscr{U})$. Since $F \cap \varphi(\kappa^+ \times \{n\})$ and $F \cap \varphi(\omega_1 \times \{n\})$ are finite for each $n \in \omega$, hence F is discrete closed in X, which shows that X is absolutely star countable discrete closed. □

Remark 2.11. The definition of S_1 in the proof of Example 2.10 is more complicated than it is necessary. In fact, S_1 is the subspace $(Y \times (\omega_1 + 1)) \setminus ((\omega \times \{\omega_1\}) \cup (L \times \omega_1))$ of the product space $Y \times (\omega_1 + 1)$. But, for the convenience of the proof of Example 2.10, we use the definition from [11].

Remark 2.12. Example 2.10 shows that regular-closed subspaces of normal star countable discrete closed (star countable, absolutely star countable, absolutely star countable discrete closed) spaces need not be star countable discrete closed (star countable, absolutely star countable, absolutely star countable discrete closed, respectively) under $2^{\aleph_0} = 2^{\aleph_1}$. The author does not know if there is a ZFC counterexample

Remark 2.13. As far as the author knows, it is open whether there exists a normal star countable space containing an uncountable discrete closed subspace within ZFC. By contrast, we discuss the cardinality of the discrete closed subspaces of normal absolutely star countable discrete closed (star countable discrete closed) spaces. Assuming Martin's axiom and the negation of CH, it is known ([13]) that there exists a separable normal space Y with a closed discrete subset B with $|B| = \kappa$ for $\omega_1 \leqslant \kappa < \mathfrak{c}$. Then, by Theorem 2.7, the space $X = S(Y)$ is a normal absolutely star countable discrete closed (star countable discrete closed) space containing a closed discrete subset B with $|B| = \kappa$. Assuming $2^{\aleph_0} = 2^{\aleph_1}$, let $Y = L \cup \omega$ be the same space Y as in the proof of Example 2.10. Then, by Theorem 2.7, the space $X = S(Y)$ is a normal absolutely star countable discrete closed (star countable discrete closed) space containing an uncountable discrete closed subspace. It is trivial that $2^{\aleph_0} = 2^{\aleph_1}$ implies $\neg$CH. Thus Examples above show the existence of a normal absolutely star countable discrete closed (star countable discrete closed) space containing an uncountable discrete closed subspace under certain set-theoretic assumption. The author does not know if there exists an example within ZFC.

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